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SOLUTION OF INEQUALITIES WITH POWER-EXPONENTIAL FUNCTIONS BY CÎRTOAJE

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ABSTRACT. In this paper, we prove the open inequality $a^{2b} + b^{2a} \leq 1$ for all nonnegative real numbers a and b such that $a + b = 1$.

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1. PRELIMINARIES

In the paper [2], V. Cîrtoaje proved the following inequality

$$a^{2b} + b^{2a} \leq 2$$

for all nonnegative real numbers a and b such that $a + b = 2$ and conjectured the following inequality

$$a^{2b} + b^{2a} \leq 1$$

for all nonnegative real numbers a and b such that $a + b = 1$. In this paper, we will prove this open inequality.

2. MAIN RESULT

Without loss of generality, assume that $a \leq b$.

Theorem 2.1.

$$a^{2b} + b^{2a} \leq 1$$

for all nonnegative real numbers a *and* b *such that* $a + b = 1$.

Put $f(a) = a^{2-2a} + (1-a)^{2a}$ for $0 \leq a \leq \frac{1}{2}$. Then we prove the case $f(a) \leq 1$ for $0 \leq a \leq \frac{1}{2}$. If $a = 0$or $\frac{1}{2}$, then it is easy to see $f(a) = 1$ we assume that $0 < a < \frac{1}{2}$.

Proof. (i) Case $0 < a \leq \frac{1}{4}$. By Bernoulli's inequality, we have $f(a) < a^{2-2a} + 1 - 2a^2 = 1 - a^2(2 - a^{-2a})$. Since a^{-2a} is strictly increasing on an interval $[0, e^{-1}]$, we have $1 - a^2(1 - a^{-2a}) \leq 1 - (\frac{1}{4})^2(2 - (\frac{1}{4})^{-\frac{1}{2}}) = 1$, this condition fulfilled.

(ii) Case $\frac{1}{4} < a < \frac{1}{2}$. Put $a = \frac{1-x}{2}$, then $0 \leq x \leq \frac{1}{2}$ and

$$\begin{aligned} f(a) = g(x) &= \left(\frac{1-x}{2}\right)^{1+x} + \left(\frac{1+x}{2}\right)^{1-x} \\ &= \frac{1}{2}\{e^{x \ln 2}(1+x)^{1-x} + e^{-x \ln 2}(1-x)^{1+x}\}. \end{aligned}$$

Put $c = \ln 2$. It is well known that

$$\ln \frac{1-x}{1+x} = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}.$$

Put $x = \frac{1}{3}$, we have

$$c = \ln 2 = 2\left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \cdots\right).$$

Then by $2(\frac{1}{3} + \frac{1}{3^4}) < c < 2(\frac{1}{3} + \frac{1}{3^4} + \frac{1}{3^6} + \frac{1}{3^8} + \cdots)$, we have $0.69 < c < 0.7$. Since $0 \leq x \leq \frac{1}{2}$ we have $0 < cx < \frac{2}{5}$. Then

$$\begin{aligned} e^{x \ln 2} &= 1 + cx + \frac{(cx)^2}{2!}\left(1 + \frac{cx}{3} + \frac{(cx)^2}{3 \cdot 4} + \cdots\right) \\ &\leq 1 + cx + \frac{(cx)^2}{2}\left(1 + \frac{2}{15} + (\frac{2}{15})^2 + \cdots\right) \\ &= 1 + cx + \frac{15}{26}(cx)^2, \end{aligned}$$

$$\begin{aligned} e^{-x\ln 2} &= 1 - cx + \frac{(cx)^2}{2} - \frac{(cx)^3}{3!} + \frac{(cx)^4}{4!} - \cdots \\ &\le 1 - cx + \frac{(cx)^2}{2}, \\ (1+x)^{1-x} &\le 1 + (1-x)x + \frac{x-1}{2}x^3, \\ (1-x)^{1+x} &\le 1 - (1+x)x + \frac{x+1}{2}x^3 + \frac{1-x^2}{6}x^4\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right) \\ &= 1 - (1+x)x + \frac{1+x}{2}x^3 + \frac{2(1-x^2)}{9}x^4. \end{aligned}$$

Then we have

$$\begin{aligned} 2g(x) &\le \left[1 + cx + \frac{15}{26}(cx)^2\right]\left[1 + (1-x)x + \frac{x-1}{2}x^3\right] \\ &+ \left[1 - cx + \frac{(cx)^2}{2}\right]\left[1 - (1+x)x + \frac{1+x}{2}x^3 + \frac{2(1-x^2)}{9}x^4\right] \\ &= 2 + (\frac{14c^2}{13} + 2c - 2)x^2 + \frac{c^2}{13}x^3 \\ &+ (-\frac{14c^2}{13} - c + \frac{11}{9})x^4 - (\frac{c^2}{26} + \frac{2c}{9})x^5 + (\frac{4c^2}{3} - \frac{2}{9})x^6 + \frac{2c}{9}x^7 - \frac{c^2}{9}x^8. \end{aligned}$$

We have

$$-(\frac{c^2}{26} + \frac{2c}{9})x^5 + \frac{2c}{9}x^7 \le -(\frac{c^2}{26} + \frac{2c}{9})x^5 + \frac{1}{4}\cdot\frac{2c}{9}x^5 = -(\frac{c^2}{26} + \frac{c}{6})x^5 \le 0,$$

moreover since $-\frac{14c^2}{13} - c + \frac{11}{9} > 0$ and $\frac{4c^2}{3} - \frac{2}{9} > 0$, we have

$$\begin{aligned} (-\frac{14c^2}{13} - c + \frac{11}{9})x^4 + (\frac{4c^2}{3} - \frac{2}{9})x^6 &\le \frac{1}{4}(-\frac{14c^2}{13} - c + \frac{11}{9})x^2 + \frac{1}{16}(\frac{4c^2}{3} - \frac{2}{9})x^2 \\ &= (\frac{-29c^2}{156} - \frac{c}{4} + \frac{7}{24})x^2, \end{aligned}$$

and

$$\frac{c^2}{13}x^3 - \frac{c^2}{9}x^8 \le \frac{c^2}{26}x^2.$$

By all, we have

$$\begin{aligned} 2g(x) &\le 2 + \left[(\frac{14c^2}{13} + 2c - 2) + \frac{c^2}{26} + (\frac{-29c^2}{156} - \frac{c}{4} + \frac{7}{24})\right]x^2 \\ &= 2 + \frac{290c^2 + 546c - 533}{312}x^2 \end{aligned}$$

Since $0.69 < c < 0.7$, we have $290c^2 + 546c - 533 < 0$, which completes the proof. ∎

By the proof of this theorem, we have the following:

Theorem 2.2. $a^{2b} + b^{2a} = 1$ *if and only if* $(a,b) = (0,1), (\frac{1}{2}, \frac{1}{2}), (1,0)$ *for all nonnegative real numbers* a *and* b *such that* $a + b = 1$.

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