The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 9, Issue 2, Article 5, pp. 1-13, 2012

POSITIVE SOLUTIONS TO A SYSTEM OF BOUNDARY VALUE PROBLEMS FOR HIGHER-DIMENSIONAL DYNAMIC EQUATIONS ON TIME SCALES

ILKAY YASLAN KARACA

Received 4 August, 2010; accepted 14 August, 2012; published 13 November, 2012.

DEPARTMENT OF MATHEMATICS, EGE UNIVERSITY, 35100 BORNOVA, IZMIR, TURKEY.
ilkay.karaca@ege.edu.tr
URL: http://ege.edu.tr

ABSTRACT. In this paper, we consider the system of boundary value problems for higher-dimensional dynamic equations on time scales. We establish criteria for the existence of at least one or two positive solutions. We shall also obtain criteria which lead to nonexistence of positive solutions. Examples applying our results are also given.

Key words and phrases: Positive solutions, Fixed-point theorems, Time scales, Dynamic equations, Cone.

2000 *Mathematics Subject Classification.* 39A10.

ISSN (electronic): 1449-5910

1. INTRODUCTION

We are concerned with the following system of boundary value problems on time scales $\mathbb{T}$:

$$
\left\{\begin{array}{l}
y^{\Delta^n}(t)+\lambda f(y(\sigma(t)))=0, \quad t \in[a, b] \subset \mathbb{T}, \\
y^{\Delta^i}(a)=0, \quad 0 \leq i \leq n-2, \\
\sum_{i=1}^m \alpha_i y^{\Delta^{n-2}}\left(\xi_i\right)=y^{\Delta^{n-2}}(\sigma(b)),
\end{array}\right. \tag{1.1}
$$

where $\lambda>0$ is a parameter, $n \geq 3, m \geq 1$ are integers, $a<\xi_1<\xi_2<\ldots<\xi_m<b$, $\alpha_i \in(0,+\infty)$ for $1 \leq i \leq m$ and $\sum_{i=1}^m \alpha_i<1$. In addition $f=\left[f_1, f_2, \ldots f_N\right]^T$, where $f_i \in \mathcal{C}([0, \infty),[0, \infty)),\ 1 \leq i \leq N$. We assume that $D=\sigma(b)-a-\sum_{i=1}^m \alpha_i\left(\xi_i-a\right)>0$ and $\sigma(b)$ is right dense so that $\sigma^j(b)=\sigma(b)$ for $j \geq 1$.

The study of dynamic equations on time scales goes back to its founder Stefan Hilger [10]. Some preliminary definitions and theorems on time scales can be found in the books [3] and [4] which are excellent references for the calculus of time scales.

Recently, existence results for positive solutions of higher-order multi-point boundary value problems was studied by some authors, see [5], [6], [7] and [8].

A few papers can be found in the literature on higher-dimensional dynamic equations [1] and [2].

We were, in particular, motivated by Anderson and Hoffacker [2]. They were interested in the following functional dynamic equations on time scales

$$
x^{\Delta}(t)=-A(t) x^\sigma(t)+\lambda h(t) f\left(t, x_t\right). \tag{1.2}
$$

They obtained sufficient conditions for the existence of multiple positive periodic solutions of the system of (1.2) by using Krasnosel'skii fixed point theorem.

This paper is organized as follows. Section 2 introduces some notation and several lemmas which play important roles in this paper. Section 3 gives nonexistence and multiplicity results for positive solutions to the system of (1.1). In this article, the main tool is the following well-known Krasnosel'skii fixed point theorem in a cone [9].

Theorem 1.1. *([9]). Let B be a Banach space, and let $P \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1,\ \bar{\Omega_1} \subset \Omega_2$, and let*

$$
A: P \cap\left(\bar{\Omega_2} \setminus \Omega_1\right) \rightarrow P
$$

be a completely continuous operator such that, either

(i) $|A y| \leq|y|, y \in P \cap \partial \Omega_1,$ *and* $|A y| \geq|y|, y \in P \cap \partial \Omega_2$*; or*
(ii) $|A y| \geq|y|, y \in P \cap \partial \Omega_1,$ *and* $|A y| \leq|y|, u \in P \cap \partial \Omega_2$.

Then A has at least one fixed point in $P \cap\left(\bar{\Omega_2} \setminus \Omega_1\right)$.

2. PRELIMINARIES AND LEMMAS

Let $G_2(t, s)$ be Green's function for the boundary value problems

$$
\left\{\begin{array}{l}
y^{\Delta^2}(t)+\lambda f(t, y(\sigma(t)))=0, \quad t \in[a, b], \\
y(a)=0, \\
\sum_{i=1}^m \alpha_i y\left(\xi_i\right)=y(\sigma(b)).
\end{array}\right. \tag{2.1}
$$

Then,

$$(2.2)\qquad G_2(t,s)=\begin{cases}\frac{(\sigma(b)-t)(\sigma(s)-a)-\sum_{j=i}^{m}\alpha_j\left(\xi_j-t\right)(\sigma(s)-a)+\sum_{j=1}^{i-1}\alpha_j\left(\xi_j-a\right)(t-\sigma(s))}{\sigma(b)-a-\sum_{i=1}^{m}\alpha_i(\xi_i-a)},\\ a\le t\le\sigma(b),\ \ \xi_{i-1}\le\sigma(s)\le\min\{\xi_i,t\},\ \ i=\overline{1,m+1},\\ \frac{(t-a)\left[\sigma(b)-\sigma(s)-\sum_{j=i}^{m}\alpha_j\left(\xi_j-\sigma(s)\right)\right]}{\sigma(b)-a-\sum_{i=1}^{m}\alpha_i(\xi_i-a)},\\ a\le t\le\sigma(b),\ \ \max\{\xi_{i-1},t\}\le\sigma(s)\le\xi_i,\ \ i=\overline{1,m+1}.\end{cases}$$

Lemma 2.1. *There exist a number $k\in(0,1)$ and a continuous function $\psi:[a,b]\to\mathbb{R}^+$ such that*

$$G_2(t,s)\le\psi(s)\ \textit{for}\ t\in[a,\sigma(b)],\ s\in[a,b]$$

and

$$G_2(t,s)\ge k\psi(s)\ \textit{for}\ t\in[\xi_1,\sigma(b)],\ s\in[a,b],$$

where

$$\psi(s)=\frac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D},$$

$$(2.3)\qquad k=\min_{2\le i\le m}\left\{\frac{1}{\sigma(b)}\sum_{j=i}^{m}\alpha_j\left(\sigma(b)-\xi_j\right),\ \frac{\xi_1-a}{\sigma(s)-a}\left[1-\sum_{j=i}^{m}\alpha_j\right]\right\}.$$

Proof. Now, we will show that we may take $\psi(s)=\frac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D}$.

Upper bounds:

Case 1. Consider $0\le\sigma(s)\le\xi_1,\ \sigma(s)\le t$. Then

$$\begin{aligned}G_2(t,s)&=\frac{\sigma(b)-t-\sum_{j=1}^{m}\alpha_j\left(\xi_j-t\right)}{D}(\sigma(s)-a)\\&=\frac{\sigma(b)-\sum_{j=1}^{m}\alpha_j\xi_j+t\left(\sum_{j=1}^{m}\alpha_j-1\right)}{D}(\sigma(s)-a).\end{aligned}$$

Since $\sum_{j=1}^{m}\alpha_j<1$, the maximum occurs when $t=\sigma(s)$ and then

$$\begin{aligned}G_2(t,s)&\le\frac{\sigma(b)-\sigma(s)+\sum_{j=1}^{m}\alpha_j\left(\sigma(s)-\xi_j\right)}{D}(\sigma(s)-a)\\&\le\frac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D},\end{aligned}$$

since $\sum_{j=1}^{m}\alpha_j\left(\sigma(s)-\xi_j\right)\le0$ for $\sigma(s)\le\xi_1$ and $\xi_j\in(a,b)$ with $a<\xi_1<\xi_2<...<\xi_{m-2}<b$.

Case 2. For $\xi_{r-1}\le t\le\xi_r,\ 2\le r\le m+1,\ \xi_{i-1}\le\sigma(s)\le\xi_i,\ \ 2\le i\le r,\ \sigma(s)\le t$, we have

$$\begin{aligned}
&G_2(t,s) \\
&= \frac{(\sigma(b)-t)(\sigma(s)-a) - \sum_{j=i}^{m}\alpha_j\left(\xi_j - t\right)(\sigma(s)-a) + \sum_{j=1}^{i-1}\alpha_j\left(\xi_j - a\right)(t-\sigma(s))}{D} \\
&= \frac{(\sigma(b)-t)(\sigma(s)-a) - \sum_{j=i}^{m}\alpha_j\left(\xi_j - \sigma(s)\right)(\sigma(s)-a)}{D} \\
&\quad + \frac{\sum_{j=1}^{m}\alpha_j(t-\sigma(s))(\sigma(s)-a) + \sum_{j=1}^{i-1}\alpha_j\left(\xi_j-\sigma(s)\right)(t-\sigma(s))}{D} \\
&\le \frac{\sigma(b) - t + \sum_{j=1}^{m}\alpha_j(t-\sigma(s))}{D}(\sigma(s)-a) \\
&\le \frac{\sigma(b) - \sigma(s)\sum_{j=1}^{m}\alpha_j + t\left(\sum_{j=1}^{m}\alpha_j - 1\right)}{D}(\sigma(s)-a),
\end{aligned}$$

since $\sum_{j=i}^{m}\alpha_j\left(\sigma(s)-\xi_j\right) \le 0$ and $\sum_{j=1}^{i-1}\alpha_j\left(\xi_j-\sigma(s)\right) \le 0$ for $\xi_{i-1} \le \sigma(s) \le \xi_i,\ 2 \le i \le m+1.$

Since $\sum_{j=1}^{m}\alpha_j < 1$, the maximum occurs when $t = \sigma(s)$ so

$$G_2(t,s) \le \frac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D}.$$

Case 3. For $\xi_{r-1} \le t \le \xi_r,\ 2 \le r \le m,\ \xi_{i-1} \le \sigma(s) \le \xi_i,\ \ r \le i \le m,\ t \le \sigma(s)$, we obtain

$$\begin{aligned}
G_2(t,s) &= \frac{(t-a)\left[\sigma(b) - \sigma(s) - \sum_{j=i}^{m}a_j\left(\xi_j - \sigma(s)\right)\right]}{\sigma(b) - a - \sum_{i=1}^{m}a_i(\xi_i - a)} \\
&\le \frac{(\sigma(b)-\sigma(s))(t-a)}{D} \\
&\le \frac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D}
\end{aligned}$$

since $\sum_{j=i}^{m}\alpha_j\left(\xi_j - \sigma(s)\right) \ge 0$ for $\xi_{i-1} \le \sigma(s) \le \xi_i,\ 2 \le i \le m.$
Case 4. For $\xi_m \le \sigma(s) \le \sigma(b),\ \ t \le \sigma(s)$, we clearly have

$$G_2(t,s) \le \frac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D}.$$

Lower bounds: We shall show that we may take an arbitrary interval $[\xi_1, \sigma(b)] \subset (a, \sigma(b)]$.

We are looking for $\min\{G_2(t,s) : t \in [\xi_1, \sigma(b)]\}$ as a function of s of the same form as the upper bound.
Case 1. Consider $0 \le \sigma(s) \le \xi_1,\ \sigma(s) \le t$, we get

$$\begin{aligned} G_2(t,s) &= \frac{\sigma(b) - t - \sum_{j=1}^{m} \alpha_j \left(\xi_j - t\right)}{D} (\sigma(s) - a) \\ &= \frac{\sigma(b) - \sum_{j=1}^{m} \alpha_j \xi_j + t\left(\sum_{j=1}^{m} \alpha_j - 1\right)}{D} (\sigma(s) - a). \end{aligned}$$

Since $\sum_{j=1}^{m} \alpha_j < 1$, the minimum occurs when $t = \sigma(b)$ and then

$$\begin{aligned} G_2(t,s) &\geq \frac{\sigma(b) - \sum_{j=1}^{m} \alpha_j \xi_j + \sigma(b)\left(\sum_{j=1}^{m} \alpha_j - 1\right)}{D} (\sigma(s) - a) \\ &> \frac{(\sigma(b) - \sigma(s))(\sigma(s) - a)}{D} \frac{1}{\sigma(b)} \sum_{j=1}^{m} \alpha_j \left(\sigma(b) - \xi_j\right). \end{aligned}$$

Case 2. For $\xi_{r-1} \leq t \leq \xi_r,\ 2 \leq r \leq m+1,\ \xi_{i-1} \leq \sigma(s) \leq \xi_i,\ \ 2 \leq i \leq r,\ \sigma(s) \leq t$, we have

$$\begin{aligned} &G_2(t,s) \\ &= \frac{(\sigma(b) - t)(\sigma(s) - a) - \sum_{j=i}^{m} \alpha_j \left(\xi_j - \sigma(s)\right)(\sigma(s) - a)}{D} \\ &\quad + \frac{\sum_{j=1}^{m} \alpha_j (t - \sigma(s))(\sigma(s) - a) + \sum_{j=1}^{i-1} \alpha_j \left(\xi_j - \sigma(s)\right)(t - \sigma(s))}{D} \\ &= \frac{t\left[\left(\sum_{j=1}^{m} \alpha_j - 1\right)(\sigma(s) - a) + \sum_{j=1}^{i-1} \alpha_j \left(\xi_j - \sigma(s)\right)\right]}{D} \\ &\quad + \frac{\left[\sigma(b) - \sigma(s) \sum_{j=1}^{m} \alpha_j - \sum_{j=i}^{m} \alpha_j \left(\xi_j - \sigma(s)\right)\right](\sigma(s) - a) - \sigma(s) \sum_{j=1}^{i-1} \alpha_j \left(\xi_j - \sigma(s)\right)}{D}. \end{aligned}$$

Since $\left(\sum_{j=1}^{m} \alpha_j - 1\right)(\sigma(s) - a) + \sum_{j=1}^{i-1} \alpha_j \left(\xi_j - \sigma(s)\right) < 0$, the minimum occurs when $t = \sigma(b)$, then

$$\begin{aligned} G_2(t,s) &\geq \frac{-\sum_{j=i}^{m} \alpha_j \left(\xi_j - \sigma(b)\right)(\sigma(s) - a) + \sum_{j=1}^{i-1} \alpha_j \left(\xi_j - a\right)(\sigma(b) - \sigma(s))}{D} \\ &\geq \frac{1}{D} \sum_{j=i}^{m} \alpha_j \left(\sigma(b) - \xi_j\right)(\sigma(s) - a) \\ &> \frac{(\sigma(b) - \sigma(s))(\sigma(s) - a)}{D} \frac{1}{\sigma(b)} \sum_{j=i}^{m} \alpha_j \left(\sigma(b) - \xi_j\right). \end{aligned}$$

Case 3. For $\xi_{r-1} \leq t \leq \xi_r,\ 2 \leq r \leq m,\ \xi_{i-1} \leq \sigma(s) \leq \xi_i,\ \ r \leq i \leq m,\ t \leq \sigma(s)$, we obtain

$$\begin{aligned}
G_2(t,s) &= \frac{(t-a)\left[\sigma(b)-\sigma(s)-\sum_{j=i}^{m}\alpha_j\left(\xi_j-\sigma(s)\right)\right]}{D} \\
&= \frac{(t-a)\left[(\sigma(b)-\sigma(s))\left(1-\sum_{j=i}^{m}\alpha_j\right)-\sum_{j=i}^{m}\alpha_j\left(\xi_j-\sigma(b)\right)\right]}{D} \\
&\geq \frac{(t-a)(\sigma(b)-\sigma(s))}{D}\left[1-\sum_{j=i}^{m}\alpha_j\right] \\
&\geq \frac{(\xi_1-a)(\sigma(b)-\sigma(s))}{D}\left[1-\sum_{j=i}^{m}\alpha_j\right] \\
&= \frac{(\sigma(s)-a)(\sigma(b)-\sigma(s))}{D}\frac{\xi_1-a}{\sigma(s)-a}\left[1-\sum_{j=i}^{m}\alpha_j\right].
\end{aligned}$$

Case 4. For $\xi_m \leq \sigma(s) \leq \sigma(b)\,,\ \ t \leq \sigma(s)$, we have

$$\begin{aligned}
G_2(t,s) &= \frac{(t-a)(\sigma(b)-\sigma(s))}{D} \\
&\geq \frac{(\xi_1-a)(\sigma(b)-\sigma(s))}{D} \\
&= \frac{(\sigma(s)-a)(\sigma(b)-\sigma(s))}{D}\frac{\xi_1-a}{\sigma(s)-a}.
\end{aligned}$$

Thus we can take

$$k = \min_{2\leq i\leq m}\left\{\frac{1}{\sigma(b)}\sum_{j=i}^{m}\alpha_j\left(\sigma(b)-\xi_j\right),\ \frac{\xi_1-a}{\sigma(s)-a}\left[1-\sum_{j=i}^{m}\alpha_j\right]\right\}.$$

∎

Lemma 2.2. *If y satisfies the boundary conditions*

$$\begin{cases} y^{\Delta^i}(a)=0, & 0\leq i\leq n-2, \\ \sum_{i=1}^{m}a_i y^{\Delta^{n-2}}(\xi_i)=y^{\Delta^{n-2}}(\sigma(b)) \end{cases}$$

and

$$y^{\Delta^n}(t)\leq 0, \quad t\in[a,b],$$

then

$$y^{\Delta^{n-2}}(t)\geq 0.$$

Proof. Let $P(t)=y^{\Delta^{n-2}}(t)\,,\ t\in[a,\sigma(b)]$. Then we have

$$P^{\Delta^2}(t)\leq 0, \quad t\in[a,b],$$

$$P(a)=0 \text{ and } \sum_{i=1}^{m}a_i P(\xi_i)=P(\sigma(b)).$$

It must be true that $P(\sigma(b)) \geq 0$. To see this, assume to the contrary that $P(\sigma(b)) < 0$. Since $P(a) = 0$ and $P(t)$ is concave downward, we have

$$P(t) \geq \frac{t-a}{\sigma(b)-a} P(\sigma(b)), \quad t \in [a, \sigma(b)].$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{m} a_i P(\xi_i) - P(\sigma(b)) &\geq \sum_{i=1}^{m} a_i \frac{\xi_i - a}{\sigma(b) - a} P(\sigma(b)) - P(\sigma(b)) \\ &> \sum_{i=1}^{m} a_i P(\sigma(b)) - P(\sigma(b)) \\ &> P(\sigma(b)) - P(\sigma(b)) = 0, \end{aligned}$$

which is a contradiction.

Now, $P(a) = 0, \ \ P(\sigma(b)) \geq 0$, and $P(t)$ is concave downward, so we have

$$P(t) = y^{\Delta^{n-2}}(t) \geq 0, \quad t \in [a, \sigma(b)].$$

This completes the proof of the lemma. ∎

Let $\mathbb{B}$ be the Banach space defined by

$$\mathbb{B} = \left\{ y : y^{\Delta^n} \text{ is continuous on } [a, \sigma(b)], \ y^{\Delta^i}(a) = 0, \ 0 \leq i \leq n-3 \right\},$$

with the norm $\|y\| = \max_{1 \leq i \leq n} |y_i|_0$, where $|y_i|_0 = \sup_{t \in [a, \sigma(b)]} \left| y_i^{\Delta^{n-2}}(t) \right|$ and let

$$\mathcal{P} = \left\{ y \in \mathbb{B} : y_i^{\Delta^{n-2}}(t) \geq 0, \quad \min_{t \in [\xi_1, \sigma(b)]} y_i^{\Delta^{n-2}}(t) \geq k \|y\| \right\}$$

where k is as in (2.3).

Solving the system (1.1) is equivalent to finding fixed points of the operator $L_\lambda : \mathcal{B} \to \mathcal{B}$ defined by

$$L_\lambda y(t) = \lambda \int_a^{\sigma(b)} G_n(t,s) f(s, y^\sigma(s)) \Delta s, \quad t \in [a, \sigma(b)] \tag{2.4}$$

and denote

$$L_\lambda y = \left(L_\lambda^1 y, \ L_\lambda^2 y, ..., \ L_\lambda^n y \right)^T.$$

It can be verified that

$$G_2(t,s) = G_n^{\Delta^{n-2}}(t,s). \tag{2.5}$$

From (2.5) it follows that

$$(L_\lambda y)^{\Delta^{n-2}}(t) = \lambda \int_a^{\sigma(b)} G_2(t,s) f(s, y^\sigma(s)) \Delta s \tag{2.6}$$

where

$$(L_\lambda y)^{\Delta^{n-2}} = \left(\left(L_\lambda^1 y\right)^{\Delta^{n-2}}, \ \left(L_\lambda^2 y\right)^{\Delta^{n-2}}, ..., \ \left(L_\lambda^n y\right)^{\Delta^{n-2}} \right)^T.$$

Solving the system (1.1) in $\mathbb{B}$ is equivalent to finding fixed points of the operator $L_\lambda^{\Delta^{n-2}}$ defined by (2.6).

Lemma 2.3. *The operator L_λ is completely continuous such that $L_\lambda(\mathcal{P}) \subset \mathcal{P}$.*

Proof. From the continuity of $G_2(t,s)$ and $f(t,\xi)$ it follows that the operator L_λ defined by (2.4) is completely continuous in $\mathbb{B}$. By Lemma 2.1, Lemma 2.2 and definition of $\mathcal{P}$, we get $L_\lambda \mathcal{P} \subset \mathcal{P}$. ∎

3. EXISTENCE OF POSITIVE SOLUTIONS

Now we are ready to establish a few sufficient conditions for the existence of at least one or two positive solutions and the nonexistence of positive solutions of (1.1).
Now we define

$$l_i^0 = \lim_{\|u\| \to 0} \frac{f_i(u)}{\|u\|}, \ \ l_i^\infty = \lim_{\|u\| \to \infty} \frac{f_i(u)}{\|u\|},$$

for $1 \le i \le n$.

Theorem 3.1. *For each λ satisfying*

$$\frac{1}{k l_i^\infty \int_a^{\sigma(b)} \psi(s)\,\Delta s} < \lambda < \frac{1}{l_i^0 \int_a^{\sigma(b)} \psi(s)\,\Delta s} \tag{3.1}$$

there exists at least one positive solution of (1.1).

Proof. Let λ be given as in (3.1). Now, let $\epsilon > 0$ be chosen such that

$$\frac{1}{k\left(l_i^\infty - \epsilon\right) \int_a^{\sigma(b)} \psi(s)\,\Delta s} \le \lambda \le \frac{1}{\left(l_i^0 + \epsilon\right) \int_a^{\sigma(b)} \psi(s)\,\Delta s}.$$

Now, turning to l_i^0, there exists an $a_i > 0$ such that $f_i(y) \le (l_i^0 + \epsilon)\|y\|$ for $0 < \|y\| \le a_i$. So, for $y \in \mathcal{P}$ with $\|y\| = a_i$, we have from the fact that $0 \le G_2(t,s) \le \psi(s)$ for $t \in [a, \sigma(b)]$, $s \in [a,b]$,

$$\begin{aligned} \left(L_\lambda^i y\right)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s)\, f_i(y^\sigma(s))\,\Delta s \\ &\le \lambda \int_a^{\sigma(b)} \psi(s)\, f_i(y^\sigma(s))\,\Delta s \\ &\le \lambda \left(l_i^0 + \epsilon\right) \int_a^{\sigma(b)} \psi(s)\,\Delta s\, \|y\| \\ &\le \|y\| = a_i. \end{aligned}$$

Next, considering l_i^∞, there exists $\hat{b_i} > 0$ such that $f_i(y) \ge (l_i^\infty - \epsilon)\|y\|$ for $\|y\| \ge \hat{b_i}$. Let $b_i = \max\left\{2a_i, \frac{\hat{b_i}}{k}\right\}$. Then $y \in \mathcal{P}$ and $\|y\| = b_i$ implies

$$\|y\| \ge \left| y_i^{\Delta^{n-2}}(t) \right|_0 \ge \min_{t \in [\xi_1, \sigma(b)]} y_i^{\Delta^{n-2}}(t) \ge k\|y\| \ge \hat{b_i},$$

and so for $t \in [\xi_1, \sigma(b)]$,

$$\begin{aligned}\left(L_\lambda^i y\right)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s)\, f_i\left(y^\sigma(s)\right)\Delta s\\ &\geq \lambda k \int_a^{\sigma(b)} \psi(s)\,\Delta s\,\left(l_i^\infty - \epsilon\right)\|y\|\\ &\geq \|y\| = b_i.\end{aligned}$$

If we take $a = \max\{a_i : 1 \leq i \leq n\}$, $b = \min\{b_i : 1 \leq i \leq n\}$, by Theorem 1.1, L_λ has a fixed point y such that $\min\{a,b\} \leq \|y\| \leq \max\{a,b\}$. The proof is complete. ∎

Theorem 3.2. *For each λ satisfying*

$$\frac{1}{k l_i^0 \int_a^{\sigma(b)} \psi(s)\,\Delta s} < \lambda < \frac{1}{l_i^\infty \int_a^{\sigma(b)} \psi(s)\,\Delta s}, \tag{3.2}$$

there exists at least one positive solution of (1.1).

Proof. Let λ be given as in (3.2), and choose let $\epsilon > 0$ such that

$$\frac{1}{k\left(l_i^0 - \epsilon\right)\int_a^{\sigma(b)} \psi(s)\,\Delta s} \leq \lambda \leq \frac{1}{\left(l_i^\infty + \epsilon\right)\int_a^{\sigma(b)} \psi(s)\,\Delta s}.$$

Beginning with l_i^0, there exists an $a_i > 0$ such that $f_i(y) \geq \left(l_i^0 - \epsilon\right)\|y\|$ for $0 < \|y\| \leq a_i$. So, for $y \in \mathcal{P}$ with $\|y\| = a_i$, and $t \in [\xi_1, \sigma(b)]$ we have

$$\begin{aligned}\left(L_\lambda^i y\right)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s)\, f_i\left(y^\sigma(s)\right)\Delta s\\ &\geq \lambda k \int_a^{\sigma(b)} \psi(s)\, f_i\left(y^\sigma(s)\right)\Delta s\\ &\geq \lambda k\left(l_i^0 - \epsilon\right)\int_a^{\sigma(b)} \psi(s)\,\Delta s\,\|y\|\\ &\geq \|y\| = a_i.\end{aligned}$$

It remains to consider l_i^∞. There exists $\hat{b}_i > 0$ such that $f_i(y) \leq \left(l_i^\infty + \epsilon\right)\|y\|$ for $\|y\| \geq \hat{b}_i$. There are two cases:

For case (a), suppose $N > 0$ is such that $f_i(y) \leq N$, for all $0 \leq y < \infty$. Let $b_i = \max\left\{2a_i, \lambda N \int_a^{\sigma(b)} \psi(s)\,\Delta s\right\}$. Then $y \in \mathcal{P}$ and $\|y\| = b_i$, we have

$$\begin{aligned}\left(L_\lambda^i y\right)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s)\, f_i\left(y^\sigma(s)\right)\Delta s\\ &\leq \lambda N \int_a^{\sigma(b)} \psi(s)\,\Delta s\\ &\leq \|y\| = b_i.\end{aligned}$$

For case (b), let $g_i(h) := \max\{f_i(y) : 0 \leq y \leq h\}$. The function g_i is nondecreasing and $\lim_{h\to\infty} g_i(h) = \infty$. Choose $b_i = \max\left\{2a_i, \hat{b}_i\right\}$ so that $g_i(b_i) \geq g_i(h)$ for $0 \leq h \leq b_i$. For $y \in \mathcal{P}$ and $\|y\| = b_i$, we have

$$\begin{aligned}\left(L_\lambda^i y\right)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s)\, f_i\left(y^\sigma(s)\right)\Delta s \\ &\leq \lambda g_i(b_i)\int_a^{\sigma(b)} \psi(s)\,\Delta s \\ &\leq \lambda\left(l_i^\infty+\epsilon\right) b_i \int_a^{\sigma(b)} \psi(s)\,\Delta s \\ &\leq \|y\| = b_i.\end{aligned}$$

If we take $a = \min\{a_i : 1 \leq i \leq n\}$, $b = \max\{b_i : 1 \leq i \leq n\}$, by Theorem 1.1, L_λ has a fixed point y such that $\min\{a,b\} \leq \|y\| \leq \max\{a,b\}$. The proof is complete. ∎

The rest of the paper we assume $f(y) > 0$ on $\mathbb{R}^+$.
Set

$$A = \int_a^{\sigma(b)} \psi(s)\,\Delta s.$$

Theorem 3.3. (a) *If either* $l_i^0 = \infty$ *or* $l_i^\infty = \infty$ *for* $1 \leq i \leq n$, *then for all* $0 < \lambda \leq \lambda_0$, *where*

$$\lambda_0 := \frac{1}{A}\min_{1\leq i\leq n}\sup_{r>0}\frac{r}{\max_{0<\|u\|\leq r} f_i(u)},$$

(1.1) has at least one positive solution.

(b) *If either* $l_i^0 = 0$ *or* $l_i^\infty = 0$ *for* $1 \leq i \leq n$, *then for all* $\lambda \geq \lambda_0$, *where*

$$\lambda_0 := \frac{1}{A}\max_{1\leq i\leq n}\inf_{r>0}\frac{r}{\min_{0<\|u\|\leq r} f_i(u)},$$

(1.1) has at least one positive solution.

Proof. We now prove the part (a) of Theorem 3.3. Let $r > 0$ given.
If $\|y\| = r$, it follows that

$$\|L_\lambda y\| = \max_{1\leq i\leq n}\left(L_\lambda^i y\right)^{\Delta^{n-2}}(t) \leq \lambda_0 \int_a^{\sigma(b)} G_2(t,s)\, f_i\left(y^\sigma(s)\right)\Delta s \leq r.$$

So for all $0 < \lambda \leq \lambda_0$ we have

$$\|L_\lambda y\| \leq \|y\|.$$

Fix $\lambda \leq \lambda_0$. Choose $R > 0$ sufficiently large so that

$$\lambda R k \int_a^{\sigma(b)} \psi(s)\,\Delta s \geq 1. \tag{3.3}$$

Since $l_i^0 = \infty$, there is $a_i > 0$ $(1 \leq i \leq n)$ such that

$$\frac{f_i(y)}{\|y\|} \geq R$$

for $t \in [a, \sigma(b)]$, $0 < \|y\| \leq a_i$. Hence we have that

$$f_i(y) \geq R\|y\|$$

for $t \in [a, \sigma(b)]$, $0 < \|y\| \leq a_i$. For $y \in \mathcal{P}$, $\|y\| = a_i$ and $t \in [\xi_1, \sigma(b)]$, we get

$$\left(L_\lambda^i y\right)^{\Delta^{n-2}}(t) \geq \lambda R k \int_a^{\sigma(b)} \psi(s)\,\Delta s\,\|y\| \geq \|y\| = a_i$$

by (3.3). If we choose $a = \min\{a_i : 1 \leq i \leq n\}$, then L_λ has a fixed point y such that $\min\{a, r\} \leq \|y\| \leq \max\{a, r\}$.

Next, we use the assumption that $l_i^\infty = \infty$. Since $l_i^\infty = \infty$ there is a $b_i > 0$, $(1 \leq i \leq n)$ such that

$$\frac{f_i(y)}{\|y\|} \geq R$$

for $\|y\| \geq b_i$ and R is chosen so that (3.3) holds. It follows that

$$f_i(y) \geq R\|y\|$$

for $\|y\| \geq b_i$.
For $y \in \mathcal{P}$, $\|y\| = b_i$ and $t \in [\xi_1, \sigma(b)]$, we have

$$\begin{aligned}\left(L_\lambda^i y\right)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s) f_i(y^\sigma(s)) \Delta s \\ &\geq \lambda R k \int_a^{\sigma(b)} \psi(s) \Delta s \|y\| \\ &\geq b_i = \|y\|\end{aligned}$$

by (3.3). If we choose $b = \min\{b_i : 1 \leq i \leq n\}$, then L_λ has a fixed point y such that $\min\{b, r\} \leq \|y\| \leq \max\{b, r\}$. This completes the proof of part (a). Part (b) holds in an analogous way. ∎

Theorem 3.4. $a)$ *If* $l_i^0 = l_i^\infty = \infty$ *for* $1 \leq i \leq n$, *then there is a* $\lambda_0 > 0$ *such that for all* $0 < \lambda \leq \lambda_0$, *(1.1) has two positive solutions.*
$b)$ *If* $l_i^0 = l_i^\infty = 0$ *for* $1 \leq i \leq n$, *then there is a* $\lambda_0 > 0$ *such that for all* $\lambda \geq \lambda_0$, *(1.1) has two positive solutions.*

Now, we give a nonexistence result as follows.

Theorem 3.5. (a) *If there is a constant* $c > 0$ *such that* $f_i(y) \geq c\|y\|$, *then there is a* $\lambda_0 > 0$ *such that (1.1) has no positive solutions for* $\lambda \geq \lambda_0$.
(b) *If there is a constant* $c > 0$ *such that* $f_i(y) \leq c\|y\|$, *then there is a* $\lambda_0 > 0$ *such that (1.1) has no positive solutions for* $0 < \lambda \leq \lambda_0$.

Proof. We now prove the part (a) of this theorem. Assume there is a constant $c > 0$ such that $f_i(y) \geq c\|y\|$. Assume $y(t)$ is a solution of the system (1.1). We will show that for λ sufficiently large that this leads to a contradiction. We have for $t \in [\xi_1, \sigma(b)]$,

$$y_i^{\Delta^{n-2}}(t) = \lambda \int_a^{\sigma(b)} G_2(t,s) f_i(y^\sigma(s)) \Delta s \geq ck\lambda_0 \int_a^{\sigma(b)} \psi(s) \Delta s \|y\|.$$

If we pick λ_0 sufficiently large so that $ck\lambda_0 \int_a^{\sigma(b)} \psi(s) \Delta s > 1$ for all $\lambda \geq \lambda_0$, then we have $y^{\Delta^{n-2}} > \|y\|$ which is a contradiction. The proof of part (b) is similar. ∎

Example 3.6. We illustrate Theorem 3.2 with specific time scale $\mathbb{T} = \mathbb{Z} \cup [5, 7]$.
Consider the system:

$$(3.4)\qquad \begin{cases} y^{\Delta^n}(t) + \lambda f(y^\sigma(t)) = 0, & t \in [0,4] \subset \mathbb{T}, \\ y^{\Delta^i}(0) = 0, \quad 0 \le i \le n-2, \\ 1/2y(1) + 1/3y(2) + 1/10y(3) = y(5) \end{cases}$$

where $f = \left[1 + (y_1 + y_2)^{1/5}, \ \ e^{-(y_1+y_2)}\right]^T, \ \alpha_1 = 1/2, \ \alpha_2 = 1/3, \alpha_3 = 1/10, \ a = 0, \ b = 4, \ \xi_1 = 1, \ \xi_2 = 2, \ \xi_3 = 3.$
Since $f_1(y) = 1 + (y_1 + y_2)^{1/5}, \ \ f_2(y) = e^{-(y_1+y_2)}$, we have

$$l_i^0 = \infty, \quad l_i^\infty = 0,$$

for $i = 1, 2$. We get $\psi(s) = \frac{15}{53}(4-s)(s+1), \ \int_0^5 \psi(s)\Delta s = \frac{300}{53}$. Therefore the assumptions of Theorem 3.2 are satisfied. By Theorem 3.2, for all $\lambda \in (0,\infty)$, (3.4) has at least one positive solution.

Example 3.7. We illustrate Theorem 3.4 with specific time scale $\mathbb{T} = \mathbb{R}$.
Consider the system:

$$(3.5)\qquad \begin{cases} y'''(t) + \lambda f(y(t)) = 0, & t \in [0,5], \\ y(0) = y'(0) = 0, \\ 1/2y(1) + 1/4y(2) = y(5) \end{cases}$$

where $f = \left[e^{y_1+y_2+y_3+1}, \ \ (y_1+y_2+y_3)^2 + 1, \ \ (y_1y_2y_3)^2 + 5\right]^T$.

Since $f_1(y) = e^{y_1+y_2+y_3+1}, \ \ f_2(y) = (y_1+y_2+y_3)^2 + 1, \ \ f_3(y) = (y_1y_2y_3)^2 + 5$, we get $l_i^\infty = \infty$, for $i = 1, 2$.

Since

$$A = \int_0^5 \psi(s)\,ds = \frac{125}{24},$$

$$\sup_{r>0} \frac{r}{\max_{0<\|(y_1,y_2,y_3)\| \le r} e^{y_1+y_2+y_3+1}} = \sup_{r>0} \frac{r}{e^{3r+1}} = \frac{1}{3e^2},$$

$$\sup_{r>0} \frac{r}{\max_{0<\|(y_1,y_2,y_3)\| \le r} (y_1+y_2+y_3)^2 + 1} = \sup_{r>0} \frac{r}{9r^2+1} = \frac{1}{6},$$

$$\sup_{r>0} \frac{r}{\max_{0<\|(y_1,y_2,y_3)\| \le r} (y_1y_2y_3)^2 + 5} = \sup_{r>0} \frac{r}{r^6+5} = \frac{1}{6},$$

we have

$$\lambda_0 = \frac{1}{A} \min_{1\le i\le 3} \sup_{r>0} \frac{r}{\max_{0<\|y\|\le r} f_i(y)} = \frac{8}{125} e^{-2}.$$

So, by Theorem 3.4, for all $\lambda \in \left(0, \frac{8}{125}e^{-2}\right]$, (3.5) has one positive solution.

REFERENCES

[1] D. R. ANDERSON and J. HOFFACKER, Positive periodic time scale solutions for functional dynamic equations, *Aust. J. Math. Anal. Appl.*, **3** (2006), pp. 1–14.

[2] D. R. ANDERSON and J. HOFFACKER, Higher-dimensional functional dynamic equations on periodic time scales, *J. Difference Equ. Appl.*, **14** (2008), pp. 83–89.

[3] M. BOHNER and A. PETERSON, *Dynamic Equations on Time scales, An Introduction with Applications*, Birkhäuser Boston, 2001.

[4] M. BOHNER and A. PETERSON, *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, 2003.

[5] P. CHANGCI, D. WEI and W. ZHONGLI, Green's function and positive solutions of nth order m-point boundary value problem, *Appl. Math. Comput.*, **182** (2006), pp. 1231–1239.

[6] P. W. ELOE and B. AHMAD, Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions, *Appl. Math. Letters*, **18** (2005), pp. 521–527.

[7] J. R. GRAEF and B. YANG, Positive solutions to a multi-point higher order boundary value problem, *J. Math. Anal. Appl.*, **316** (2006), pp. 409–421.

[8] J. R. GRAEF, J. HENDERSON, P. J. Y. WONG and B. YANG, Three solutions of an nth order three-point focal type boundary value problem, *Nonlinear Anal.*, **69** (2008), pp. 3386–3404.

[9] D. GUO and V. LAKSHMKANTHAM, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.

[10] S. HILGER, Analysis on measure chains a unifed approach to continuous and discrete calculus, *Results Math.*, **18** (1990), pp. 18–56.