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EXPECTED UTILITY WITH SUBJECTIVE EVENTS

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ABSTRACT. We provide a new theory of expected utility with subjective events modeled by a lattice of projections. This approach allows us to capture the notion of a "small world" as a context dependent or local state space embedded into a subjective set of events, the "grand world". For each situation the decision makers' subjective "small world" reflects the events perceived to be relevant for the act under consideration. The subjective set of events need not be representable by a classical state space. Maintaining preference axioms similar in spirit to the classical axioms, we obtain an expected utility representation which is consistent across local state spaces and separates subjective probability and utility. An added benefit is that this alternative expected utility representation allows for an intuitive distinction between risk and uncertainty.

Key words and phrases: Subjective expected utility, Subjective events, Decision making under uncertainty, Uncertainty aversion, Ellsberg's paradox.

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We dedicate this paper to the memory of Birgit Grodal and Karl Vind.

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1. INTRODUCTION

The notion of contextuality in human cognition is a very appealing idea that most people will accept as a guiding principle when it comes to modeling decision making or other cognitive processes. It is therefore surprising that the analytical apparatus commonly used overwhelmingly is tuned to an artificial situation, where human subjects are assumed the ability to accurately perform global analysis before arriving at a conclusion. This is most notably the case when human behavior is modeled using the classical concepts of a state space and a probability distribution.

In [10] the second author explored the use of a lattice of projections to model events and the use of "density matrices" to generate probabilities. It turns out that these techniques allow for a natural way of representing contextuality, and in the present paper, which is based on the earlier papers [8] and [9], we use these ideas to develop a new axiomatic theory of subjective expected utility. A decision maker is only able to make informed decisions based on classical notions of a state space and a probability distribution in a given context. The preferences over acts defined in different contexts are only loosely knit together by few and natural conditions. Nevertheless, the surprising mathematical tool contained in Gleason's theorem allows for a common description in term of a single density matrix generating each probability distribution in every given context.

We assume, similar to Savage, that the decision maker facing the "grand world", for each group of related decisions, creates a "small world" or local state space of only those events which are considered relevant in the given context. This may be interpreted as a cognitive process, where, before a decision is taken, it is grouped together with other decisions in a small and more manageable world. Events belonging to a local state space are in our model only risky, while Knightian uncertainty is related to the comparison of events across different local state spaces. In each local state space we rely on the axioms in [4] together with an additional axiom which lead to a Savage-type expected utility description. The application of Fishburn's axioms is not crucial to the theory, but Fishburn's generalization of the Savage theory ensures that the decision maker is able to make decisions by taking into account objective probabilities in the same way as suggested in [28] for game theory and only assign subjective probabilities to the (local) states on which the acts are defined. In order to lift the local resolutions of utility and probability to the grand world and there obtain a description in term of a single density matrix, we introduce two additional axioms which put relatively mild restrictions on preferences across local state spaces.

An important innovation is that we allow the decision maker to create and use a subjective set of events that do not (necessarily) have counterparts in the physical world. The underlying idea is that the decision maker creates local state spaces from a (global) set of subjective events which may be unaccessible to direct measurements, see also [17]. The proposition that a decision maker creates small worlds from the set of all possible events was already discussed in [23]. Savage however only considered "small worlds" created by the partitioning of a state space, and this way of integrating the local worlds into the grand world tacitly put assumptions on the decision maker's preferences that essentially lead to a description not different from what is obtained by using a single state space with a probability distribution.

Both the classical state space formalism and our theory based on a subjective lattice of events lead to expected subjective utility theories with separation between subjective probability and utility. But it is important to realize that the separation only applies to models satisfying the full set of specified axioms. In contrast, if one considers an experiment with only a limited number of acts to consider, then both frameworks allow non-equivalent models consistent with the preferences revealed in the coarse experiment. An implication of this observation is that additional questions put to the decision maker in a coarse experiment may be answered in non-unique ways without compromising consistency. Additional questions may therefore lead to non-isomorphic models with different resolutions of subjective probability and utility that still are consistent with the preferences revealed in the initial (coarse) experiment.

One can view our theory as a way to capture Savage's notion of a small world in a way flexible enough to allow for the introduction of a notion of uncertainty aversion. One may alternatively view our paper as a generalization of existing models of state-dependent utilities where, as is pointed out in [24], local or small world acts can be ranked in a consistent manner in the grand world by multiplication by suitable constants.

This phenomenon may be even more pronounced in our model as demonstrated in Example 6.1 where the two-color Ellsberg experiment is studied. In this example we present different assignments of projections to events all leading to an accurate representation of the observed preferences, but with different subjective probabilities and different levels of uncertainty aversion. This finding may be attributed to the lack of information provided by a coarse experiment that does not reveal all aspects of the decision makers preferences.

The standard expected utility model is presented in section 2, and the notion of an event space is introduced in section 3. The preference relations in the "grand world" are discussed in subsection 3.2. In section 4 the main representation result is proved and a measure of uncertainty aversion is introduced. We reconsider the Ellsberg paradox, in our framework, in section 5. Finally, we compare the approach taken in this paper with the literature in section 6.

2. THE STANDARD EXPECTED UTILITY MODEL

The standard subjective expected utility model is well-known to most readers, but since the underlying assumptions come in slightly different versions we shall take the effort to specify the axioms underpinning our use of the model. Here we rely on Fishburn's rendition of the Luce-Krantz axioms for two reasons. First, we make sure that a decision maker uses the utility function provided by the subjective expected utility theorem to evaluate also objective lotteries not associated with acts. This is nicely provided for in Fishburn's setup and is used in our analysis across local state spaces. Secondly, Fishburn's setup elucidates the non-uniqueness of the standard model in fairly general situations.

Definition 2.1. An *act* (basic act) is a measurable map $x\colon \Omega \to C$ defined on a state space Ω equipped with a σ-algebra $\mathcal{E}$, where C is the set of consequences. The elements in $\mathcal{E}$ are called events, and the set of non-empty events is denoted by $\mathcal{E}'$.

The set of consequences is equipped with an affine structure and is convex.

Definition 2.2. We consider for any consequence $c \in C$ and any event A the *constant act* c defined by setting $c(A) = c$ for every event $A.$

Some authors see it as a problem if there are too many constant acts. The reason is that some consequences may be so dire, that it is inconceivable that they may be chosen regardless of the obtaining events. These kind of considerations will be ignored and may at most limit the usage of the theory.

Definition 2.3. A convex combination of (basic) acts $x_1, \dots, x_n$ given by

$$x(s) = \sum_{i=1}^{n} t_i x_i(s),$$

where $t_i \geq 0$ and $t_1 + \cdots + t_n = 1$, is called a *mixed act.* The factors t_i are sometimes interpreted as probabilities.

Convex combinations of mixed acts are again naturally interpreted as mixed acts. The set of mixed acts is a mixture set in the sense of [11]. A basic act x may be thought of as a mixed act that assigns probability 1 to x.

Definition 2.4. A mixed conditional act $x|_A$ is the restriction $x\colon A \to C$ of a mixed act x to an event $A \in \mathcal{E}'$.

Let X denote a non-empty convex set of mixed acts (here we simplify Fishburns's model slightly). The primary datum in Fishburn's version of the standard model is a binary preference relation $\succeq$ over $L = \{x|_A \mid x \in X,\ A \in \mathcal{E}'\}$ that satisfies the following axioms:

(i) **Totality**: For all $x|_A$ and $y|_B$ we have either $x|_A \succeq y|_B$ or $y|_B \succeq x|_A$.
(ii) **Transitivity**: If $x|_A \succeq y|_B$ and $y|_B \succeq z|_G$ then $x|_A \succeq z|_G$.

A total and transitive order relation is also called a *weak ordering*.

(iii) **Archimedean continuity**: The sets

$$\{t \in [0,1] \mid (tx + (1-t)y)|_A \succeq z|_B\} \quad \text{and} \quad \{t \in [0,1] \mid z|_B \succeq (tx + (1-t)y)|_A\}$$

are closed for arbitrary $x, y, z \in X$ and $A, B \in \mathcal{E}'$.

(iv) **Mixture indifference**: If $x|_A \sim z|_B$ and $y|_A \sim w|_B$ then

$$\frac{1}{2}x|_A + \frac{1}{2}y|_A \sim \frac{1}{2}z|_B + \frac{1}{2}w|_B$$

for arbitrary $x, y, z, w \in X$ and $A, B \in \mathcal{E}'$.

(v) **Averaging condition**: If $A \cap B = \emptyset$ and $x|_A \succeq x|_B$ then

$$x|_A \succeq x|_{A\cup B} \succeq x|_B$$

for $x \in X$ and $A, B \in \mathcal{E}'$.

(vi) **Non-degeneracy**: There exist $x, y \in X$ such that $x \succ y$.

(vii) **Weak act richness**: If $A \cap B = \emptyset$ then

$$x|_A \succ x|_B \quad \text{and} \quad y|_B \succ y|_A$$

for some acts x and y in X.

(viii) **Strong act richness**: If A, B and C are mutually disjoint, and if there is an act $x \in X$ such that $x|_A \sim x|_B$ then there is an act $y \in X$ such that exactly two of the acts $y|_A$, $y|_B$, and $y|_C$ are equivalent.

It is a main feature of the model that the decision maker only need to have preferences over a restricted set X of mixed acts and their restrictions to the non-empty events. The state space may be finite, and the set of events $\mathcal{E}$ may be a "small" σ-algebra on the state space.

Theorem 2.1 (Fishburn 1973)**.** *Assume that the axioms (i) through (viii) are satisfied. Then there exists a map $u\colon L \times \mathcal{E}' \to \mathbf{R}$ and for each $A \in \mathcal{E}'$ a finitely additive probability measure P_A on $\{A \cap B \mid B \in \mathcal{E}\}$ such that*

(i) $x|_A \succ y|_B$ *if and only if* $u(x|_A) > u(y|_B)$

for all acts $x|_A$ and $y|_B$ in $L \times \mathcal{E}'$.

(ii) $x \to u(x|_A)$ is a linear function on X for each $A \in \mathcal{E}'$.

(iii) $P_C(A) = P_C(B)P_B(A)$,

whenever $A \subseteq B \subseteq C$ for $A \in \mathcal{E}$ and $B, C \in \mathcal{E}'$.

(iv) $u(x|_{A\cup B}) = P_{A\cup B}(A)u(x|_A) + P_{A\cup B}(B)u(x|_B)$

whenever $x \in X$, $A, B \in \mathcal{E}'$, and $A \cap B = \emptyset$.

The map u is uniquely defined up to an increasing affine transformation, and the probability measures P_A are uniquely defined for each $A \in \mathcal{E}'$.

The statement in (iv) is extended by induction to

$$u(x) = u(x|_\Omega) = \sum_{j=1}^{n} u(x|_{A_j})P(A_j)$$

for a (mixed) act x and a finite partition $A_1, \dots, A_n$ of Ω with each $A_j \in \mathcal{E}'$. If in addition $x = \sum_{i=1}^m \lambda_i x_i$ we obtain from (ii) the formula

$$u\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{n} u(x_i|_{A_j})P(A_j). \tag{2.1}$$

This is more flexible than in Savage's theory. If for example $c \in X$ is the constant act with consequence $c \in C$ then

$$u(c) = \sum_{j=1}^{n} u(c|_{A_j})P(A_j). \tag{2.2}$$

We can therefore model that the constant act of getting an umbrella is more utile when it is raining than otherwise. But we retain the attractive property, to be used later, that the utility of an unconditional constant act is the subjectively weighted average of utilities of the corresponding conditional constant acts. We will eventually add two more axioms to Fishburn's list. The first is straightforward although controversial in some settings.

(ix) **Richness of constant acts**: The set of acts X contains the constant act c associated with each consequence $c \in C$.

Fishburn's model allows different acts to be subjectively indistinguishable to the decision maker. Consider an (unconditional) act x with finite many consequences $c_1, \dots, c_n$ and set $A_j = \{s \in \Omega \mid x(s) = c_j\}$ for $j = 1, \dots, n$. The corresponding constant acts (also denoted by $c_1, \dots, c_n$) are in X by axiom (ix), hence the mixed act

$$\tilde{x} = \sum_{j=1}^{n} P(A_j)c_j$$

is also in X, since X is a convex set. The two acts x and $\tilde{x}$ are subjectively indistinguishable to the decision maker. Indeed, $\tilde{x}$ is an objective lottery between the consequences $c_1, \dots, c_n$ with probabilities $P(A_1), \dots, P(A_n)$, and this is exactly how x is perceived by the decision maker who subjectively assigns the same probabilities $P(A_1), \dots, P(A_n)$ to the events $A_1, \dots, A_n$ with outcomes $c_1, \dots, c_n$.

It seems natural to assume that the decision maker is indifferent between two acts which are subjectively indistinguishable.

(x) **Equivalence**: Subjectively indistinguishable acts are equivalent.

It is worthwhile to discuss whether such a condition is behavioral or functional. We would argue that it is behavioral since the decision maker knows by his own perceptions whether two given acts are indistinguishable. It is only an outside observer that need to calculate probabilities before it can be established analytically whether two acts are subjectively indistinguishable to the decision maker.

The equivalence axiom (x) states that the two acts x and $\tilde{x}$ considered above are equivalent. The utility of x is given by

$$u(x) = \sum_{j=1}^{n} u(x|_{A_j})P(A_j)$$

according to (2.1), and the utility of $\tilde{x}$ is given by

$$\begin{aligned}
\tilde{u}(x) &= \sum_{j=1}^{n} u(\tilde{x}|_{A_j})P(A_j) \\
&= \sum_{j=1}^{n} u\left(\sum_{i=1}^{n} P(A_i)c_i\Big|_{A_j}\right)P(A_j) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} P(A_i)u(c_i|_{A_j})P(A_j) \\
&= \sum_{i=1}^{n} u(c_i)P(A_i),
\end{aligned}$$

where we first used the linearity (ii) and then (2.2). The equivalence axiom thus leads to the formula

$$u(x) = \sum_{i=1}^{n} u(c_i)P(A_i). \tag{2.3}$$

But this is exactly Savage's expected utility function where the (state independent) utility of consequences are weighted with the subjective probabilities of the events leading to the consequences. Fishburn [4] shows that the subjective probabilities are not necessarily uniquely determined if the strong act richness axiom $(viii)$ is dropped from the list. We finally introduce the following axiom that only serves to facilitate subsequent proofs.

(xi) **Certainty equivalent**: To each act in X there is an equivalent constant act.

3. SUBJECTIVE EVENTS AS A LATTICE OF PROJECTIONS

We are now ready to introduce the subjective set of events which we model as a lattice of projections. We demonstrate that a lattice of projections satisfies the same logical rules one naturally associates with the hierarchy of events.

3.1. The subjective event space.

Definition 3.1 (Subjective event space)**.** An event space is a pair $(\mathcal{F}, H)$ of a (separable) Hilbert space H and a family $\mathcal{F}$ of projections on H satisfying:

(i) The zero projection on H (denoted 0) and the identity projection on H (denoted 1) are both in $\mathcal{F}$.

(ii) $1 - P \in \mathcal{F}$ for arbitrary $P \in \mathcal{F}$.

(iii) The minorant projection $P \wedge Q \in \mathcal{F}$ for arbitrary $P, Q \in \mathcal{F}$.

(iv) $\sum_{i\in I} P_i \in \mathcal{F}$ for any family $(P_i)_{i\in I}$ of mutually orthogonal projections in $\mathcal{F}$.

- The family $\mathcal{F}$ inherits the natural (partial) order relation $P \le Q$ for projections on a Hilbert space. Notice that $0 \le P \le 1$ for arbitrary events $P \in \mathcal{F}$.
- We define a bijective mapping $P \to P^\perp$ of $\mathcal{F}$ onto itself by setting $P^\perp = 1 - P$. The event $P^\perp$ is called the event complementary to P.
- The minorant projection $P \wedge Q$ is the projection on the intersection of the ranges of P and Q. It has the property that $R \le P \wedge Q$ for any event $R \in \mathcal{F}$ such that both $R \le P$ and $R \le Q$.

  The majorant projection $P \vee Q$ is the projection on the closure of the sum of the ranges of P and Q. It has the property that $P \vee Q \le R$ for any event $R \in \mathcal{F}$ with

$P \leq R$ and $Q \leq R$. Since

$$P \vee Q = 1 - (1 - P) \wedge (1 - Q)$$

it follows that $\mathcal{F}$ is closed also under majorant formation.

- Condition (iv) in the definition is a technical requirement which ensures that $\mathcal{F}$ is closed under arbitrary formation of minorants or majorants. The condition corresponds to the requirement that a σ-algebra is complete. Thus to any family $(P_i)_{i \in I}$ of events in $\mathcal{F}$ there is a minorant event $\wedge_{i \in I} P_i$ and a majorant event $\vee_{i \in I} P_i$ both contained in $\mathcal{F}$.

A subjective event space possesses a number of properties that are natural even crucial in any representation of events.

- An event space contains the projections 0 and 1 corresponding respectively to the vacuous (empty) event and the universal (sure) event.
- There is a partial order relation $\leq$ defined in $\mathcal{F}$ such that any event $P \in \mathcal{F}$ is placed between the vacuous and the universal events, that is $0 \leq P \leq 1$. More generally, for two events P and Q in $\mathcal{F}$ we consider Q to be a larger, more comprehensive event than P if $P \leq Q$. This corresponds to the statement $A \subseteq B$ for measurable subsets A and B of a state space. The interpretation is that we know for sure that the event Q occurs (obtains) if P occurs.
- The joining of two events P and Q in $\mathcal{F}$ is represented by the projection $P \wedge Q$ and the union is represented by the projection $P \vee Q$, and these are both included in the event space $\mathcal{F}$. We express this by saying that $\mathcal{F}$ is a lattice. It follows from (iv) that $\mathcal{F}$ is even closed under the joining or union of arbitrary families of events.

The bijective mapping $P \to P^\perp = 1 - P$ of $\mathcal{F}$ which associates an event with its complementary event has the following natural properties:

- $P \leq Q \Rightarrow Q^\perp \leq P^\perp \quad$ for all $P, Q \in \mathcal{F}$.
  (more comprehensive events have smaller complementary events)
- $P \wedge P^\perp = 0 \quad$ for all $P \in \mathcal{F}$.
  (the joining between an event and its complementary event is the empty event)
- $P \vee P^\perp = 1 \quad$ for all $P \in \mathcal{F}$.
  (the union between an event and its complementary event is the sure event)
- $P^{\perp\perp} = P \quad$ for all $P \in \mathcal{F}$.
  (the event complementary to the complementary event to an event is the event itself)

Suppose that the complementary event to a given event Q is more comprehensive than another event P, meaning that if P obtains then so does the complement to Q. If the events are represented by projections (here also denoted by P and Q) on a Hilbert space H, then the condition is equivalent to the requirement $P \leq 1 - Q = Q^\perp$ which means that the ranges of P and Q are orthogonal subspaces of H. For this reason it becomes natural to say that such events are orthogonal.

Definition 3.2. We say that events $P, Q \in \mathcal{F}$ are mutually exclusive if the minorant $P \wedge Q = 0$, and we say that P and Q are orthogonal if $P \leq Q^\perp$.

Note that the definition is symmetric in P and Q, that is $P \leq Q^\perp$ if and only if $Q \leq P^\perp$.

It readily follows that orthogonal events are mutually exclusive. However, it may happen that mutually exclusive events are not orthogonal. It is exactly because of this possibility, that a subjective event space generally differs from a state space. It is demonstrated in [10] that every state space with a σ-algebra is (under very mild conditions) isomorphic to an event space. Furthermore, an event space is isomorphic to a state space with a σ-algebra (satisfying the same mild conditions as in the first result), if and only if mutually exclusive events are orthogonal.

Note that the multiplicative structure plays no direct role in the theory, cf. [10, Theorem 4.3]. Therefore, if an event space only contains commuting projections then it is isomorphic to a state space with a σ-algebra. On the other hand, if an event space contains non-commuting projections then it cannot be associated with a state space. In the remainder of the paper we assume, to avoid unnecessary technical difficulties, that the Hilbert space H is of finite dimension. This corresponds to assuming a finite state space in the standard model.

Given a subjective event space, a local state space or context is a subdivision of the sure event into the risky events which are pertinent for a particular set of acts. As such it fits neatly into Savage's concept of "neglecting some distinctions between states".

Definition 3.3. A local state space is a set of projections $\{P_1, \ldots, P_n\}$ in $\mathcal{F}$ which satistfy the condition $P_1 + \cdots + P_n = 1$, where 1 denotes the identity projection representing the sure event. The projections in a local state space are thus orthogonal. The set of local state spaces (or small worlds) is denoted by $P(H)$.

The events (projections) given by a local state space as specified above are mutually exclusive and their majorant event is the sure event. Therefore exactly one of these events obtains. The events specified in a local state space thus serve as local states and the obtaining event is the "true state of nature". In this way $P(H)$ becomes a collection of state spaces, each describing a certain part of the subjective event space $(\mathcal{F}, H)$.

3.2. **Preferences with a subjective event space.** Let C be a common set of consequences. We consider, for each local state space $\alpha \in P(H)$, a set L_α of (local) acts defined in α with consequences in C. The set of (global) acts is then defined by setting

$$L = \bigcup_{\alpha \in P(H)} L_\alpha$$

and the "grand world" preferences are specified by a weak order relation on L. The totality and transitivity conditions earlier considered only in Fishburn's model (and in any other model based on a state space formalism) are thus extended to the event space L.

(A) **Totality**: For any pair of acts (α, f) and (β, g) in L we have either $(\alpha, f) \succeq (\beta, g)$ or $(\beta, g) \succeq (\alpha, f)$.
(B) **Transitivity**: If $(\alpha, f) \succeq (\beta, g)$ and $(\beta, g) \succeq (\gamma, h)$ for acts in L, then $(\alpha, f) \succeq (\gamma, h)$.

Every act in L is local in the sense that it belongs to a specific local state space but the preference relation $\succeq$ is given over L. We indicate that a constant act corresponding to a consequence $c \in C$ is defined relative to a local state space $\alpha \in P(H)$ by writing (α, c). Notice that one may consider an objective lottery with consequences $c = (c_1, \ldots, c_n)$ and probabilities $p = (p_1, \ldots, p_n)$ as a mixed act in α and denote it by $(\alpha, (c, p))$.

We assume that the restriction of $\succeq$ to each set of (local) acts L_α satisfies the axioms (i) through (xi). Notice that the totality and transitivity axioms (i) and (ii) are already satisfied by restricting the conditions (A) and (B) to a set of local acts. Since every act is equivalent to a constant act by the certainty equivalent axiom (xi) and constant acts are totally ordered we realize that the totality axiom (A) is redundant in this situation. We choose to maintain the axiom for clarity and as a preparation for future developments where the certainty equivalent axiom may be relaxed.

We introduce two new axioms for the preferences in the "grand world".

(xii) **Indifference**: Let $(\alpha, (c, p))$ and $(\alpha, (d, q))$ be lotteries between constant acts in a local state space $\alpha \in P(H)$. Then

$$(\alpha, (c, p)) \succeq (\alpha, (d, q)) \quad \Rightarrow \quad (\beta, (c, p)) \succeq (\beta, (d, q))$$

for any other local state space $\beta \in P(H)$.

The axiom states that the ordering of lotteries of constant acts does not depend on the local state space in which they are considered. It may be interpreted as the requirement that an objective lottery should be equally attractive, independent of the context in which it is available.

(xiii) **Separation**: Let $\alpha, \beta \in P(H)$ be "small worlds" with a common event $P \in \alpha \cap \beta$. There exist equivalent actions (α, f) and (β, g) in L and non-equivalent consequences $a, b \in C$ such that

$$f(P) \sim g(P) \sim a \quad \text{and} \quad f(Q) \sim g(R) \sim b.$$

for every $Q \in \alpha \backslash \{P\}$ and $R \in \beta \backslash \{P\}$.

In the axiom the common event P functions as a local state in both α and β. The equivalent actions (α, f) and (β, g) may be interpreted as two bets, one in each of the two local states spaces, on the local state P. If P obtains then both bets have outcomes equivalent to consequence a. If P does not obtain then both bets have outcomes equivalent to consequence b.

4. EXPECTED UTILITY

Before we state and prove the main result, we proceed by demonstrating that the introduced axioms give rise to a common utility function across all local state spaces. We also demonstrate that the decision maker assigns subjective probability in a consistent way across different state spaces.

4.1. **Common utility.** We first note that for each local state space $\alpha \in P(H)$, the preference relation on L induces a preference relation $\succeq_\alpha$ on C by setting

$$c \succeq_\alpha d \qquad \text{if} \quad (\alpha, c) \succeq (\alpha, d)$$

for consequences c and d in C. The indifference axiom entails that all of the order relations $\succeq_\alpha$ induced on C in this way are equivalent. We may therefore suppress the subscript in $\succeq_\alpha$ and just write

$$c \succeq d \qquad \text{if} \quad (\alpha, c) \succeq (\beta, d)$$

for consequences c and d in C, and small worlds $\alpha, \beta \in P(H)$.

Since axioms (i) through (xi) are assumed there exist, for each "small world" $\alpha \in P(H)$, a subjective probability measure E_α and a (local) utility function u_α such that the preferences in L_α are represented by the (local) subjective expected utility function

$$U_\alpha(\alpha, f) = \sum_{i=1}^{n} E_\alpha(P_i) u_\alpha(f(P_i)), \tag{4.1}$$

cf. equation (2.3).

Lemma 4.1. *There exists a common utility function $u : C \to \mathbf{R}$, unique up to an increasing affine transformation, such that*

$$c \succ d \quad \textit{if and only if} \quad u(c) > u(d)$$

for consequences $c, d \in C$. For each $\alpha \in P(H)$ the local utility function u_α is an increasing affine transformation of the common utility function u.

Proof. Consider acts $(\alpha, f), (\alpha, g) \in L_\alpha$ for a small world $\alpha = (P_1, \ldots, P_n)$. Since the subjective expected utility is given by

$$U_\alpha(\alpha, f) = \sum_{i=1}^{n} E_\alpha(P_i) u_\alpha(f(P_i)),$$

we may also consider $U_\alpha(\alpha, f)$ as the expected utility of a lottery between constant acts

$$(\alpha, f(P_1)), \ldots, (\alpha, f(P_n))$$

with probabilities $(E_\alpha(P_1), \ldots, E_\alpha(P_n))$. Since such a lottery is equally attractive in any other context we derive that

$$U_\alpha(\alpha, f) \geq U_\alpha(\alpha, g) \quad \text{if and only if} \quad \sum_{i=1}^n E_\alpha(P_i) u_\beta(f(P_i)) \geq \sum_{i=1}^n E_\alpha(P_i) u_\beta(g(P_i))$$

for any other $\beta \in P(H)$. This means that the function

$$V(\alpha, f) = \sum_{i=1}^n E_\alpha(P_i) u_\beta(f(P_i))$$

also represents the ordering in L_α. Accordingly, u_β is an increasing affine transformation of u_α and we may replace u_β with u_α without changing the ordering in L_β. ∎

4.2. **Subjective probabilities.** It is essential for the theory that a decision maker assigns subjective probability to an event independent of the local state space in which it is considered.

Lemma 4.2. *If two "small worlds" $\alpha, \beta \in P(H)$ share a common event $P \in \alpha \cap \beta$ then necessarily $E_\alpha(P) = E_\beta(P)$, where E_α and E_β are the subjective probability measures, derived from the decision maker's preferences, in each of the two local state spaces.*

Proof. Consider two state spaces $\alpha, \beta \in P(H)$ with a common event $P \in \alpha \cap \beta$. We may write the state spaces on the form

$$\alpha = \{P, Q_1, \ldots, Q_n\} \quad \text{and} \quad \beta = \{P, R_1, \ldots, R_m\}.$$

By the separation axiom there exist equivalent actions (α, f) and (β, g) in L and non-equivalent consequences $a, b \in C$ such that

$$f(P) \sim g(P) \sim a \quad \text{and} \quad f(Q_i) \sim g(R_j) \sim b$$

for $i = 1, \ldots n$ and $j = 1, \ldots, m$. The certainty equivalent axiom (xi) ensures the existence of constant acts (α, c) and (β, d) such that

$$\begin{aligned} u(c) &= U_\alpha(\alpha, c) = U_\alpha(\alpha, f) \\ &= E_\alpha(P) u(f(P)) + \sum_{i=1}^n E_\alpha(Q_i) u(f(Q_i)) \\ &= E_\alpha(P) u(a) + (1 - E_\alpha(P)) u(b) \end{aligned}$$

and similarly

$$u(d) = U_\beta(\beta, d) = U_\beta(\beta, g) = E_\beta(P) u(a) + (1 - E_\beta(P)) u(b).$$

Since the constant acts (α, c) and (β, d) are equivalent by the (global) transitivity axiom (B), we conclude that $u(c) = u(d)$. We have thus written $u(c)$ as two convex combinations of $u(a)$ and $u(b)$. Since $u(a) \neq u(b)$ we conclude that $E_\alpha(P) = E_\beta(P)$. ∎

4.3. **Main theorem.** Lemma 4.2 ensures that we can unambiguously define a function

$$E : \mathcal{F} \to [0, 1]$$

by setting $E(P) = E_\alpha(P)$ for any local state space $\alpha \in P(H)$ containing P. This function has the property that

$$E(P_1) + \cdots + E(P_n) = 1$$

for any sequence $P_1, \ldots, P_n$ of projections in $\mathcal{F}$ with sum $P_1 + \cdots + P_n = 1$. A function with this property is called a frame function, and such functions were intensitively studied by a number of authors [18, 7, 26, 21]. The following remarkable result was conjectured by Mackey and proved by Gleason.

Gleasons' theorem. *Let $\mathcal{F}$ be the event space of projections on a (real or complex) separable Hilbert space H of dimension greater than or equal to three, and let $F : \mathcal{F} \to [0, 1]$ be a frame function. Then there exists a uniquely defined positive semi-definite trace class operator h on H with unit trace such that*

$$F(P) = \mathrm{Tr}(hP)$$

for any $P \in \mathcal{F}$.

Note that a frame function automatically is continuous by Gleason's theorem.

We are now ready to state and prove the main result.

Theorem 4.3. *Let $(\mathcal{F}, H)$ be the event space consisting of all projections on a (real or complex) Hilbert space of finite dimension greater than or equal to three, let C be a common set of consequences, and let L be a set of actions. The primitive datum of the utility theory is a weak ordering $\succeq$ over the set L satisfying the local axioms* (i) *through* (xi) *in each local state space together with the global axioms* (xii) *and* (xiii). *Then there exists a map $u\colon C \to \mathbf{R}$, unique up to an increasing affine transformation, and a positive semi-definite operator h on H with unit trace such that*

$$(\alpha, f) \succ (\beta, g) \quad \textit{if and only if} \quad U(\alpha, f) > U(\beta, g)$$

for arbitrary acts (α, f) and (β, g) in L, where the expected utility function U is defined by setting

$$U(\alpha, f) = \sum_{i=1}^{n} \mathrm{Tr}(hP_i) u(f(P_i))$$

for any act $(\alpha, f) \in L$ where $\alpha = \{P_1, \ldots, P_n\}$.

Proof. We first notice that the (local) subjective expected utility function in (4.1) may be written

$$\begin{aligned} U_\alpha(\alpha, f) &= \sum_{i=1}^{n} E_\alpha(P_i) u_\alpha(f(P_i)) \\ &= \sum_{i=1}^{n} E(P_i) u(f(P_i)) \\ &= \sum_{i=1}^{n} \mathrm{Tr}(hP_i) u(f(P_i)) \\ &= U(\alpha, f), \end{aligned}$$

where we first used Lemma 4.2, the common utility function derived in Lemma 4.1, Gleason's Theorem and the definition of $U(\alpha, f)$ as given in the theorem.

The acts (α, f) and (β, g) in L are by the certainty equivalent axiom (xi) equivalent to constant acts (α, c) and (β, d) respectively, therefore we obtain

$$U(\alpha, f) = U(\alpha, c) = u(c) \quad \text{and} \quad U(\beta, g) = U(\beta, d) = u(d).$$

Suppose first that $(\alpha, f) \succ (\beta, g)$. Since

$$(\alpha, c) \simeq (\alpha, f) \succ (\beta, g) \simeq (\beta, d),$$

we obtain by the global transitivity axiom (B) that $(\alpha, c) \succ (\beta, d)$ and thus

$$U(\alpha, f) = u(c) > u(d) = U(\beta, g).$$

If on the other hand $U(\alpha, f) > U(\beta, g)$ then

$$(\alpha, f) \simeq (\alpha, c) \succ (\beta, d) \simeq (\beta, g),$$

and thus $(\alpha, f) \succeq (\beta, g)$. If they were equivalent we could deduce $u(c) = u(d)$, hence necessarily $(\alpha, f) \succ (\beta, g)$ and the statement follows. ∎

Note that the statement in the main result entails that the indifference axiom for preferences across local state spaces is satisfied. The implication is that this axiom must be satisfied in any expected utility formulation of the given form.

4.4. **Measuring uncertainty aversion.** In this subsection we introduce a numerical measure of uncertainty aversion. With this purpose in mind, consider two events P and Q in an event space $(\mathcal{F}, H)$ and a decision maker with preferences as given in Theorem 4.3. If the number

$$\nu(P, Q) = E(P \vee Q) - (E(P) + E(Q))$$

is positive, this is interpreted as a reflection of the decision maker's uncertainty aversion. We may think of an experiment in which a ball is drawn from an urn with an unknown distribution of red and black balls. The event P represents the drawing of a red ball while the event Q represents the drawing of a black ball. The union (majorant) of the two events $P \vee Q$ is the sure event so $E(P \vee Q) = 1$. The decision maker may assign so low probabilities to the individual events that their sum is less than the probability of the union, and hereby exhibit uncertainty aversion.

Let now $P_1, \ldots, P_n$ be events in $\mathcal{F}$ with no further assumptions and consider the number

$$\nu(P_1, \ldots, P_n) = E(P_1 \vee \cdots \vee P_n) - \sum_{i=1}^{n} E(P_i).$$

This number is obviously less or equal to one and it may be negative. But if the events are part of a local state space, then $P_1 \vee \cdots \vee P_n = P_1 + \cdots + P_n$ and thus $\nu(P_1, \ldots, P_n) = 0$.

Definition 4.1. The number

$$\nu = \sup\{\nu(P_1, \ldots, P_n) \mid P_1, \ldots, P_n \in \mathcal{F},\ n = 1, 2, \ldots\}.$$

is defined as the decision maker's uncertainty aversion.

Note that the decision maker's uncertainty aversion ν satisfies $0 \le \nu \le 1$. It is determined as the largest possible difference between the weight attached to the union and the sum of the weights of the individual events. Note that by focusing on the "worst possible" situation the introduced measure of uncertainty aversion is linked to that of [25].

Proposition 4.4. *Suppose that $\mathcal{F}$ is the event space of all projections on a Hilbert space H, and let h be the positive semi-definite operator (matrix) on H with unit trace such that $E(P) = \operatorname{Tr}(hP)$ for any event $P \in \mathcal{F}$. Then*

$$\nu = 1 - \lambda_{min} \cdot \dim H,$$

where $\dim H$ is the finite dimension of the Hilbert space H and λ_{min} is the minimal eigenvalue of the operator h.

Proof. Consider the expression $\nu(P_1, \dots, P_n)$ for events $P_1, \dots, P_n$. Since E is additive we may without loss of generality assume the majorant event $P_1 \vee \cdots \vee P_n = 1$ and that all the constituent projections are one-dimensional. We may then discard events until all remaining events are needed to maintain the sure event as majorant. In this situation $n = \dim H$ and the remaining events are necessarily projections on a set of basis vectors in H. The supremum is then obtained by choosing a sequence of bases of H with each basis vector converging to an eigenvector for the minimal eigenvalue of h. ∎

If the decision maker's uncertainty aversion $\nu = 0$, then the proposition entails that h is the identity operator on H (the identity matrix) divided by $\dim H$, hence

$$E(P) = \frac{\dim R(P)}{\dim H} \qquad P \in \mathcal{F},$$

where $R(P)$ denotes the range of P. An uncertainty neutral $(\nu = 0)$ decision maker is thus assigning likelihood to an event solely according to the dimension of the representing projection.

5. THE ELLSBERG PARADOX

Below we model Ellsberg's experiment using our framework. We consider two versions, a two-color variation taken from [19] as well as the original three-color thought experiment in [2]. This variation was mentioned already in [15].

5.1. Two-color variation.

There are two urns, denoted urn 1 and urn 2. Each urn contains 100 balls that are either white or black. Urn 1 contains 49 white balls and 51 black balls while Urn 2 contains an unspecified assortment of white and black balls. A ball has been picked randomly from each urn; we call them the 1-ball and the 2-ball, respectively. The colors of the chosen balls have not been disclosed. Now we consider two consecutive choice situations or experiments in which the decision maker must choose either the 1-ball or the 2-ball. After both choices have been made, the color will be disclosed. In the first choice situation, a prize is won if the chosen ball is black. In the second choice situation, the same prize is won if the ball is white.

With this information, most people will chose the 1-ball in the first experiment where the objective probability of winning is 0.51. There is no information available concerning the proportion of balls in urn 2, hence there is objectively complete symmetry between the two colors, white and black. One might therefore expect that most people would choose the 2-ball in the second experiment since the likelihood that the 1-ball is white is less than half. However, it turns out that this does not happen overwhelmingly in actual experiments. The decision maker understands that by choosing the 1-ball, he only has a 49 percent chance of winning. But this chance is "safe" and well understood. The uncertainties incurred are much less clear if the 2-ball is chosen.

The combined likelihood of the two possible outcomes of drawing a ball from urn 2 is considered to be less than one although the two outcomes are mutually exclusive.

We may model this behavior by assigning the event "the 1-ball is black" to the projection P and the event "the 1-ball is white" to the projection $1-P$, where

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The two events are thus understood to be complementary. The matrices

$$\begin{aligned} Q_a &= \begin{pmatrix} a & 0 & a^{1/2}(1-a)^{1/2} \\ 0 & 0 & 0 \\ a^{1/2}(1-a)^{1/2} & 0 & 1-a \end{pmatrix} \\ Q_b &= \begin{pmatrix} b & 0 & b^{1/2}(1-b)^{1/2} \\ 0 & 1 & 0 \\ b^{1/2}(1-b)^{1/2} & 0 & 1-b \end{pmatrix} \end{aligned}$$

are projections for $0 \le a, b \le 1$. We assign the "2-ball is black" event to Q_a and the "2-ball is white" event to Q_b for some a, b with $0 < a, b < 1$ and $a \ne b$. With these assignments the joined event is vacuous, and the union event is the sure event.

Note that Q_a and Q_b are mutually exclusive but not complementary events. In addition, none of the four projections introduced above are related by inclusion. We are therefore not forcing the decision maker to assume that the result of one experiment determines the outcome of the other.

Since the pay-offs are equal in the two experiments the subjective utility is proportional to the subjective likelihood of the outcomes in both experiments.

As already discussed, the likelihood $E(X)$ is calculated by $E(X) = \operatorname{Tr}(hX)$, where h is determined by the decision makers preferences. We use in the example the positive semi-definite unit trace matrix h defined by

$$h = \begin{pmatrix} 0.49 & 0 & -0.2 \\ 0 & 0.25 & 0 \\ -0.2 & 0 & 0.26 \end{pmatrix}.$$

We have $E(P) = 0.49$ and $E(1-P) = 0.51$ as anticipated. In addition, we calculate

$$\begin{aligned} E(Q_a) &= 0.26 + 0.23a - 0.4a^{1/2}(1-a)^{1/2}, \\ E(Q_b) &= 0.51 + 0.23b - 0.4b^{1/2}(1-b)^{1/2}. \end{aligned}$$

If we choose $0 < a < 1/2 \le b < 1$, then by an elementary calculation we obtain

$$E[P] > E[Q_a] \quad \text{and} \quad E[1-P] > E[Q_b].$$

The "1-ball is white" is thus preferred to the "2-ball is white" and the "1-ball is black" is preferred to the "2-ball is black" events as in the experiment. This phenomenon is not possible with a state space description. Different choices of the parameter values a and b (corresponding to different assignments of events to projections) lead in general to non-isomorphic models. This is most easily realized by calculating the uncertainty aversion ν associated with drawing a ball from urn 2 which is given by

$$\nu = \max\{0, E(Q_a \vee Q_b) - E(Q_a) - E(Q_b)\}.$$

A small calculation shows that ν for $0 < a < 1/2 \le b < 1$ may take any value in the interval $[0, c_0]$ where approximately $c_0 = 0.430705$. The maximum value c_0 is obtained in approximately $a = 0.250764$ and $b = 1/2$ with the corresponding subjective probabilities $E(Q_a) = 0.144295$ and $E(Q_b) = 0.425$.

It demonstrates that decision makers with different subjective probabilities and different degrees of uncertainty aversion may well make identical choices in the two-color experiment that cannot possibly reveal all aspects of the decision makers behaviour.

5.2. **Three-color variation.** A decision maker is presented with an urn containing 90 balls. He is told that 30 of the balls are red and the remaining 60 balls are either black or yellow, but he is given no information about the distribution of the black and yellow balls. The decision maker is first asked to state his preferences between three bets, each on the exact color of a single drawn ball. We may consider the bet on the "red ball" as an act where the local state space only contains the pertinent events "red ball" and "not red ball".

The prize is 1 if one wins the bet and 0 otherwise. To simplify further the utility function is chosen as the identity such that the expected utility of a bet on the "red ball" becomes

$$E(R) \cdot 1 + E(1-R) \cdot 0 = E(R)$$

which is simply the expected likelihood $E(R)$ of the associated event R. The same approach is taken to the five other bets.

The decision maker is asked to state his preferences between three bets in which he is given a choice between two colors of a single drawn ball. All six bets pay out the same amounts, conditional on the outcome of the draw. In the first choice situation the decision maker is found to prefer a bet on a "red ball" to a bet on a "black ball" and is indifferent between a bet on a "black ball" and a bet on a "yellow ball", that is

$$\text{Bet}(R) \succ \text{Bet}(B) \sim \text{Bet}(Y). \tag{5.1}$$

In the second choice situation the decision maker is found to prefer a bet on a "black or yellow ball" to a bet on either a "red or yellow ball" or a bet on a "black or red ball", and is indifferent between these two last bets, that is

$$\text{Bet}(B \vee Y) \succ \text{Bet}(R \vee Y) \sim \text{Bet}(B \vee R). \tag{5.2}$$

These preferences display uncertainty aversion in the sense that uncertain events or bets are seen as less attractive.

To model these preferences, we choose an event space with three projections R, B and Y (corresponding to balls of color read, black or yellow respectively) and a likelihood function E such that

$$E(R) > E(B) = E(Y) \quad \text{and} \quad E(B \vee Y) > E(R \vee Y) = E(B \vee R).$$

As the decision maker has exact information about the fraction of the red balls, he considers a bet on the red ball to be a simple lottery described by a probability distribution given the weight 1/3 to the event "the ball is red" and the weight 2/3 to the event "the ball is not red", and this last event is recognized to be the same event as "the ball is either black or yellow". This may be modeled by letting the event "red ball" be represented by the projection R and the event "the ball is not red" or "black or yellow ball" be represented by the projection $1 - R$.

As in the two-color variation, the decision maker is, in the absence of further information, not able to subdivide the "black or yellow ball" event into two single-color events with a probability distribution; they belong to different contexts. We capture this by assigning non-orthogonal projections B and Y to the two events. See Appendix A for a set of projections which may be used. Note that the three single color events have minorant 0, that is

$$R \wedge B = 0, \quad B \wedge Y = 0, \quad R \wedge Y = 0,$$

and the majorant event $B \vee Y = 1 - R$. The three two-color events $B \vee Y$, $R \vee B$ and $R \vee Y$ are thus endogenously given by the lattice operations once the one-color events R, B and Y are specified. In the appendix we choose h such that

$$\begin{aligned} E(R) &= \frac{1}{3} & E(B) &= \frac{1}{6} & E(Y) &= \frac{1}{6} \\ E(B \vee Y) &= \frac{2}{3} & E(R \vee B) &= \frac{1}{2} & E(R \vee Y) &= \frac{1}{2}\,. \end{aligned}$$

In this way we obtain the relations

$$E(R) > E(B) = E(Y) \quad \text{and} \quad E(B \vee Y) > E(R \vee Y) = E(B \vee R),$$

and they accurately reflect the preferences in Ellsberg's paradox.

6. Concluding remarks

The Ellsberg paradox has inspired a substantial literature in axiomatic models of decision making. This literature contains alternative suggestions as to how one can model the appropriate subjective conditions that characterize self-contained local state spaces such that the decision maker's preferences over acts, restricted to any one domain, exhibit probabilistic sophistication. Focus has been on modeling decision making under uncertainty, while at the same time allowing for a clear distinction between risk and uncertainty in the spirit of [16]. See [14] and more recently [31] for comprehensive surveys of this literature. Early contributions include [3] and [22]. In general, this literature has weakened the Savage/Anscombe-Aumann axioms. Some authors have chosen to abandon the Savage axioms - in the case of [27] the notion of totality of preferences - to construct more flexible expected utility models.

6.1. **State space models with non-additive probabilities.** Our paper is obviously related to the influential contribution of [25] which models uncertainty and uncertainty aversion in a state space formalism by assuming that decision makers assign non-additive probabilities to some events as a reflection of uncertainty aversion. By imposing slightly weaker versions of the Anscombe-Aumann axioms on preferences, it is possible to capture preferences towards uncertainty and risk aversion in an expected utility formulation. Clearly, this work demonstrates that it is possible to formulate expected utility theories which capture a notion of uncertainty-aversion while still relying on the use of a state space. Several researchers have applied this type of framework to analyze economic situations. See [20] for a survey of this literature.

The primitive datum in Schmeidler's theory consists of the space space, the acts, and the preferences, and it is the modelers task to specify this datum in such a way that it adequately reflects the problem at hand. In our theory projections are used to model events. They are taken from an infinite source of projections in an event space, but only a few that adequately corresponds to the problem at hand will be considered by the modeler. The acts and the preferences will then by Gleason's theorem, when applicable, provide a unique representation of the likelihood function and define the measure of uncertainty aversion. The basic problem of choosing states in a state space or choosing events (projections) in an event space are similar in nature. Our approach does however have the advantage that it provides an intuitive representation of uncertainty aversion. Secondly, it retains linearity of the grand likelihood functional - even across local state spaces. Our framework also allows for easy generalizations of a given model by adding additional local state spaces.

One may consider our model as a bundle of local state spaces loosely knit together by some global requirements and consistency conditions. Then why do we need the formalism of projections and an event space? The practical reason is that our theory provides an easy way of ensuring consistency across small worlds. Once a model is specified and preferences given by a utility function and a unit trace positive semi-definite matrix as provided for by Gleason's theorem, then we know for sure that the model is consistent.

Our paper is also related to [32] in which a state space model with non-linear capacities relaxes conditions of richness of the state space in [5], or richness of the outcome space in [29, 30]. It is argued that structural restrictions are not mere technical but add content of an unknown nature to models that most naturally have a small finite number of states and outcomes. In our framework the set of consequences is convex and therefore naturally rich. Thus, even though the lattice of events may be small, the preferences must be extendable to the full lattice of projections in order to accommodate Gleason's theorem, which at present has only been considered for the full lattice of projections. As such the method proposed in this paper is more closely related to those suggested by Schmeidler and Gilboa than that of Wakker. We suspect, however, that the different approaches are genuine alternatives. It seems unlikely that they are isomorphic in any precise meaning of the word.

6.2. **State-dependent utilities.** One may interpret our paper as an attempt to generalize models of state-dependent utilities where, as is pointed out in [24], one can ensure that small world acts are ranked in a consistent manner within the grand world by multiplying by suitable constants. In this paper we propose a set of axioms which result in a grand likelihood functional that provides probability distributions in every small world and simultaneously ensures that the ranking of acts is consistent in the grand world.

A more recent paper [12] also discusses preferences that cannot be expressed by state-independent utilities. The author motivates his approach by considering the example of a man who would rather bet on his wife's survival than on her death, even when the probabilities and the pay-offs are the same in the two situations. Axioms are proposed that allow for a situation dependent factor γ that modifies the state independent utilities without compromising the elucidation of subjective probabilities. The example in [12] cannot be described as taking place in a local state space in our model. One may, however, reconcile our approach with the one in [12] by modeling the survival or death of the wife as events taking place in different contexts rather than being complementary events. This would allow for a description, similar to our description of the two urn variation of Ellsberg's paradox, where the total subjective probability of either death or survival is less than one due to uncertainty aversion. Alternatively, one can try to retain the flexibility in Fishburn's model [4] by restricting the application of the equivalence axiom and hence the description in (2.3) to acts where it is meaningful (or acceptable) to the decision maker to separate the consequences from the events leading to the consequences. This is accounted for in Fishburn's setup where the utility of consequences may be conditioned on an event. However in our setup this flexibility is offset by the equivalence axiom (x) to the extent that the decision maker is indifferent between acts with the same set of consequences and associated probabilities. The benefit of axiom (x) is that it allows for a Savage expected utility representation (2.3) which only involves the utility of unconditioned consequences.

6.3. **Models that do not rely on a state space.** To our knowledge, there are only a few papers that do not rely on the explicit presence of a state space. See [6] for a model with subjective distributions that does not rely on a state space. The authors model preferences over acts conditional on bets and assume the existence of an outcome-independent linear utility on bets. Subjective probabilities on outcomes, consistent with expected value maximizing behavior, are then derived. An axiomatic theory of decision making under uncertainty that dispenses with the Savage state space is developed in [13]. A subjective expected utility theory, which does not invoke the notion of states of the world to resolve uncertainty, is formulated. Importantly, this approach does not rule out that decision makers may mentally construct a state space to help organize their thoughts - but it does not require that they do. Thus, the traditional approach may also be embedded into this framework.

In [1] the authors assume a Savage state space, but provide a set of axioms which allow for domains of events that arise endogenously according to the preferences of the decision maker and the manner in which sources of uncertainty are treated. The authors show, given weak assumptions, that preferences restricted to a domain exhibit probabilistic sophistication. This allows for an endogenous formulation of a two-stage approach and a distinction between risk and uncertainty in a setting with a Savage state space. However, as opposed to Savage's formulation, the approach taken is to model decisions as generally taking place at the local state space level, hence leaving the question of consistent extension of decision making across state spaces unanswered.

Finally, our approach has links to discussions of the foundation of quantum physics; in particular to quantum mechanical derivations of probability, cf. [33, 34] for a discussion of possible applications of decision theory in quantum mechanics.

APPENDIX A. PROJECTIONS FOR MODEL OF ELLSBERG PARADOX

$$
\begin{aligned}
R &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad & 1 - R &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
B &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad & Y &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}, \\
R \vee B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad & R \vee Y &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}.
\end{aligned}
$$

$$
h = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/6 & -1/6 \\ 0 & -1/6 & 1/2 \end{pmatrix}.
$$

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