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ON SOME RELATIONS AMONG THE SOLUTIONS OF THE LINEAR VOLTERRA INTEGRAL EQUATIONS WITH CONVOLUTION KERNEL

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ABSTRACT. The sufficient conditions for $y_1(x) \leq y_2(x)$ were given in [1] such that $y_m(x) = f_m(x) + \int_a^x K_m(x,t)y_m(t)dt, (m = 1, 2)$ and $x \in [a, b]$. Some properties such as positivity, boundedness and monotonicity of the solution of the linear Volterra integral equation of the form $f(t) = 1 - \int_0^t K(t-\tau)f(\tau)d\tau = 1 - K * f, (0 \leq t < \infty)$ were obtained, without solving this equation, in [3, 4, 5, 6]. Also, the boundaries for functions $f', f'', \dots, f^{(n)}$, $(n \in \mathbb{N})$ defined on the infinite interval $[0, \infty)$ were found in [7, 8].

In this work, for the given equation $f(t) = 1 - K * f$ and $n \geq 2$, it is derived that there exist the functions $L_2, L_3, \dots, L_n$ which can be obtained by means of K and some inequalities among the functions $f, h_2, h_3, \dots, h_i$ for $i = 2, 3, \dots, n$ are satisfied on the infinite interval $[0, \infty)$, where h_i is the solution of the equation $h_i(t) = 1 - L_i * h_i$ and n is a natural number.

Key words and phrases: Linear Volterra integral equations with convolution kernel, Equivalence theorem, Convolution theorem.

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1. INTRODUCTION

An integral equation of the form

$$f(t) = \phi(t) - \int_0^t K(t-\tau)\, f(\tau)\, d\tau = \phi(t) - K * f \tag{1.1}$$

is known as the second type linear Volterra integral equation with convolution kernel. Here, ϕ is the source term and K is kernel which are the known functions, f is an unknown function, [9, p.23].

The way of obtaining a new equation which is equivalent to (1.1) is given by Theorem A, as follows:

Theorem A. [3, Theorem 1.1.1] *If*
(1) $K \in C^1[0,\infty)$,
(2) ϕ *is locally integrable,*
then (1.1) *is equivalent to*

$$f(t) = \psi(t) - \int_0^t L(t-\tau)\, f(\tau)\, d\tau,$$

where

$$\psi(t) = \phi(t) + \int_0^t g'(t-\tau)\, \phi(\tau)\, d\tau,$$

$$L(t) = g'(t) + ag(t) + \int_0^t g(t-\tau)\, K'(\tau)\, d\tau,$$

where $a = K(0)$, g *is any function such that* $g \in C^1[0,\infty)$ *and* $g(0) = 1$.

The sufficient conditions for obtaining the solution of (1.1) in terms of the solution of the equation

$$g(t) = 1 - \int_0^t K(t-\tau) g(\tau) d\tau = 1 - K * g \tag{1.2}$$

were given by the following Convolution Theorem:

Theorem B. [2, pp. 229-230] *Let the conditions*

$$(1) \quad \phi'(t) \text{ exists for } 0 \le t \le T, \quad \int_0^T |\phi'(t)|\, dt < \infty,\ (T > 0),$$

$$(2) \quad \int_0^T |K(t)|\, dt < \infty$$

hold, then the solution of (1.1) *is given by*

$$f(t) = g(t)\, \phi(0) + \int_0^t g(t-\tau)\, \phi'(\tau) d\tau = g(t)\, \phi(0) + g * \phi',\ (0 \le t \le T), \tag{1.3}$$

where $g(t)$ *is the solution of* (1.2).

Therefore, if g is known, so is f. In the other words, if the properties of g are known, we may obtain certain properties of f by (1.3).

We assume throughout that $t \in [0,\infty)$ and $n \in \mathbb{N}_2 = \{2, 3, \ldots\}$.

2. THE MAIN RESULTS

Theorem C. [3, Theorem 1.2.1] *Let us consider the equation given of the form*

$$f(t) = 1 - \int_0^t K(t-\tau)\, f(\tau) d\tau = 1 - K * f. \tag{2.1}$$

If the conditions

$$\begin{aligned} &(1) \quad K(0) = a < 0, \\ &(2) \quad K'(0) = b, \\ &(3) \quad K \in C^2[0,\infty), K''(t) < 0 \text{ for all } t, \\ &(4) \quad 4b \leq a^2 \end{aligned}$$

hold, then the solution of (2.1) *satisfies the inequalities* $f^{(n)}(t) > 0$ *for* $n = 0, 1, 2, 3$ *and all* t.

Theorem 1. *Let us consider the equation*

$$f_1(t) = 1 - \int_0^t K_1(t-\tau)\, f_1(\tau) d\tau = 1 - K_1 * f_1. \tag{2.2}$$

If

$$\begin{aligned} &(1) \quad K_1 \in C^3[0,\infty), K_1'''(t) \leq 0 \text{ for all } t, \\ &(2) \quad K_1'(0) = a_{11} < 0, \\ &(3) \quad K_1(0) = a_{10}, K_1''(0) = a_{12} \text{ and } b_1(a_{10} - b_1)^2 - 4a_{12} > 0, \\ &\qquad \text{where } b_1 = \frac{1}{3}\left(a_{10} + 2\sqrt{a_{10}^2 - 3a_{11}}\right), \end{aligned}$$

then the solution of (2.2) *satisfies the inequalities*

$$\begin{cases} f_1'(t) + b_1 f_1(t) > 0, \\ f_1''(t) + b_1 f_1'(t) > 0, \\ f_1'''(t) + b_1 f_1''(t) > 0 \end{cases} \tag{2.3}$$

for all t.

Proof. By taking $g(t) = e^{-\gamma t}$ in Theorem A, we get the equivalent equation to (2.2) of the form

$$f_1(t) = e^{-\gamma t} - \int_0^t L_1(t-\tau)\, f_1(\tau) d\tau = e^{-\gamma t} - L_1 * f_1, \tag{2.4}$$

where

$$L_1(t) = (a_{10} - \gamma)e^{-\gamma t} + K_1' * e^{-\gamma t}. \tag{2.5}$$

Thus,

$$L_1'(t) = (\gamma^2 - a_{10}\gamma + a_{11})e^{-\gamma t} + K_1'' * e^{-\gamma t}$$

and

$$L_1''(t) = (-\gamma^3 + a_{10}\gamma^2 - a_{11}\gamma + a_{12})e^{-\gamma t} + K_1''' * e^{-\gamma t}.$$

If we choose $\gamma = b_1 = \left(a_{10} + 2\sqrt{a_{10}^2 - 3a_{11}}\right)/3$, the equality

$$(a_{10} - \gamma)^2 = 4(\gamma^2 - a_{10}\gamma + a_{11}) \tag{2.6}$$

which is equivalent to

$$[L_1(0)]^2 = 4L_1'(0)$$

is satisfied and so, L_1 verifies condition (4) of Theorem C.

Also, since $a_{11} < 0$, the inequality $L_1(0) = a_{10} - \gamma < 0$ holds for $\gamma = b_1$. Namely, L_1 satisfies condition (1) of Theorem C.

On the other hand, by (2.6), it is obtained that the equality

$$-\gamma^3 + a_{10}\gamma^2 - a_{11}\gamma + a_{12} = -\gamma(\gamma^2 - a_{10}\gamma + a_{11}) + a_{12} = -\frac{\gamma(a_{10} - \gamma)^2}{4} + a_{12}$$

holds for $\gamma = b_1$. Thus, $L_1 \in C^2[0,\infty)$ and $L_1''(t) < 0$ for all t by conditions (1) and (3) of Theorem 1.

Therefore, L_1 satisfies all of the conditions of Theorem C. Thus, by Theorem C, the solution of the equation

$$h_1(t) = 1 - \int_0^t L_1(t-\tau)h_1(\tau)d\tau = 1 - L_1 * h_1 \tag{2.7}$$

satisfies the inequalities $h_1^{(n)}(t) > 0, (n = 1, 2, 3)$. By using Theorem B, the solution f_1 of (2.4) can be written by means of h_1 of the form

$$f_1(t) = h_1(t) - \gamma h_1 * e^{-\gamma t}. \tag{2.8}$$

Thus, we have by (2.8) that

$$h_1'(t) = f_1'(t) + b_1 f_1(t) \tag{2.9}$$

which yields

$$h_1''(t) = f_1''(t) + b_1 f_1'(t),\ h_1'''(t) = f_1'''(t) + b_1 f_1''(t). \tag{2.10}$$

Hence, (2.3) holds for all t. This completes the proof. ∎

Example 1. The function K_1 of the form

$$K_1(t) = c_0 t^3 + \left(\frac{a_{12}}{2}\right) t^2 + a_{11} t + a_{10}, (c_0 \le 0) \tag{2.11}$$

corresponding to any constants a_{10}, a_{11}, a_{12} satisfying the inequalities in conditions (2) and (3) of Theorem 1 also satisfies condition (1) of Theorem 1.

For example, if a_{10} and a_{11} are taken as $a_{10} = 1$ and $a_{11} = -1$, then b_1 is found as

$$b_1 = \frac{1}{3}(1 + 2\sqrt{4}) = \frac{5}{3}.$$

Thus, a_{12} satisfying the inequality

$$a_{12} < \frac{b_1(a_{10} - b_1)^2}{4} = \frac{5}{27}$$

can be chosen as $a_{12} = 1/9$. In this case, K_1 is obtained as

$$K_1(t) = c_0 t^3 + \frac{1}{18} t^2 - t + 1, (c_0 \le 0).$$

Theorem 2. *Let us consider the equation given as*

$$f_2(t) = 1 - \int_0^t K_2(t-\tau)\, f_2(\tau) d\tau = 1 - K_2 * f_2. \tag{2.12}$$

If

$$\begin{aligned}
&(1)\quad K_2 \in C^4[0,\infty), K_2^{(4)}(t) \le 0 \textit{ for all } t \textit{ and } K_2^{(i)}(0) = a_{2i},\ (0 \le i \le 3),\\
&(2)\quad a_{20}^2 > 4a_{21},\\
&(3)\quad 4a_{20}a_{21} - 8a_{22} + 2b_2\left(\frac{a_{20}}{2} - b_2\right)^2 - a_{20}^3 > 0,\\
&\qquad \textit{where } b_2 = \frac{a_{20}}{6} + \frac{2}{3}\sqrt{a_{20}^2 - 3a_{21}},\\
&(4)\quad -a_{20}^4 + 4a_{20}^2 a_{21} - 8a_{20}a_{22} + 16a_{23} \le 0,
\end{aligned}$$

then there exists the function L_2 satisfying all of the conditions of Theorem 1 and can be written by means of K_2 such that the inequalities

$$h_2'(t) + b_2h_2(t) > 0, h_2''(t) + b_2h_2'(t) > 0, h_2'''(t) + b_2h_2''(t) > 0$$

and

$$\begin{cases} f_2'(t) + \gamma_2 f_2(t) + b_2h_2(t) > 0,\\ f_2''(t) + (b_2+\gamma_2)f_2'(t) + b_2\gamma_2 f_2(t) > 0,\\ f_2'''(t) + (b_2+\gamma_2)f_2''(t) + b_2\gamma_2 f_2'(t) > 0 \end{cases} \tag{2.13}$$

hold, where h_2 is the solution of the equation $h_2(t) = 1 - L_2 * h_2$ and $\gamma_2 = a_{20}/2$.

Proof. By taking $g(t) = e^{-\gamma t}$ and $\gamma \in \mathbb{R}$ in the Equivalence Theorem, we obtain the equivalent equation of (2.12) as

$$f_2(t) = e^{-\gamma t} - \int_0^t L_2(t-\tau)\, f_2(\tau)d\tau = e^{-\gamma t} - L_2 * f_2, \tag{2.14}$$

where

$$L_2(t) = (a_{20} - \gamma)e^{-\gamma t} + K_2' * e^{-\gamma t}. \tag{2.15}$$

Thus, from (2.15)

$$\begin{aligned}
L_2'(t) &= (\gamma^2 - a_{20}\gamma + a_{21})e^{-\gamma t} + K_2'' * e^{-\gamma t},\\
L_2''(t) &= (-\gamma^3 + a_{20}\gamma^2 - a_{21}\gamma + a_{22})e^{-\gamma t} + K_2''' * e^{-\gamma t}
\end{aligned}$$

and

$$L_2'''(t) = (\gamma^4 - a_{20}\gamma^3 + a_{21}\gamma^2 - a_{22}\gamma + a_{23})e^{-\gamma t} + K_2^{(4)} * e^{-\gamma t}.$$

Now, we show that L_2 satisfies all of the conditions of Theorem 1.

The discriminant of $\gamma^2 - a_{20}\gamma + a_{21} = 0$ is positive by condition (2). So, if γ is chosen as $\gamma = \gamma_2 = a_{20}/2$, then $L_2'(0) = \gamma_2^2 - a_{20}\gamma_2 + a_{21} < 0$, that is, L_2 satisfies condition (2) of Theorem 1. Also, if $L_2(0)$ and $L_2'(0)$ are taken, respectively instead of the constants a_{10} and a_{11} in b_1 of Theorem 1, then

$$\begin{aligned}
\frac{1}{3}\left[L_2(0) + 2\sqrt{[L_2(0)]^2 - 3L_2'(0)}\right] &= \frac{1}{3}\left[a_{20} - \gamma + 2\sqrt{(a_{20}-\gamma)^2 - 3\left(\gamma^2 - a_{20}\gamma + a_{21}\right)}\right]\\
&= \frac{a_{20}}{6} + \frac{2}{3}\sqrt{a_{20}^2 - 3a_{21}} = b_2
\end{aligned}$$

is found for $\gamma = \gamma_2$. Hence, we have the equality

$$\begin{aligned}
4L_2''(0) - b_2\left[L_2(0) - b_2\right]^2 &= 4(-\gamma^3 + a_{20}\gamma^2 - a_{21}\gamma + a_{22}) - b_2\left(a_{20} - \gamma - b_2\right)^2\\
&= -\frac{1}{2}\left[4a_{20}a_{21} - 8a_{22} + 2b_2\left(\frac{a_{20}}{2} - b_2\right)^2 - a_{20}^3\right]
\end{aligned}$$

for $\gamma = \gamma_2$. So, from condition (3) of Theorem 2, the inequality

$$4L_2''(0) - b_2 \left[L_2(0) - b_2\right]^2 < 0$$

holds. This means that L_2 satisfies condition (3) of Theorem 1.

Since

$$\gamma^4 - a_{20}\gamma^3 + a_{21}\gamma^2 - a_{22}\gamma + a_{23} = \frac{1}{2^4}\left(-a_{20}^4 + 4a_{20}^2 a_{21} - 8a_{20}a_{22} + 16a_{23}\right)$$

for $\gamma = \gamma_2$, we get $L_2 \in C^3[0,\infty)$ and $L_2'''(t) \le 0$ for all t by conditions (1) and (4) of Theorem 2. Namely, L_2 also satisfies condition (1) of Theorem 1.

By using Theorem 1, one can see that the solution of the equation

$$h_2(t) = 1 - L_2 * h_2$$

satisfies the inequalities

$$\begin{cases} h_2'(t) + b_2 h_2(t) > 0, \\ h_2''(t) + b_2 h_2'(t) > 0, \\ h_2'''(t) + b_2 h_2''(t) > 0. \end{cases} \tag{2.16}$$

From Theorem B, the solution f_2 of equation (2.14) can be written by means of h_2 as the form

$$f_2(t) = h_2(t) - \gamma_2 h_2 * e^{-\gamma_2 t}. \tag{2.17}$$

Thus, we have by (2.17) that

$$h_2'(t) = f_2'(t) + \gamma_2 f_2(t) \tag{2.18}$$

which yields

$$h_2''(t) = f_2''(t) + \gamma_2 f_2'(t),\ h_2'''(t) = f_2'''(t) + \gamma_2 f_2''(t). \tag{2.19}$$

So, we obtain by (2.16), (2.18) and (2.19) that (2.13) is satisfied for all t. ∎

Example 2. The function K_2 given of the form

$$K_2(t) = c_0 t^4 + \left(\frac{a_{23}}{6}\right) t^3 + \left(\frac{a_{22}}{2}\right) t^2 + a_{21} t + a_{20}, (c_0 \le 0) \tag{2.20}$$

corresponding to any constants a_{20}, a_{21}, a_{22} and a_{23} satisfying the inequalities of conditions $(2) - (4)$ of Theorem 2 also satisfies condition (1) of Theorem 2.

Now, let us try to show that there exist the constants a_{20}, a_{21}, a_{22} and a_{23} satisfying conditions $(2) - (4)$ of Theorem 2. First, let us choose the constants a_{20} and a_{21} verifying the inequality

$$a_{20}^2 > 4a_{21}.$$

Thus, it is seen by the proof of Theorem 2 that the constants a_{10}, a_{11}, b_1 and a_{12} obtained by means of a_{20}, a_{21} and defined by

$$a_{10} = a_{20} - \gamma_2,\ a_{11} = \gamma_2^2 - a_{20}\gamma_2 + a_{21},\ \left(\gamma_2 = \frac{a_{20}}{2}\right),\ b_1 = \frac{1}{3}\left(a_{10} + 2\sqrt{a_{10}^2 - 3a_{11}}\right)$$

and $a_{12} < \dfrac{b_1(a_{10} - b_1)^2}{4}$

satisfy conditions (2) and (3) of Theorem 1.

Since the inequality in condition (3) of Theorem 2 is equivalent to new inequality obtained by taking

$$a_{20} - \gamma_2, \gamma_2^2 - a_{20}\gamma_2 + a_{21}, -\gamma_2^3 + a_{20}\gamma_2^2 - a_{21}\gamma_2 + a_{22} \text{ and } b_2,$$

respectively instead of the constants a_{10}, a_{11}, a_{12} and b_1 in condition (3) of Theorem 1, if a_{12} is chosen as

$$a_{12} = -\gamma_2^3 + a_{20}\gamma_2^2 - a_{21}\gamma_2 + a_{22} = -\gamma_2\left(\gamma_2^2 - a_{20}\gamma_2 + a_{21}\right) + a_{22} = -\gamma_2 a_{11} + a_{22},$$

then a_{22} is found as

$$a_{22} = a_{12} + \gamma_2 a_{11}.$$

In that case, the constants a_{20}, a_{21} and a_{22} hold condition (3) of Theorem 2.

Because condition (4) of Theorem 2 is equivalent to

$$\gamma_2^4 - a_{20}\gamma_2^3 + a_{21}\gamma_2^2 - a_{22}\gamma_2 + a_{23} \leq 0,$$

a_{23} can be chosen as

$$-\gamma_2\left(-\gamma_2^3 + a_{20}\gamma_2^2 - a_{21}\gamma_2 + a_{22}\right) + a_{23} = -\gamma_2 a_{12} + a_{23} \leq 0.$$

That is, $a_{23} \leq \gamma_2 a_{12}$. So, the constants a_{20}, a_{21}, a_{22} and a_{23} satisfy condition (4) of Theorem 2.

Additionally, every function K_2 given by (2.20) also satisfies condition (1) of Theorem 2.

For example, if a_{20} and a_{21} are taken as $a_{20} = -3$ and $a_{21} = 1$, then a_{10}, a_{11} and b_1 are found as

$$a_{10} = a_{20} - \gamma_2 = -\frac{3}{2},\ a_{11} = \gamma_2^2 - a_{20}\gamma_2 + a_{21} = -\frac{5}{4}$$

and

$$b_1 = \frac{a_{10} + 2\sqrt{a_{10}^2 - 3a_{11}}}{3} = \frac{1}{3}\left(2\sqrt{6} - \frac{3}{2}\right).$$

Thus, from Example 1, a_{12} satisfying the inequality

$$a_{12} < \frac{b_1(a_{10} - b_1)^2}{4}$$

can be taken as $a_{12} = 0$.

Hence,

$$a_{22} = a_{12} + \gamma_2 a_{11} = \frac{15}{8}$$

and a_{23} satisfying the inequality

$$a_{23} \leq \gamma_2 a_{12} = 0$$

can be chosen as $a_{23} = -1$.

Therefore,

$$K_2(t) = c_0 t^4 - \frac{1}{6}t^3 + \frac{15}{16}t^2 + t - 3, (c_0 \leq 0).$$

Theorem 3. *Let us consider the equation*

$$(2.21) \qquad f_3(t) = 1 - \int_0^t K_3(t-\tau) f_3(\tau) d\tau = 1 - K_3 * f_3.$$

If

(1) $K_3 \in C^5[0,\infty), K_3^{(5)}(t) \leq 0$ *for all* t *and* $K_3^{(i)}(0) = a_{3i}$, $(0 \leq i \leq 4)$,

(2) $a_{30}^2 > 3a_{31}$,

(3) $144a_{30}a_{31} - 216a_{32} + 6b_3\left(a_{30} - 3b_3\right)^2 - 40a_{30}^3 > 0$,
where $b_3 = \dfrac{1}{9}\left(a_{30} + 2\sqrt{10a_{30}^2 - 27a_{31}}\right)$,

(4) $-112a_{30}^4 + 432a_{30}^2 a_{31} - 864a_{30}a_{32} + 1296a_{33} \leq 0$,

(5) $2a_{30}^5 - 9a_{30}^3 a_{31} + 27a_{30}^2 a_{32} - 81a_{30}a_{33} + 243a_{34} \leq 0$,

then there exist the functions L_2 and L_3 satisfying all of the conditions of Theorem 1 and Theorem 2, respectively and can be written by means of K_3 such that the following inequalities hold for all t:

$$h_2'(t) + b_3h_2(t) > 0,\ h_2''(t) + b_3h_2'(t) > 0,\ h_2'''(t) + b_3h_2''(t) > 0,$$
$$h_3'(t) + \gamma_3h_3(t) + b_3h_2(t) > 0,\ h_3''(t) + (b_3 + \gamma_3)h_3'(t) + b_3\gamma_3h_3(t) > 0,$$
$$h_3'''(t) + (b_3 + \gamma_3)h_3''(t) + b_3\gamma_3h_3'(t) > 0$$

and

$$\begin{cases} f_3'(t) + \gamma_3 \left[f_3(t) + h_3(t)\right] + b_3h_2(t) > 0, \\ f_3''(t) + (b_3 + 2\gamma_3)\, f_3'(t) + (b_3 + \gamma_3)\, \gamma_3 f_3(t) + b_3\gamma_3h_3(t) > 0, \\ f_3'''(t) + (b_3 + 2\gamma_3)\, f_3''(t) + (2b_3 + \gamma_3)\, \gamma_3 f_3'(t) + b_3\gamma_3^2 f_3(t) > 0, \end{cases} \tag{2.22}$$

where h_i for $i = 2, 3$ is the solution of the equation $h_i(t) = 1 - L_i * h_i$ and $\gamma_3 = a_{30}/3$.

Proof. By taking $g(t) = e^{-\gamma t}$ $(\gamma \in \mathbb{R})$ in Theorem A, the equivalent equation of (2.21) is derived as

$$f_3(t) = e^{-\gamma t} - \int_0^t L_3(t-\tau)\, f_3(\tau)d\tau = e^{-\gamma t} - L_3 * f_3, \tag{2.23}$$

where

$$L_3(t) = (a_{30} - \gamma)e^{-\gamma t} + K_3' * e^{-\gamma t}. \tag{2.24}$$

Thus, from (2.24)

$$\begin{aligned} L_3'(t) &= (\gamma^2 - a_{30}\gamma + a_{31})e^{-\gamma t} + K_3'' * e^{-\gamma t}, \\ L_3''(t) &= (-\gamma^3 + a_{30}\gamma^2 - a_{31}\gamma + a_{32})e^{-\gamma t} + K_3''' * e^{-\gamma t}, \\ L_3'''(t) &= (\gamma^4 - a_{30}\gamma^3 + a_{31}\gamma^2 - a_{32}\gamma + a_{33})e^{-\gamma t} + K_3^{(4)} * e^{-\gamma t} \end{aligned}$$

and

$$L_3^{(4)}(t) = (-\gamma^5 + a_{30}\gamma^4 - a_{31}\gamma^3 + a_{32}\gamma^2 - a_{33}\gamma + a_{34})e^{-\gamma t} + K_3^{(5)} * e^{-\gamma t}.$$

Now, we show that L_3 satisfies all of the conditions of Theorem 2.

The inequality corresponding to condition (2) of Theorem 2 is

$$\left[L_3(0)\right]^2 > 4L_3'(0)$$

which is equivalent to

$$(a_{30} - \gamma)^2 > 4\left(\gamma^2 - a_{30}\gamma + a_{31}\right). \tag{2.25}$$

(2.25) is equivalent to

$$3\gamma^2 - 2a_{30}\gamma + 4a_{31} - a_{30}^2 < 0. \tag{2.26}$$

The discriminant of $3\gamma^2 - 2a_{30}\gamma + 4a_{31} - a_{30}^2 = 0$ is $16(a_{30}^2 - 3a_{31})$ which is positive by condition (2) of Theorem 3. So, if γ is chosen as $\gamma = \gamma_3 = a_{30}/3$, then (2.26) holds. Thus, L_3 satisfies condition (2) of Theorem 2.

Also, if $L_3(0)$ and $L_3'(0)$ are taken, respectively instead of the constants a_{20} and a_{21} in b_2 of Theorem 2, then

$$\begin{aligned} \frac{L_3(0)}{6} + \frac{2}{3}\sqrt{\left[L_3(0)\right]^2 - 3L_3'(0)} &= \frac{a_{30} - \gamma}{6} + \frac{2}{3}\sqrt{(a_{30} - \gamma)^2 - 3\left(\gamma^2 - a_{30}\gamma + a_{31}\right)} \\ &= \frac{a_{30}}{9} + \frac{2}{9}\sqrt{10a_{30}^2 - 27a_{31}} = b_3 \end{aligned}$$

is found for $\gamma = \gamma_3$.

Furthermore, the inequality

$$\begin{aligned} & 4L_3(0)L_3'(0) - 8L_3''(0) + 2b_3\left[\frac{L_3(0)}{2} - b_3\right]^2 - [L_3(0)]^3 \\ = \ & 4(a_{30}-\gamma)(\gamma^2 - a_{30}\gamma + a_{31}) - 8(-\gamma^3 + a_{30}\gamma^2 - a_{31}\gamma + a_{32}) \\ & +2b_3\left(\frac{a_{30}-\gamma}{2} - b_3\right)^2 - (a_{30}-\gamma)^3 \\ = \ & \frac{1}{27}\left[-40a_{30}^3 + 144a_{30}a_{31} - 216a_{32} + 6b_3\left(a_{30} - 3b_3\right)^2\right] > 0 \end{aligned}$$

holds for $\gamma = \gamma_3$ by condition (3) of Theorem 3. Thus, L_3 satisfies condition (3) of Theorem 2.

The inequality corresponding to condition (4) of Theorem 2 for L_3 is

$$-\left[L_3(0)\right]^4 + 4\left[L_3(0)\right]^2 L_3'(0) - 8L_3(0)L_3''(0) + 16L_3'''(0) \le 0$$

which is equivalent to

$$\begin{aligned} & -\left(a_{30}-\gamma\right)^4 + 4\left(a_{30}-\gamma\right)^2\left(\gamma^2 - a_{30}\gamma + a_{31}\right) \\ & -8\left(a_{30}-\gamma\right)\left(-\gamma^3 + a_{30}\gamma^2 - a_{31}\gamma + a_{32}\right) \\ & +16(\gamma^4 - a_{30}\gamma^3 + a_{31}\gamma^2 - a_{32}\gamma + a_{33}) \le 0. \end{aligned} \tag{2.27}$$

(2.27) for $\gamma = \gamma_3$ is equivalent to

$$\frac{1}{81}\left(-112a_{30}^4 + 432a_{30}^2a_{31} - 864a_{30}a_{32} + 1296a_{33}\right) \le 0. \tag{2.28}$$

It is obvious that (2.28) holds by condition (4) of Theorem 3. Hence, L_3 satisfies condition (4) of Theorem 2.

Finally, since $K_3 \in C^5[0,\infty)$, $L_3 \in C^4[0,\infty)$. Additionally, $L_3^{(4)}(t) \le 0$ holds, since the inequality

$$\begin{aligned} & -\gamma^5 + a_{30}\gamma^4 - a_{31}\gamma^3 + a_{32}\gamma^2 - a_{33}\gamma + a_{34} \\ = \ & \frac{1}{3^5}\left(2a_{30}^5 - 9a_{30}^3a_{31} + 27a_{30}^2a_{32} - 81a_{30}a_{33} + 243a_{34}\right) \le 0 \end{aligned}$$

holds by condition (5) for $\gamma = \gamma_3$. So, L_3 also satisfies condition (1) of Theorem 2.

Therefore, L_3 satisfies all of the conditions of Theorem 2.

If $L_3(0), L_3$ and $L_3(0)/2$ are taken, respectively instead of a_{20}, K_2 and γ_2 in (2.15), L_2 is found by means of L_3 as

$$\begin{aligned} L_2(t) &= \left[L_3(0) - \frac{L_3(0)}{2}\right]e^{-\frac{L_3(0)}{2}t} + L_3' * e^{-\frac{L_3(0)}{2}t} \\ &= \left(a_{30} - \gamma - \frac{a_{30}-\gamma}{2}\right)e^{-\frac{a_{30}-\gamma}{2}t} + L_3' * e^{-\frac{a_{30}-\gamma}{2}t} \\ &= \frac{a_{30}}{3}e^{-\frac{a_{30}}{3}t} + L_3' * e^{-\frac{a_{30}}{3}t} \end{aligned} \tag{2.29}$$

for $\gamma = \gamma_3$ and it is understood by the proof of Theorem 2 that L_2 satisfies all of the conditions of Theorem 1. Thus, because b_3 and γ_3 are replaced, respectively by b_2 and γ_2, it is seen by (2.16) and (2.13) that the solutions of the equations $h_i(t) = 1 - L_i * h_i$ for $i = 2, 3$ satisfy the inequalities

$$h_2'(t) + b_3h_2(t) > 0,\ h_2''(t) + b_3h_2'(t) > 0,\ h_2'''(t) + b_3h_2''(t) > 0 \tag{2.30}$$

and

$$\begin{cases} h_3'(t) + \gamma_3 h_3(t) + b_3 h_2(t) > 0, \\ h_3''(t) + (b_3+\gamma_3)h_3'(t) + b_3\gamma_3 h_3(t) > 0, \\ h_3'''(t) + (b_3+\gamma_3)h_3''(t) + b_3\gamma_3 h_3'(t) > 0 \end{cases} \tag{2.31}$$

for all t.

On the other hand, by using the Convolution Theorem, the solution f_3 of the equivalent equation (2.23) can be written by means of h_3 as

$$f_3(t) = h_3(t) - \gamma_3 h_3 * e^{-\gamma_3 t}. \tag{2.32}$$

By (2.32), we have

$$h_3'(t) = f_3'(t) + \gamma_3 f_3(t) \tag{2.33}$$

which yields

$$h_3''(t) = f_3''(t) + \gamma_3 f_3'(t),\ h_3'''(t) = f_3'''(t) + \gamma_3 f_3''(t). \tag{2.34}$$

So, we derive the inequalities

$$f_3'(t) + \gamma_3\left[f_3(t) + h_3(t)\right] + b_3 h_2(t) > 0,$$
$$f_3''(t) + (b_3 + 2\gamma_3)\, f_3'(t) + (b_3+\gamma_3)\,\gamma_3 f_3(t) + b_3\gamma_3 h_3(t) > 0$$

and

$$f_3'''(t) + (b_3 + 2\gamma_3)\, f_3''(t) + (2b_3+\gamma_3)\,\gamma_3 f_3'(t) + b_3\gamma_3^2 f_3(t) > 0$$

for all t from (2.31), (2.33) and (2.34). ∎

Example 3. The function K_3 defined by

$$K_3(t) = c_0 t^5 + \left(\frac{a_{34}}{24}\right) t^4 + \left(\frac{a_{33}}{6}\right) t^3 + \left(\frac{a_{32}}{2}\right) t^2 + a_{31} t + a_{30}, (c_0 \le 0) \tag{2.35}$$

corresponding to any constants $a_{3i},\ (0 \le i \le 4)$ satisfying conditions $(2) - (5)$ of Theorem 3 also satisfies condition (1) of Theorem 3.

Now, let us try to show that there exist the constants $a_{3i},\ (0 \le i \le 4)$ satisfying the conditions of Theorem 3. First, let us choose the constants a_{30} and a_{31} verifying the inequality

$$a_{30}^2 > 3a_{31}.$$

In this case, it can be followed from the proof of Theorem 3 that the constants a_{20}, a_{21} derived by means of a_{30}, a_{31} and defined by

$$a_{20} = a_{30} - \gamma_3,\ a_{21} = \gamma_3^2 - a_{30}\gamma_3 + a_{31},\ \left(\gamma_3 = \frac{a_{30}}{3}\right)$$

satisfy condition (2) of Theorem 2.

The constants a_{20} and a_{21} together with a_{22}, a_{23} which can also be found by the way in Example 2 satisfy conditions (3) and (4) of Theorem 2.

Since the inequality in condition (3) of Theorem 3 is equivalent to the new inequality obtained by taking

$$a_{30} - \gamma_3,\ \gamma_3^2 - a_{30}\gamma_3 + a_{31},\ -\gamma_3^3 + a_{30}\gamma_3^2 - a_{31}\gamma_3 + a_{32} \text{ and } b_3,$$

respectively instead of the constants a_{20}, a_{21}, a_{22} and b_2 in condition (3) of Theorem 2, if a_{22} is chosen as

$$a_{22} = -\gamma_3^3 + a_{30}\gamma_3^2 - a_{31}\gamma_3 + a_{32} = -\gamma_3\left(\gamma_3^2 - a_{30}\gamma_3 + a_{31}\right) + a_{32} = -\gamma_3 a_{21} + a_{32},$$

then a_{32} is found as

$$a_{32} = a_{22} + \gamma_3 a_{21}.$$

In that case, the constants a_{30}, a_{31} and a_{32} derived, above, satisfy condition (3) of Theorem 3.

Similarly, because condition (4) of Theorem 3 is equivalent to the new inequality obtained by taking

$$a_{30} - \gamma_3,\ \gamma_3^2 - a_{30}\gamma_3 + a_{31},\ -\gamma_3^3 + a_{30}\gamma_3^2 - a_{31}\gamma_3 + a_{32}$$

and

$$\gamma_3^4 - a_{30}\gamma_3^3 + a_{31}\gamma_3^2 - a_{32}\gamma_3 + a_{33},$$

respectively instead of the constants a_{20}, a_{21}, a_{22} and a_{23} in condition (4) of Theorem 2, if a_{23} is taken as

$$a_{23} = \gamma_3^4 - a_{30}\gamma_3^3 + a_{31}\gamma_3^2 - a_{32}\gamma_3 + a_{33} = -\gamma_3(-\gamma_3^3 + a_{30}\gamma_3^2 - a_{31}\gamma_3 + a_{32}) + a_{33} = -\gamma_3 a_{22} + a_{33},$$

then a_{33} is found as

$$a_{33} = a_{23} + \gamma_3 a_{22}.$$

So, the constants a_{30}, a_{31}, a_{32} and a_{33} obtained, above, satisfy condition (4) of Theorem 3.

Furthermore, since condition (5) of Theorem 3 is equivalent to

$$-\gamma_3\left(\gamma_3^4 - a_{30}\gamma_3^3 + a_{31}\gamma_3^2 - a_{32}\gamma_3 + a_{33}\right) + a_{34} = -\gamma_3 a_{23} + a_{34} \leq 0,$$

the constants a_{30}, a_{31}, a_{32} and a_{33} together with the constant a_{34} which is chosen as

$$a_{34} \leq \gamma_3 a_{23}$$

also satisfy condition (5) of Theorem 3.

Thus, every function K_3 of the form (2.35) corresponding to constants a_{3i}, $(0 \leq i \leq 4)$ and obtained by our method also satisfies condition (1) of Theorem 3.

For example, if a_{30} and a_{31} are taken as $a_{30} = 1$ and $a_{31} = -1$, then a_{20}, a_{21} are found as

$$a_{20} = a_{30} - \gamma_3 = \frac{2}{3}, a_{21} = \gamma_3^2 - a_{30}\gamma_3 + a_{31} = -\frac{11}{9}.$$

From Example 2,

$$\gamma_2 = \frac{1}{3}, a_{10} = a_{20} - \gamma_2 = \frac{1}{3}, a_{11} = \gamma_2^2 - a_{20}\gamma_2 + a_{21} = -\frac{4}{3}.$$

Thus, from Example 1, a_{12} can be taken as $a_{12} = 0$ because of

$$b_1 = \frac{1}{3}\left(a_{10} + 2\sqrt{a_{10}^2 - 3a_{11}}\right) = \frac{1}{9}\left(1 + 2\sqrt{37}\right) \text{ and } a_{12} < \frac{b_1}{4}(a_{10} - b_1)^2.$$

Hence, from Example 2,

$$a_{22} = a_{12} + \gamma_2 a_{11} = -\frac{4}{9}$$

and a_{23} satisfying the inequality

$$a_{23} \leq \gamma_2 a_{12} = 0$$

can be chosen as $a_{23} = -1$. So,

$$a_{32} = a_{22} + \gamma_3 a_{21} = -\frac{23}{27},\ a_{33} = a_{23} + \gamma_3 a_{22} = -\frac{31}{27}$$

and a_{34} satisfying the inequality

$$a_{34} \leq \gamma_3 a_{23} = -\frac{1}{3}$$

can be taken as $a_{34} = -1$. Therefore,

$$K_3(t) = c_0 t^5 - \frac{1}{24}t^4 - \frac{31}{162}t^3 - \frac{23}{54}t^2 - t + 1, (c_0 \leq 0).$$

Theorem 4. *Let us consider the equation*

$$f_4(t) = 1 - \int_0^t K_4(t-\tau) f_4(\tau) d\tau = 1 - K_4 * f_4. \tag{2.36}$$

If

(1) $K_4 \in C^6[0,\infty), K_4^{(6)}(t) \le 0$ *for all* t *and* $K_4^{(i)}(0) = a_{4i}, (0 \le i \le 5)$,

(2) $a_{40}^2 > \frac{8}{3} a_{41}$,

(3) $648 a_{40} a_{41} - 864 a_{42} + 24 b_4 \left(\frac{3a_{40}}{4} - 3b_4 \right)^2 - 189 a_{40}^3 > 0$,

where $b_4 = \frac{1}{12}\left(a_{40} + 2\sqrt{19 a_{40}^2 - 48 a_{41}} \right)$,

(4) $-2025 a_{40}^4 + 7776 a_{40}^2 a_{41} - 15552 a_{40} a_{42} + 20736 a_{43} \le 0$,

(5) $1701 a_{40}^5 - 7776 a_{40}^3 a_{41} + 23328 a_{40}^2 a_{42} - 62208 a_{40} a_{43} + 124416 a_{44} \le 0$,

(6) $-3a_{40}^6 + 16 a_{40}^4 a_{41} - 64 a_{40}^3 a_{42} + 256 a_{40}^2 a_{43} - 1024 a_{40} a_{44} + 4096 a_{45} \le 0$,

then there exist the functions L_2, L_3 *and* L_4 *satisfying all of the conditions of Theorem 1, Theorem 2 and Theorem 3, respectively and can be written by the means of* K_4*. Additionally, the inequalities*

$$h_2'(t) + b_4 h_2(t) > 0, h_2''(t) + b_4 h_2'(t) > 0, h_2'''(t) + b_4 h_2''(t) > 0,$$

$$h_3'(t) + \gamma_4 h_3(t) + b_4 h_2(t) > 0, h_3''(t) + (b_4 + \gamma_4) h_3'(t) + b_4 \gamma_4 h_3(t) > 0,$$

$$h_3'''(t) + (b_4 + \gamma_4) h_3''(t) + b_4 \gamma_4 h_3'(t) > 0,$$

$$h_4'(t) + \gamma_4 \left[h_4(t) + h_3(t) \right] + b_4 h_2(t) > 0,$$

$$h_4''(t) + (b_4 + 2\gamma_4) h_4'(t) + (b_4 + \gamma_4) \gamma_4 h_4(t) + b_4 \gamma_4 h_3(t) > 0,$$

$$h_4'''(t) + (b_4 + 2\gamma_4) h_4''(t) + (2b_4 + \gamma_4) \gamma_4 h_4'(t) + b_4 \gamma_4^2 h_4(t) > 0$$

and

$$\begin{cases} f_4'(t) + \gamma_4 \left[f_4(t) + h_4(t) + h_3(t) \right] + b_4 h_2(t) > 0, \\ f_4''(t) + (b_4 + 3\gamma_4) f_4'(t) + (b_4 + 2\gamma_4) \gamma_4 f_4(t) \\ \qquad + (b_4 + \gamma_4) \gamma_4 h_4(t) + b_4 \gamma_4 h_3(t) > 0, \\ f_4'''(t) + (b_4 + 3\gamma_4) f_4''(t) + (3b_4 + 3\gamma_4) \gamma_4 f_4'(t) \\ \qquad + (2b_4 + \gamma_4) \gamma_4^2 f_4(t) + b_4 \gamma_4^2 h_4(t) > 0 \end{cases} \tag{2.37}$$

are satisfied for all t*, where* h_i *is the solution of the equation* $h_i(t) = 1 - L_i * h_i, (i = 2, 3, 4)$ *and* $\gamma_4 = a_{40}/4$.

Proof. By taking $g(t) = e^{-\gamma t}$ with $\gamma \in \mathbb{R}$ in the Equivalence Theorem, we get the equivalent equation to (2.36) as

$$f_4(t) = e^{-\gamma t} - \int_0^t L_4(t-\tau) f_4(\tau) d\tau = e^{-\gamma t} - L_4 * f_4, \tag{2.38}$$

where

$$L_4(t) = (a_{40} - \gamma) e^{-\gamma t} + K_4' * e^{-\gamma t}. \tag{2.39}$$

Thus, from (2.39)

$$\begin{aligned} L_4'(t) &= (\gamma^2 - a_{40}\gamma + a_{41})e^{-\gamma t} + K_4'' * e^{-\gamma t}, \\ L_4''(t) &= (-\gamma^3 + a_{40}\gamma^2 - a_{41}\gamma + a_{42})e^{-\gamma t} + K_4''' * e^{-\gamma t}, \\ L_4'''(t) &= (\gamma^4 - a_{40}\gamma^3 + a_{41}\gamma^2 - a_{42}\gamma + a_{43})e^{-\gamma t} + K_4^{(4)} * e^{-\gamma t}, \\ L_4^{(4)}(t) &= (-\gamma^5 + a_{40}\gamma^4 - a_{41}\gamma^3 + a_{42}\gamma^2 - a_{43}\gamma + a_{44})e^{-\gamma t} + K_4^{(5)} * e^{-\gamma t} \end{aligned}$$

and

$$L_4^{(5)}(t) = (\gamma^6 - a_{40}\gamma^5 + a_{41}\gamma^4 - a_{42}\gamma^3 + a_{43}\gamma^2 - a_{44}\gamma + a_{45})e^{-\gamma t} + K_4^{(6)} * e^{-\gamma t}.$$

Now, we show that L_4 satisfies all of the conditions of Theorem 3.

The inequality corresponding to condition (2) of Theorem 3 is

$$[L_4(0)]^2 > 3L_4'(0)$$

which is equivalent to

$$(a_{40} - \gamma)^2 > 3\left(\gamma^2 - a_{40}\gamma + a_{41}\right). \tag{2.40}$$

(2.40) is equivalent to

$$2\gamma^2 - a_{40}\gamma + 3a_{41} - a_{40}^2 < 0. \tag{2.41}$$

The discriminant of $2\gamma^2 - a_{40}\gamma + 3a_{41} - a_{40}^2 = 0$ is $3(3a_{40}^2 - 8a_{41})$ which is positive by condition (2) of Theorem 4. So, if γ is chosen as $\gamma = \gamma_4 = a_{40}/4$, then (2.41) is satisfied. Thus, L_4 holds condition (2) of Theorem 3.

Also, if $L_4(0)$ and $L_4'(0)$ are taken, respectively instead of the constants a_{30} and a_{31} in b_3 of Theorem 3, then

$$\begin{aligned} &\frac{1}{9}\left[L_4(0) + 2\sqrt{[10L_4(0)]^2 - 27L_4'(0)}\right] \\ &= \frac{1}{9}\left[a_{40} - \gamma + 2\sqrt{10\left(a_{40} - \gamma\right)^2 - 27\left(\gamma^2 - a_{40}\gamma + a_{41}\right)}\right] \\ &= \frac{1}{12}\left(a_{40} + 2\sqrt{19a_{40}^2 - 48a_{41}}\right) = b_4 \end{aligned}$$

for $\gamma = \gamma_4$.

Furthermore, the inequality

$$\begin{aligned} &144L_4(0)L_4'(0) - 216L_4''(0) + 6b_4\left[L_4(0) - 3b_4\right]^2 - 40\left[L_4(0)\right]^3 \\ &= 144(a_{40} - \gamma)(\gamma^2 - a_{40}\gamma + a_{41}) - 216(-\gamma^3 + a_{40}\gamma^2 - a_{41}\gamma + a_{42}) \\ &\quad +6b_4\left(a_{40} - \gamma - 3b_4\right)^2 - 40(a_{40} - \gamma)^3 \\ &= \frac{1}{4}\left[-189a_{40}^3 + 648a_{40}a_{41} - 864a_{42} + 24b_4\left(\frac{3a_{40}}{4} - 3b_4\right)^2\right] > 0 \end{aligned}$$

holds for $\gamma = \gamma_4$ by condition (3) of Theorem 4. Thus, L_4 satisfies condition (3) of Theorem 3.

The inequality corresponding to condition (4) of Theorem 3 for L_4 is

$$-112\left[L_4(0)\right]^4 + 432\left[L_4(0)\right]^2 L_4'(0) - 864L_4(0)L_4''(0) + 1296L_4'''(0) \le 0$$

which is equivalent to

$$\begin{aligned} &-112\left(a_{40} - \gamma\right)^4 + 432\left(a_{40} - \gamma\right)^2\left(\gamma^2 - a_{40}\gamma + a_{41}\right) \\ &-864\left(a_{40} - \gamma\right)\left(-\gamma^3 + a_{40}\gamma^2 - a_{41}\gamma + a_{42}\right) \\ &+1296(\gamma^4 - a_{40}\gamma^3 + a_{41}\gamma^2 - a_{42}\gamma + a_{43}) \le 0. \end{aligned} \tag{2.42}$$

(2.42) is equivalent to

$$\frac{1}{16}\left(-2025a_{40}^4+7776a_{40}^2a_{41}-15552a_{40}a_{42}+20736a_{43}\right)\le 0 \tag{2.43}$$

for $\gamma=\gamma_4$. It is obvious that (2.43) holds by condition (4) of Theorem 4. Hence, L_4 satisfies condition (4) of Theorem 3.

On the other hand, the inequality corresponding to condition (5) of Theorem 3 for L_4 is

$$2\left[L_4(0)\right]^5-9\left[L_4(0)\right]^3L_4'(0)+27\left[L_4(0)\right]^2L_4''(0)-81L_4(0)L_4'''(0)+243L_4^{(4)}(0)\le 0$$

which is equivalent to

$$\begin{aligned}&2\left(a_{40}-\gamma\right)^5-9\left(a_{40}-\gamma\right)^3\left(\gamma^2-a_{40}\gamma+a_{41}\right)\\&+27\left(a_{40}-\gamma\right)^2\left(-\gamma^3+a_{40}\gamma^2-a_{41}\gamma+a_{42}\right)\\&-81(a_{40}-\gamma)(\gamma^4-a_{40}\gamma^3+a_{41}\gamma^2-a_{42}\gamma+a_{43})\\&+243(-\gamma^5+a_{40}\gamma^4-a_{41}\gamma^3+a_{42}\gamma^2-a_{43}\gamma+a_{44})\le 0.\end{aligned} \tag{2.44}$$

(2.44) for $\gamma=\gamma_4$ is equivalent to

$$\frac{1}{512}\left(1701a_{40}^5-7776a_{40}^3a_{41}+23328a_{40}^2a_{42}-62208a_{40}a_{43}+124416a_{44}\right)\le 0. \tag{2.45}$$

It is obvious that (2.45) holds by condition (5) of Theorem 4. Hence, L_4 satisfies condition (5) of Theorem 3.

Finally, since $K_4\in C^6[0,\infty)$, $L_4\in C^5[0,\infty)$. Additionally, $L_4^{(5)}(t)\le 0$ holds, since

$$\begin{aligned}&\gamma^6-a_{40}\gamma^5+a_{41}\gamma^4-a_{42}\gamma^3+a_{43}\gamma^2-a_{44}\gamma+a_{45}\\=\;&\frac{1}{4^6}\left(-3a_{40}^6+16a_{40}^4a_{41}-64a_{40}^3a_{42}+256a_{40}^2a_{43}-1024a_{40}a_{44}+4096a_{45}\right)\le 0\end{aligned}$$

holds for $\gamma=\gamma_4$ by condition (6). So, L_4 also satisfies condition (1) of Theorem 3.

As a result, L_4 satisfies all of the conditions of Theorem 3.

If $L_4(0),L_4$ and $L_4(0)/3$ are taken, respectively instead of a_{30},K_3 and γ_3 in (2.24), L_3 is found by means of L_4 as

$$\begin{aligned}L_3(t)&=\left[L_4(0)-\frac{L_4(0)}{3}\right]e^{-\frac{L_4(0)}{3}t}+L_4'*e^{-\frac{L_4(0)}{3}t}\\&=\left(a_{40}-\gamma-\frac{a_{40}-\gamma}{3}\right)e^{-\frac{a_{40}-\gamma}{3}t}+L_4'*e^{-\frac{a_{40}-\gamma}{3}t}\\&=\frac{a_{40}}{2}e^{-\frac{a_{40}}{4}t}+L_4'*e^{-\frac{a_{40}}{4}t}\end{aligned}$$

for $\gamma=\gamma_4$ and it is understood by the proof of Theorem 3 that L_3 satisfies all of the conditions of Theorem 2.

Similarly, if $L_4(0)$ is taken instead of a_{30} in (2.29), L_2 is found as

$$\begin{aligned}L_2(t)&=\frac{L_4(0)}{3}e^{-\frac{L_4(0)}{3}t}+L_3'*e^{-\frac{L_4(0)}{3}t}\\&=\frac{a_{40}}{4}e^{-\frac{a_{40}}{4}t}+L_3'*e^{-\frac{a_{40}}{4}t}\end{aligned}$$

for $\gamma=\gamma_4$ and also it is understood by the proof of Theorem 3 that L_2 satisfies all of the conditions of Theorem 1.

Thus, because b_4 and γ_4 are replaced, respectively by b_3 and γ_3, it is seen by (2.22), (2.30) and (2.31) that the solutions of the equations $h_i(t) = 1 - L_i * h_i, (i = 2, 3, 4)$ satisfy the inequalities

$$h_2'(t) + b_4 h_2(t) > 0, h_2''(t) + b_4 h_2'(t) > 0, h_2'''(t) + b_4 h_2''(t) > 0, h_3'(t) + \gamma_4 h_3(t) + b_4 h_2(t) > 0,$$
$$h_3''(t) + (b_4 + \gamma_4) h_3'(t) + b_4 \gamma_4 h_3(t) > 0, h_3'''(t) + (b_4 + \gamma_4) h_3''(t) + b_4 \gamma_4 h_3'(t) > 0$$

and

$$(2.46) \qquad \begin{cases} h_4'(t) + \gamma_4 \left[h_4(t) + h_3(t)\right] + b_4 h_2(t) > 0, \\ h_4''(t) + (b_4 + 2\gamma_4)\, h_4'(t) + (b_4 + \gamma_4)\, \gamma_4 h_4(t) + b_4 \gamma_4 h_3(t) > 0, \\ h_4'''(t) + (b_4 + 2\gamma_4)\, h_4''(t) + (2b_4 + \gamma_4)\, \gamma_4 h_4'(t) + b_4 \gamma_4^2 h_4(t) > 0 \end{cases}$$

for all t.

On the other hand, by using the Convolution Theorem, the solution f_4 of (2.38) can be written by means of h_4 as the form

$$(2.47) \qquad f_4(t) = h_4(t) - \gamma_4 h_4 * e^{-\gamma_4 t}.$$

By (2.47), we have

$$(2.48) \qquad h_4'(t) = f_4'(t) + \gamma_4 f_4(t)$$

which yields

$$(2.49) \qquad h_4''(t) = f_4''(t) + \gamma_4 f_4'(t),\ h_4'''(t) = f_4'''(t) + \gamma_4 f_4''(t).$$

So, we derive (2.37) for all t by (2.46), (2.48) and (2.49). Thus, the proof is completed. ∎

Example 4. The function K_4 of the form

$$(2.50) \quad K_4(t) = c_0 t^6 + \left(\frac{a_{45}}{120}\right) t^5 + \left(\frac{a_{44}}{24}\right) t^4 + \left(\frac{a_{43}}{6}\right) t^3 + \left(\frac{a_{42}}{2}\right) t^2 + a_{41} t + a_{40}, \quad (c_0 \leq 0)$$

corresponding to any constants a_{4i}, $(0 \leq i \leq 5)$ satisfying conditions $(2) - (6)$ of Theorem 4 also satisfies condition (1) of Theorem 4.

To see the validity of this claim, first, choose the constants a_{40} and a_{41} verifying the inequality

$$a_{40}^2 > \frac{8}{3} a_{41}.$$

In this case, it can be easily seen by the proof of Theorem 4 that the constants a_{30}, a_{31} defined by

$$a_{30} = p_1(\gamma_4), a_{31} = p_2(\gamma_4),$$

where

$$p_i(\gamma) = (-1)^i \gamma^i + (-1)^{i-1} a_{40} \gamma^{i-1} + (-1)^{i-2} a_{41} \gamma^{i-2} + \cdots - a_{4(i-2)} \gamma + a_{4(i-1)}, \quad (1 \leq i \leq 6)$$

and $\gamma_4 = a_{40}/4$ satisfy condition (2) of Theorem 3.

On the other hand, the constants a_{30}, a_{31} together with a_{32}, a_{33} and a_{34} which can also be found as those were found by the method in Example 3 satisfy conditions $(3) - (5)$ of Theorem 3.

Since the inequality in condition (3) of Theorem 4 is equivalent to new inequality obtained by taking $p_1(\gamma_4), p_2(\gamma_4), p_3(\gamma_4)$ and b_4, respectively instead of the constants a_{30}, a_{31}, a_{32} and b_3 in condition (3) of Theorem 3, if a_{32} is chosen as

$$a_{32} = p_3(\gamma_4) = -\gamma_4 p_2(\gamma_4) + a_{42} = -\gamma_4 a_{31} + a_{42},$$

then a_{42} is found as

$$a_{42} = a_{32} + \gamma_4 p_2(\gamma_4) = a_{32} + \gamma_4 a_{31}.$$

In that case, the constants a_{40}, a_{41} and a_{42} which are derived, above, satisfy condition (3) of Theorem 4.

Because condition (4) of Theorem 4 is equivalent to the new inequality obtained by taking $p_1(\gamma_4), p_2(\gamma_4), p_3(\gamma_4)$ and $p_4(\gamma_4)$, respectively instead of the constants a_{30}, a_{31}, a_{32} and a_{33} in condition (4) of Theorem 3, if a_{33} is taken as

$$a_{33} = p_4(\gamma_4) = -\gamma_4 p_3(\gamma_4) + a_{43},$$

then a_{43} is found as

$$a_{43} = a_{33} + \gamma_4 p_3(\gamma_4) = a_{33} + \gamma_4 a_{32}.$$

So, the constants a_{40}, a_{41}, a_{42} and a_{43} obtained, above, satisfy condition (4) of Theorem 4.

Because condition (5) of Theorem 4 is equivalent to the new inequality obtained by taking $p_1(\gamma_4), p_2(\gamma_4), p_3(\gamma_4), p_4(\gamma_4)$ and $p_5(\gamma_4)$, respectively instead of the constants $a_{30}, a_{31}, a_{32}, a_{33}$ and a_{34} in condition (5) of Theorem 3, if a_{34} is taken as

$$a_{34} = p_5(\gamma_4) = -\gamma_4 p_4(\gamma_4) + a_{44},$$

then a_{44} is found as

$$a_{44} = a_{34} + \gamma_4 p_4(\gamma_4) = a_{34} + \gamma_4 a_{33}.$$

Hence, the constants $a_{40}, a_{41}, a_{42}, a_{43}$ and a_{44} obtained, above, satisfy condition (5) of Theorem 4.

Furthermore, since condition (6) of Theorem 4 is equivalent to

$$p_6(\gamma_4) = -\gamma_4 p_5(\gamma_4) + a_{45} \leq 0,$$

the constants $a_{40}, a_{41}, a_{42}, a_{43}$ and a_{44} together with the constant a_{45} chosen as

$$a_{45} \leq \gamma_4 a_{34}$$

also satisfy condition (6) of Theorem 4.

Thus, the function K_4 of the form (2.50) corresponding to constants a_{4i}, $(0 \leq i \leq 5)$ and obtained by our method also satisfies condition (1) of Theorem 4.

For example, if one choose $a_{40} = 3$ and $a_{41} = 3$, then γ_4, a_{30}, a_{31} are found as

$$\gamma_4 = \frac{3}{4}, a_{30} = p_1(\gamma_4) = \frac{9}{4}, a_{31} = p_2(\gamma_4) = \frac{21}{16}.$$

From Example 3,

$$\gamma_3 = \frac{a_{30}}{3} = \frac{3}{4}, a_{20} = a_{30} - \gamma_3 = \frac{3}{2}, a_{21} = \gamma_3^2 - a_{30}\gamma_3 + a_{31} = \frac{3}{16}.$$

From Example 2,

$$\gamma_2 = \frac{a_{20}}{2} = \frac{3}{4}, a_{10} = a_{20} - \gamma_2 = \frac{3}{4}, a_{11} = \gamma_2^2 - a_{20}\gamma_2 + a_{21} = -\frac{3}{8}.$$

Thus, from Example 1, a_{12} can be taken as $a_{12} = 1/64$ because of

$$b_1 = \frac{1}{3}\left(a_{10} + 2\sqrt{a_{10}^2 - 3a_{11}}\right) = \frac{1}{4}\left(1 + 2\sqrt{3}\right) \text{ and } a_{12} < \frac{b_1}{4}(a_{10} - b_1)^2 = \frac{(3\sqrt{3} - 4)}{32}.$$

Hence, from Example 2,

$$a_{22} = a_{12} + \gamma_2 a_{11} = -\frac{17}{64}$$

and a_{23} satisfying the inequality

$$a_{23} \leq \gamma_2 a_{12} = \frac{3}{256}$$

can be chosen as $a_{23} = 1/256$.

So, from Example 3,

$$a_{32} = a_{22} + \gamma_3 a_{21} = -\frac{1}{8},\ a_{33} = a_{23} + \gamma_3 a_{22} = -\frac{25}{128}$$

and a_{34} satisfying the inequality

$$a_{34} \leq \gamma_3 a_{23} = \frac{3}{1024}$$

can be taken as $a_{34} = 1/512$.

Thus,

$$a_{42} = a_{32} + \gamma_4 a_{31} = \frac{55}{64},\ a_{43} = a_{33} + \gamma_4 a_{32} = -\frac{37}{128},\ a_{44} = a_{34} + \gamma_4 a_{33} = -\frac{37}{256}$$

and a_{45} satisfying the inequality

$$a_{45} \leq \gamma_4 a_{34} = \frac{3}{2048}$$

can be taken as $a_{45} = 1/1024$.

Therefore,

$$K_4(t) = c_0 t^5 + \frac{1}{122880} t^5 - \frac{37}{6144} t^4 - \frac{37}{768} t^3 + \frac{55}{128} t^2 + 3t + 3, (c_0 \leq 0).$$

By continuing this process for $n \in \mathbb{N}_2$, we get Theorem n which can be given as follows:

Theorem n. *Let us consider the equation given of the form*

$$f_n(t) = 1 - \int_0^t K_n(t-\tau) f_n(\tau) d\tau = 1 - K_n * f_n \tag{2.51}$$

under the following assumptions:

(1) $K_n \in C^{n+2}[0,\infty), K_n^{(n+2)}(t) \leq 0$ *for all* t *and* $K_n^{(i)}(0) = a_{ni},\ (0 \leq i \leq n+1)$,

(2) $a_{n0}^2 > \left(\frac{2n}{n-1}\right) a_{n1}$.

Furthermore, conditions $(3)-(n+1)$ *of Theorem n are the inequalities obtained by taking* $p_1(\gamma_n), p_2(\gamma_n), \ldots, p_{n+1}(\gamma_n)$ *and* b_n *respectively instead of the constants* $a_{(n-1)0}, a_{(n-1)1}, \ldots, a_{(n-1)n}$ *and* b_{n-1} *in conditions* $(3)-(n+1)$ *of Theorem* $(n-1)$*, where*

$$p_i(\gamma) = (-1)^i \gamma^i + (-1)^{i-1} a_{n0} \gamma^{i-1} + (-1)^{i-2} a_{n1} \gamma^{i-2} + \cdots - a_{n(i-2)} \gamma + a_{n(i-1)}, (1 \leq i \leq n+2),$$

$$\gamma_n = \frac{a_{n0}}{n} \text{ and } b_n = \frac{1}{3n}\left[a_{n0} + 2\sqrt{\left[\frac{2+3n(n-1)}{2}\right] a_{n0}^2 - 3n^2 a_{n1}}\right].$$

Besides, let condition $(n+2)$ *of Theorem n also be* $p_{n+2}(\gamma_n) \leq 0$*. Then, there exist the functions* $L_2, L_3, \ldots, L_n$ *which satisfy all of the conditions of Theorem 1, Theorem 2,..., Theorem* $(n-1)$*, respectively and can be obtained by means of* K_n *as*

$$L_n(t) = \left(\frac{n-1}{n}\right) a_{n0}\, e^{-\frac{a_{n0}}{n} t} + K_n' * e^{-\frac{a_{n0}}{n} t} \tag{2.52}$$

for $n \geq 2$ *and*

$$
\begin{cases} L_{n-1}(t) &= \left(\frac{n-2}{n}\right) a_{n0}\, e^{-\frac{a_{n0}}{n}t} + L'_n * e^{-\frac{a_{n0}}{n}t}, \\ L_{n-2}(t) &= \left(\frac{n-3}{n}\right) a_{n0}\, e^{-\frac{a_{n0}}{n}t} + L'_{n-1} * e^{-\frac{a_{n0}}{n}t}, \\ L_{n-3}(t) &= \left(\frac{n-4}{n}\right) a_{n0}\, e^{-\frac{a_{n0}}{n}t} + L'_{n-2} * e^{-\frac{a_{n0}}{n}t}, \\ &\vdots \\ L_3(t) &= \left(\frac{2}{n}\right) a_{n0}\, e^{-\frac{a_{n0}}{n}t} + L'_4 * e^{-\frac{a_{n0}}{n}t}, \\ L_2(t) &= \left(\frac{1}{n}\right) a_{n0}\, e^{-\frac{a_{n0}}{n}t} + L'_3 * e^{-\frac{a_{n0}}{n}t} \end{cases} \tag{2.53}
$$

for $n \geq 3$*. Additionally, the inequalities*

$$
\begin{cases} h'_2(t) + b_n h_2(t) > 0, \\ h''_2(t) + b_n h'_2(t) > 0, \\ h'''_2(t) + b_n h''_2(t) > 0 \end{cases} \tag{2.54}
$$

for $n \geq 2$*,*

$$
\begin{cases} h'_i(t) + \gamma_n \left[h_i(t) + h_{i-1}(t) + \cdots + h_3(t)\right] + x_{1n} h_2(t) > 0, \\ h''_i(t) + x_{(i-1)n} h'_i(t) + \gamma_n \left[x_{(i-2)n} h_i(t) + x_{(i-3)n} h_{i-1}(t) \right. \\ \qquad \left. + \cdots + x_{1n} h_3(t)\right] > 0, \\ h'''_i(t) + x_{(i-1)n} h''_i(t) + y_{(i-1)n} h'_i(t) + \gamma_n \left[y_{(i-2)n} h_i(t) \right. \\ \qquad \left. + y_{(i-3)n} h_{i-1}(t) + \cdots + y_{1n} h_3(t)\right] > 0 \end{cases} \tag{2.55}
$$

for $i = 3, 4, \ldots, n$ *and* $n \geq 3$*,*

$$
\begin{cases} f'_2(t) + \gamma_2 f_2(t) + x_{12} h_2(t) > 0, \\ f''_2(t) + x_{22} f'_2(t) + x_{12}\gamma_2 f_2(t) > 0, \\ f'''_2(t) + x_{22} f''_2(t) + x_{12}\gamma_2 f'_2(t) > 0 \end{cases} \tag{2.56}
$$

for $n = 2$ *and*

$$
\begin{cases} f'_n(t) + \gamma_n \left[f_n(t) + h_n(t) + h_{n-1}(t) + \cdots + h_3(t)\right] + x_{1n} h_2(t) > 0, \\ f''_n(t) + x_{nn} f'_n(t) + \gamma_n \left[x_{(n-1)n} f_n(t) + x_{(n-2)n} h_n(t) + x_{(n-3)n} h_{n-1}(t) \right. \\ \qquad \left. + \cdots + x_{1n} h_3(t)\right] > 0, \\ f'''_n(t) + x_{nn} f''_n(t) + y_{nn} f'_n(t) + \gamma_n \left[y_{(n-1)n} f_n(t) + y_{(n-2)n} h_n(t) \right. \\ \qquad \left. + y_{(n-3)n} h_{n-1}(t) + \cdots + y_{1n} h_3(t)\right] > 0 \end{cases} \tag{2.57}
$$

for $n \geq 3$ *hold. Here,* x_{in}, y_{in} *for* $1 \leq i \leq n$ *are defined by the equalities*

$$
x_{in} = b_n + (i-1)\gamma_n, y_{in} = \left[(i-1)b_n + \frac{(i-1)(i-2)}{2}\gamma_n\right]\gamma_n, \tag{2.58}
$$

h_i *for* $2 \leq i \leq n$ *is the solution of the equation* $h_i(t) = 1 - \int_0^t L_i(t-\tau)h_i(\tau)d\tau$ *and* $0 \leq t < \infty$*.*

Proof. We can prove Theorem n by the mathematical induction. It is observed by Theorem 2 that there exists the function L_2 providing all of the conditions of Theorem 1 and given by

$$
L_2(t) = \left(\frac{a_{20}}{2}\right) e^{-\frac{a_{20}}{2}t} + K'_2 * e^{-\frac{a_{20}}{2}t}
$$

such that (2.54) and (2.56) are satisfied for $n = 2$. Namely, Theorem n is true for $n = 2$.

Let us suppose that Theorem n is valid for $n = m\,(m \geq 2)$. In this case, we will try to show that Theorem n is also valid for $n = m + 1$. If Theorem m is true, then there exist the functions $L_2, L_3, \ldots, L_m$ which satisfy all of the conditions of Theorem 1, Theorem 2,..., Theorem $(m-1)$, respectively and can be obtained by means of K_m as form (2.52) and (2.53).

Also, (2.54) for $m \geq 2$, (2.55) for $i = 3, 4, \ldots, m$ and $m \geq 3$, (2.56) for $m = 2$ and (2.57) for $m \geq 3$ are satisfied.

By taking $g(t) = e^{-\gamma t}$ with $\gamma \in \mathbb{R}$ in the Equivalence Theorem, we get

$$f_{m+1}(t) = e^{-\gamma t} - \int_0^t L_{m+1}(t-\tau)\, f_{m+1}(\tau) d\tau = e^{-\gamma t} - L_{m+1} * f_{m+1} \tag{2.59}$$

which is equivalent to

$$f_{m+1}(t) = 1 - \int_0^t K_{m+1}(t-\tau)\, f_{m+1}(\tau) d\tau = 1 - K_{m+1} * f_{m+1},$$

where

$$L_{m+1}(t) = (a_{(m+1)0} - \gamma) e^{-\gamma t} + K'_{m+1} * e^{-\gamma t}. \tag{2.60}$$

Let us try to see that L_{m+1} satisfies conditions $(1) - (m+2)$ of Theorem m, by considering the conditions of Theorem $(m+1)$.

Differentiation of (2.60) leads to

$$\begin{aligned} L'_{m+1}(t) &= (\gamma^2 - a_{(m+1)0}\gamma + a_{(m+1)1}) e^{-\gamma t} + K''_{m+1} * e^{-\gamma t}, \\ L''_{m+1}(t) &= (-\gamma^3 + a_{(m+1)0}\gamma^2 - a_{(m+1)1}\gamma + a_{(m+1)2}) e^{-\gamma t} + K'''_{m+1} * e^{-\gamma t}, \\ &\vdots \\ L^{(m+2)}_{m+1}(t) &= \big[(-1)^{m+3}\gamma^{m+3} + (-1)^{m+2} a_{(m+1)0}\gamma^{m+2} + (-1)^{m+1} a_{(m+1)1}\gamma^{m+1} \\ &\quad + \cdots - a_{(m+1)(m+1)}\gamma + a_{(m+1)(m+2)}\big]\, e^{-\gamma t} + K^{(m+3)}_{m+1} * e^{-\gamma t}. \end{aligned}$$

The inequality corresponding to condition (2) of Theorem m is

$$[L_{m+1}(0)]^2 > \left(\frac{2m}{m-1}\right) L'_{m+1}(0)$$

which is equivalent to

$$\left(a_{(m+1)0} - \gamma\right)^2 > \left(\frac{2m}{m-1}\right) \left(\gamma^2 - a_{(m+1)0}\gamma + a_{(m+1)1}\right). \tag{2.61}$$

(2.61) is equivalent to

$$(m+1)\gamma^2 - 2a_{(m+1)0}\gamma + 2m a_{(m+1)1} - (m-1) a^2_{(m+1)0} < 0. \tag{2.62}$$

The discriminant of $(m+1)\gamma^2 - 2a_{(m+1)0}\gamma + 2m a_{(m+1)1} - (m-1)a^2_{(m+1)0} = 0$ is $4m[m a^2_{(m+1)0} - 2(m+1) a_{(m+1)1}]$ which is positive by condition (2) of Theorem $(m+1)$. So, if γ is chosen as $\gamma = \gamma_{m+1} = a_{(m+1)0}/(m+1)$, then (2.62) holds. Thus, L_{m+1} satisfies condition (2) of Theorem m.

Besides, if $L_{m+1}(0) = a_{(m+1)0} - \gamma_{m+1}$ and $L'_{m+1}(0) = \gamma_{m+1}^2 - a_{(m+1)0}\gamma_{m+1} + a_{(m+1)1}$ are taken, respectively instead of the constants a_{m0} and a_{m1} in the constant b_m of Theorem m, then

$$\begin{aligned}
&\frac{1}{3m}\left[L_{m+1}(0) + 2\sqrt{\left[\frac{2+3m(m-1)}{2}\right][L_{m+1}(0)]^2 - 3m^2 L'_{m+1}(0)}\right] \\
= \;& \frac{1}{3m}\Bigg[a_{(m+1)0} - \gamma_{m+1} \\
& +2\sqrt{\left[\frac{2+3m(m-1)}{2}\right]\left(a_{(m+1)0} - \gamma_{m+1}\right)^2 - 3m^2\left(\gamma_{m+1}^2 - a_{(m+1)0}\gamma_{m+1} + a_{(m+1)1}\right)}\Bigg] \\
= \;& \frac{1}{3m}\Bigg[\frac{m a_{(m+1)0}}{m+1} \\
& +2\sqrt{\frac{m^2}{(m+1)^2}\left[\left(\frac{3m^2-3m+2}{2} + 3m\right)a_{(m+1)0}^2 - 3(m+1)^2 a_{(m+1)1}\right]}\Bigg] \\
= \;& \frac{1}{3(m+1)}\left[a_{(m+1)0} + 2\sqrt{\left(\frac{3m^2+3m+2}{2}\right)a_{(m+1)0}^2 - 3(m+1)^2 a_{(m+1)1}}\right] \\
= \;& b_{m+1}.
\end{aligned}$$

L_{m+1} satisfies conditions $(3) - (m+2)$ of Theorem m if and only if the new inequalities obtained by taking

$$L_{m+1}(0) = p_1(\gamma_{m+1}), L'_{m+1}(0) = p_2(\gamma_{m+1}), \ldots, L_{m+1}^{(m+1)}(0) = p_{m+2}(\gamma_{m+1}) \text{ and } b_{m+1},$$

respectively instead of the constants $a_{m0}, a_{m1}, \ldots, a_{m(m+1)}$ and b_m in conditions $(3) - (m+2)$ of Theorem m hold. The new inequalities are also conditions $(3) - (m+2)$ of Theorem $(m+1)$. Namely, L_{m+1} satisfies conditions $(3) - (m+2)$ of Theorem m.

Finally, let us try to see that L_{m+1} satisfies condition (1) of Theorem m. Since $K_{m+1} \in C^{m+3}[0,\infty), K_{m+1}^{(m+3)}(t) \leq 0$ for all t and $p_{m+3}(\gamma_{m+1}) \leq 0$ by conditions (1) and $(m+3)$ of Theorem $(m+1)$, respectively, we have $L_{m+1} \in C^{m+2}[0,\infty), L_{m+1}^{(m+2)}(t) \leq 0$. Thus, L_{m+1} also satisfies condition (1) of Theorem m.

Consequently, L_{m+1} satisfies all of the conditions of Theorem m. That is, L_{m+1} satisfies all of the conditions which are satisfied by K_m in Theorem m.

In this case, by using the fact that Theorem m is true, if $L_{m+1}(0)$ and L_{m+1} are taken, respectively instead of a_{m0} and K_m in the equalities (2.52) and (2.53),

$$(2.63) \quad \left\{\begin{array}{rcl}
L_m(t) &=& \left(\frac{m-1}{m}\right) L_{m+1}(0)\, e^{-\frac{L_{m+1}(0)}{m}t} + L'_{m+1} * e^{-\frac{L_{m+1}(0)}{m}t}, \\
L_{m-1}(t) &=& \left(\frac{m-2}{m}\right) L_{m+1}(0)\, e^{-\frac{L_{m+1}(0)}{m}t} + L'_{m} * e^{-\frac{L_{m+1}(0)}{m}t}, \\
L_{m-2}(t) &=& \left(\frac{m-3}{m}\right) L_{m+1}(0)\, e^{-\frac{L_{m+1}(0)}{m}t} + L'_{m-1} * e^{-\frac{L_{m+1}(0)}{m}t}, \\
&\vdots& \\
L_3(t) &=& \left(\frac{2}{m}\right) L_{m+1}(0)\, e^{-\frac{L_{m+1}(0)}{m}t} + L'_4 * e^{-\frac{L_{m+1}(0)}{m}t}, \\
L_2(t) &=& \left(\frac{1}{m}\right) L_{m+1}(0)\, e^{-\frac{L_{m+1}(0)}{m}t} + L'_3 * e^{-\frac{L_{m+1}(0)}{m}t}
\end{array}\right.$$

is found for $m \geq 2$. So, for $m \geq 2$, we get

$$(2.64)\quad \begin{cases} L_m(t) &= \left(\frac{m-1}{m+1}\right) a_{(m+1)0}\, e^{-\frac{a_{(m+1)0}}{m+1}t} + L'_{m+1} * e^{-\frac{a_{(m+1)0}}{m+1}t}, \\ L_{m-1}(t) &= \left(\frac{m-2}{m+1}\right) a_{(m+1)0}\, e^{-\frac{a_{(m+1)0}}{m+1}t} + L'_{m} * e^{-\frac{a_{(m+1)0}}{m+1}t}, \\ L_{m-2}(t) &= \left(\frac{m-3}{m+1}\right) a_{(m+1)0}\, e^{-\frac{a_{(m+1)0}}{m+1}t} + L'_{m-1} * e^{-\frac{a_{(m+1)0}}{m+1}t}, \\ &\vdots \\ L_3(t) &= \left(\frac{2}{m+1}\right) a_{(m+1)0}\, e^{-\frac{a_{(m+1)0}}{m+1}t} + L'_{4} * e^{-\frac{a_{(m+1)0}}{m+1}t}, \\ L_2(t) &= \left(\frac{1}{m+1}\right) a_{(m+1)0}\, e^{-\frac{a_{(m+1)0}}{m+1}t} + L'_{3} * e^{-\frac{a_{(m+1)0}}{m+1}t} \end{cases}$$

by (2.63). Hence, it is concluded that $L_2, L_3, \ldots, L_m$ satisfy all of the conditions of Theorem 1, Theorem 2,..., Theorem $(m-1)$, respectively. Therefore, it is obtained that (2.52) and (2.53) holds for $n = m+1$. Namely, $L_2, L_3, \ldots, L_{m+1}$ satisfy all of the conditions of Theorem 1, Theorem 2,..., Theorem m, respectively.

Furthermore, since the constant b_m in Theorem m is changed b_{m+1} and γ_m is changed γ_{m+1} because of $L_{m+1}(0)/m = a_{(m+1)0}/(m+1) = \gamma_{m+1}$, the constants x_{im} and y_{im} for $1 \leq i \leq m$ are changed $x_{i(m+1)}$ and $y_{i(m+1)}$, respectively by (2.58). Thus, by taking $h_{m+1}, \gamma_{m+1}, x_{i(m+1)}$ and $y_{i(m+1)}$, respectively instead of f_m, γ_m, x_{im} and y_{im} and using Theorem m, the following inequalities for all t are obtained from (2.54), (2.55), (2.56) and (2.57):

$$(2.65)\quad h'_2(t) + b_{m+1}h_2(t) > 0, h''_2(t) + b_{m+1}h'_2(t) > 0, \quad h'''_2(t) + b_{m+1}h''_2(t) > 0$$

for $m \geq 2$,

$$(2.66)\quad \begin{cases} h'_i(t) + \gamma_{m+1}\left[h_i(t) + h_{i-1}(t) + \cdots + h_3(t)\right] + x_{1(m+1)}h_2(t) > 0, \\ h''_i(t) + x_{(i-1)(m+1)}h'_i(t) + \gamma_{m+1}\big[x_{(i-2)(m+1)}h_i(t) + x_{(i-3)(m+1)}h_{i-1}(t) \\ \qquad + \cdots + x_{1(m+1)}h_3(t)\big] > 0, \\ h'''_i(t) + x_{(i-1)(m+1)}h''_i(t) + y_{(i-1)(m+1)}h'_i(t) + \gamma_{m+1}\big[y_{(i-2)(m+1)}h_i(t) \\ \qquad + y_{(i-3)(m+1)}h_{i-1}(t) + \cdots + y_{1(m+1)}h_3(t)\big] > 0 \end{cases}$$

for $i = 3, 4, \ldots, m$ and $m \geq 3$,

$$(2.67)\quad \begin{cases} h'_{m+1}(t) + \gamma_{m+1}\left[h_{m+1}(t) + h_m(t) + \cdots + h_3(t)\right] + x_{1(m+1)}h_2(t) > 0, \\ h''_{m+1}(t) + x_{m(m+1)}h'_{m+1}(t) + \gamma_{m+1}\big[x_{(m-1)(m+1)}h_{m+1}(t) \\ \qquad + x_{(m-2)(m+1)}h_m(t) + x_{(m-3)(m+1)}h_{m-1}(t) \\ \qquad + \cdots + x_{1(m+1)}h_3(t)\big] > 0, \\ h'''_{m+1}(t) + x_{m(m+1)}h''_{m+1}(t) + y_{m(m+1)}h'_{m+1}(t) \\ \qquad + \gamma_{m+1}\big[y_{(m-1)(m+1)}h_{m+1}(t) + y_{(m-2)(m+1)}h_m(t) \\ \qquad + y_{(m-3)(m+1)}h_{m-1}(t) + \cdots + y_{1(m+1)}h_3(t)\big] > 0 \end{cases}$$

for $m \geq 2$, where h_i is the solution of the equation $h_i(t) = 1 - \int_0^t L_i(t-\tau)h_i(\tau)d\tau$ for $2 \leq i \leq m+1$. So, (2.55) holds for $i = 3, 4, \ldots, m+1$ and $m \geq 2$ by (2.66) and (2.67).

On the other hand, by the Convolution Theorem, the solution f_{m+1} of the equivalent equation (2.59) can be written by means of h_{m+1} as

$$(2.68)\quad f_{m+1}(t) = h_{m+1}(t) - \gamma_{m+1}e^{-\gamma_{m+1}} * h_{m+1},$$

where h_{m+1} is the solution of the equation $h_{m+1}(t) = 1 - \int_0^t L_{m+1}(t-\tau)h_{m+1}(\tau)d\tau$. Hence, we derive

$$(2.69) \qquad \begin{cases} h'_{m+1}(t) & = f'_{m+1}(t) + \gamma_{m+1} f_{m+1}(t), \\ h''_{m+1}(t) & = f''_{m+1}(t) + \gamma_{m+1} f'_{m+1}(t), \\ h'''_{m+1}(t) & = f'''_{m+1}(t) + \gamma_{m+1} f''_{m+1}(t) \end{cases}$$

for all t by (2.68). As a result, it is seen by (2.67) and (2.69) that (2.57) holds for $n = m+1$ and $m \geq 2$.

Consequently, it is conclude that if Theorem n is true for $n = m$, then it is also true for $n = m+1$. So, Theorem n is valid for all $n \in \mathbb{N}_2 = \{2, 3, \ldots\}$. ∎

A lot of kernels K_n satisfying the conditions of Theorem n for all $n \in \mathbb{N}_2$ are obtained by our method in Example n, as follows:

Example n. The function K_n of the form

$$(2.70) \quad K_n(t) = c_0 t^{n+2} + \left[\frac{a_{n(n+1)}}{(n+1)!}\right] t^{n+1} + \left(\frac{a_{nn}}{n!}\right) t^n + \cdots + a_{n1} t + a_{n0}, (c_0 \leq 0)$$

corresponding to any constants a_{ni}, $(0 \leq i \leq n+1)$ satisfying conditions $(2) - (n+2)$ of Theorem n also satisfies condition (1) of Theorem n.

In order to see the validity of this assertion, choose the constants a_{n0} and a_{n1} such that the inequality

$$a_{n0}^2 > \left(\frac{2n}{n-1}\right) a_{n1}$$

holds. In this case, it can easily be seen by the proof of Theorem n that the constants $a_{(n-1)0}, a_{(n-1)1}$ defined by

$$a_{(n-1)0} = p_1(\gamma_n), a_{(n-1)1} = p_2(\gamma_n),$$

where

$$\begin{aligned} p_i(\gamma) \quad = \quad & (-1)^i \gamma^i + (-1)^{i-1} a_{n0} \gamma^{i-1} + (-1)^{i-2} a_{n1} \gamma^{i-2} + \cdots - a_{n(i-2)} \gamma + a_{n(i-1)}, \\ & (1 \leq i \leq n+2) \end{aligned}$$

and $\gamma_n = a_{n0}/n$ satisfy condition (2) of Theorem $(n-1)$.

On the other hand, the constants $a_{(n-1)0}, a_{(n-1)1}$ together with $a_{(n-1)2}, a_{(n-1)3}, \ldots, a_{(n-1)n}$ which can also be found as those were found by the method in Example $(n-1)$ satisfy conditions $(3) - (n+1)$ of Theorem $(n-1)$.

Since the inequalities in conditions $(3) - (n+1)$ of Theorem n are equivalent to new inequalities obtained by taking $p_1(\gamma_n), p_2(\gamma_n), \ldots, p_{n+1}(\gamma_n)$, respectively instead of the constants $a_{(n-1)0}, a_{(n-1)1}, \ldots, a_{(n-1)n}$ in conditions $(3)-(n+1)$ of Theorem $(n-1)$, if $a_{(n-1)2}, a_{(n-1)3}, \ldots,$ $a_{(n-1)n}$ are chosen as

$$\begin{aligned} a_{(n-1)2} &= p_3(\gamma_n) = -\gamma_n p_2(\gamma_n) + a_{n2}, \\ a_{(n-1)3} &= p_4(\gamma_n) = -\gamma_n p_3(\gamma_n) + a_{n3}, \\ &\vdots \\ a_{(n-1)n} &= p_{n+1}(\gamma_n) = -\gamma_n p_n(\gamma_n) + a_{nn}, \end{aligned}$$

then $a_{n2}, a_{n3}, \ldots, a_{nn}$ are found as

$$\begin{aligned} a_{n2} &= a_{(n-1)2} + \gamma_n a_{(n-1)1}, \\ a_{n3} &= a_{(n-1)3} + \gamma_n a_{(n-1)2}, \\ &\vdots \\ a_{nn} &= a_{(n-1)n} + \gamma_n a_{(n-1)(n-1)}. \end{aligned}$$

Finally, since $(n+2)$th condition of Theorem n is equivalent to

$$p_{n+2}(\gamma_n) = -\gamma_n p_{n+1}(\gamma_n) + a_{n(n+1)} \leq 0,$$

the constants $a_{n0}, a_{n1}, a_{n2}, \ldots, a_{nn}$ together with the constant $a_{n(n+1)}$ chosen as

$$a_{n(n+1)} \leq \gamma_n a_{(n-1)n}$$

also satisfy $(n+2)$th condition of Theorem n.

Any constants a_{ni}, $(0 \leq i \leq n+1)$ obtained by presented method satisfy conditions $(2) - (n+2)$ of Theorem n.

Thus, the function K_n corresponding to the constants a_{ni} and given by (2.70) also satisfies condition (1) of Theorem n.

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