The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 9, Issue 2, Article 2, pp. 1-8, 2012

FURTHER BOUNDS FOR TWO MAPPINGS RELATED TO THE HERMITE-HADAMARD INEQUALITY

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Received 6 June, 2012; accepted 21 September, 2012; published 17 October, 2012.

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ABSTRACT. Some new results concerning two mappings associated to the celebrated Hermite-Hadamard integral inequality for twice differentiable functions with applications for special means are given.

Key words and phrases: Convex functions, Hermite-Hadamard inequality, Special means.

1991 *Mathematics Subject Classification.* 26D15; 25D10.

ISSN (electronic): 1449-5910

1. INTRODUCTION

The Hermite-Hadamard integral inequality for convex functions $f:[a,b]\to\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2} \tag{HH}$$

is well known in the literature and has many applications for special means.

In order to provide various refinements of this result, the first author introduced in 1991, see [2], the following associated mapping $H:[0,1]\to\mathbb{R}$ defined by

$$H(t):=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx,$$

for a given convex function $f:[a,b]\to\mathbb{R}$.

Some of the main properties of H are explored in [2], [3], [4] and [9].

The corresponding double integral mapping in connection with the Hermite-Hadamard inequalities was considered first in [3] and is defined as

$$F:[0,1]\to\mathbb{R},F(t):=\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy.$$

Some of the main results concerning this mapping can be seen in [3] (see also [4]).

For other related results, see for instance the research papers [1], [11], [12], [13], [15], [14], [16], [17], [18], the monograph online [10] and the references therein.

In the recent paper [7] we proved the following result where upper and lower bounds for the associated functions

$$\frac{t}{b-a}\int_a^b f(x)\,dx+(1-t)f\left(\frac{a+b}{2}\right)-H(t)$$

and

$$\frac{1}{b-a}\int_a^b f(x)\,dx-F(t)$$

with $t\in[0,1]$, have been given.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. Then we have*

$$\begin{aligned} 0 &\leq 2\min\{t,1-t\} \\ &\times\left[\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)\right]-\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}f(x)\,dx\right] \\ &\leq\frac{t}{b-a}\int_a^b f(x)\,dx+(1-t)f\left(\frac{a+b}{2}\right)-H(t) \\ &\leq 2\max\{t,1-t\} \\ &\times\left[\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)\right]-\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}f(x)\,dx\right] \end{aligned} \tag{1.1}$$

and

$$0 \le 2\min\{t, 1-t\}\left[\frac{1}{b-a}\int_a^b f(x)\,dx - F\left(\frac{1}{2}\right)\right] \tag{1.2}$$
$$\le \frac{1}{b-a}\int_a^b f(x)\,dx - F(t)$$
$$\le 2\max\{t, 1-t\}\left[\frac{1}{b-a}\int_a^b f(x)\,dx - F\left(\frac{1}{2}\right)\right],$$

for any $t \in [0,1]$.

Employing a different technique, in [8] we obtained the following result as well:

Theorem 1.2. *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function on the interval* $[a,b]$*. Then we have*

$$\frac{t}{b-a}\int_a^b f(x)\,dx + (1-t) f\left(\frac{a+b}{2}\right) - H(t) \tag{1.3}$$
$$\le t(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]$$

and

$$\frac{1}{b-a}\int_a^b f(x)\,dx - F(t) \tag{1.4}$$
$$\le 2t(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]$$

for any $t \in [0,1]$.

Motivated by the above results we establish in this paper some new bounds involving these two mappings. Applications for special means are also provided.

2. THE RESULTS

Theorem 2.1. *Let* $f : [a,b] \to \mathbb{R}$ *be a twice differentiable function on the interval* (a,b) *and assume that there exists the constants* $k < K$ *such that*

$$k \le f''(s) \le K \text{ for any } s \in (a,b). \tag{2.1}$$

Then we have

$$\frac{1}{24}k(1-t)t(b-a)^2 \tag{2.2}$$
$$\le \frac{t}{b-a}\int_a^b f(x)\,dx + (1-t) f\left(\frac{a+b}{2}\right) - H(t)$$
$$\le \frac{1}{24}K(1-t)t(b-a)^2$$

and

$$\frac{1}{12}k(1-t)t(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)\,dx - F(t) \le \frac{1}{12}K(1-t)t(b-a)^2 \tag{2.3}$$

for any $t \in [0,1]$.

Proof. Consider the auxiliary function $g_k : [a,b] \to \mathbb{R}$, $g_k(s) := f(s) - \frac{1}{2}ks^2$. This function is twice differentiable and $g_k''(s) = f''(s) - k \geq 0$ by (2.1), which shows that g_k is convex on $[a,b]$.

By the definition of convexity we have

$$\begin{aligned} 0 &\leq tg_k(x) + (1-t)g_k(y) - g_k(tx + (1-t)y) \\ &= tf(x) + (1-t)f(y) - f(tx + (1-t)y) \\ &\quad - \frac{1}{2}k\left[tx^2 + (1-t)y^2 - (tx + (1-t)y)^2\right] \\ &= tf(x) + (1-t)f(y) - f(tx + (1-t)y) \\ &\quad - \frac{1}{2}k(1-t)t(x-y)^2 \end{aligned}$$

for any $x, y \in [a,b]$ and for any $t \in [0,1]$.

Therefore we have

$$\frac{1}{2}k(1-t)t(x-y)^2 \leq tf(x) + (1-t)f(y) - f(tx + (1-t)y) \tag{2.4}$$

for any $x, y \in [a,b]$ and for any $t \in [0,1]$.

By utilising the auxiliary function $g_K : [a,b] \to \mathbb{R}$, $g_K(s) := \frac{1}{2}Ks^2 - f(s)$ we also get

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \leq \frac{1}{2}K(1-t)t(x-y)^2 \tag{2.5}$$

for any $x, y \in [a,b]$ and for any $t \in [0,1]$.

Now, from (2.4) we get

$$\begin{aligned} &\frac{1}{2}k(1-t)t\left(x - \frac{a+b}{2}\right)^2 \\ &\leq tf(x) + (1-t)f\left(\frac{a+b}{2}\right) - f\left(tx + (1-t)\frac{a+b}{2}\right) \end{aligned} \tag{2.6}$$

for any $x \in [a,b]$ and for any $t \in [0,1]$.

Integrating the inequality (2.4) over $x \in [a,b]$ we have

$$\begin{aligned} &\frac{1}{2}k(1-t)t\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx \\ &\leq t\int_a^b f(x)\,dx + (1-t)f\left(\frac{a+b}{2}\right) - \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx \end{aligned} \tag{2.7}$$

and since

$$\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx = \frac{1}{12}(b-a)^3$$

then we get from (2.7) the first inequality in (2.2).

The second inequality in (2.2) follows from (2.5) by a similar argument.

Integrating the inequality (2.4) over x and y on $[a, b]$ we have

$$
\begin{aligned}
(2.8)\qquad & \frac{1}{2}k(1-t)t\int_a^b\int_a^b (x-y)^2\,dxdy \\
& \le t(b-a)\int_a^b f(x)\,dx + (1-t)(b-a)\int_a^b f(y)\,dy \\
& -\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy \\
& = (b-a)\int_a^b f(x)\,dx - \int_a^b\int_a^b f(tx+(1-t)y)\,dxdy.
\end{aligned}
$$

Since

$$\int_a^b\int_a^b (x-y)^2\,dxdy = \frac{1}{6}(b-a)^4$$

then from (2.8) we get the first inequality in (2.3).

The second inequality in (2.3) follows from (2.5) by a similar argument. ∎

The following result also holds:

Theorem 2.2. *With the assumptions of Theorem 2.1 we have*

$$(2.9)\qquad \frac{1}{12}\left(t-\frac{1}{2}\right)^2 k(b-a)^2 \le F(t) - F\left(\frac{1}{2}\right) \le \frac{1}{12}\left(t-\frac{1}{2}\right)^2 K(b-a)^2$$

for any $t \in [0,1]$.

Proof. By taking $t = \frac{1}{2}$, $x = u$ and $y = v$ in the inequalities (2.4) and (2.5) we get

$$(2.10)\qquad \frac{1}{8}k(u-v)^2 \le \frac{f(u)+f(v)}{2} - f\left(\frac{u+v}{2}\right) \le \frac{1}{8}K(u-v)^2$$

for any $u.v \in [a,b]$.

Now, if we write the inequality (2.10) for $u = tx + (1-t)y$ and $v = ty + (1-t)x$ the we get

$$
\begin{aligned}
(2.11)\qquad & \frac{1}{2}k\left(t-\frac{1}{2}\right)^2 (x-y)^2 \\
& \le \frac{f(tx+(1-t)y) + f(ty+(1-t)x)}{2} - f\left(\frac{x+y}{2}\right) \\
& \le \frac{1}{2}K\left(t-\frac{1}{2}\right)^2 (x-y)^2
\end{aligned}
$$

for any $x, y \in [a,b]$ and for any $t \in [0,1]$.

Integrating the inequality (2.11) over x and y on $[a,b]$ we have

$$\begin{aligned}(2.12)\qquad &\frac{1}{2}k\left(t-\frac{1}{2}\right)^2\int_a^b\int_a^b (x-y)^2\,dxdy \\ &\le \int_a^b\int_a^b \frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2}dxdy \\ &-\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \\ &\le \frac{1}{2}K\left(t-\frac{1}{2}\right)^2\int_a^b\int_a^b (x-y)^2\,dxdy\end{aligned}$$

and since

$$\begin{aligned}&\int_a^b\int_a^b \frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2}dxdy \\ &=\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy = F(t)\end{aligned}$$

we deduce from (2.12) the desired inequality (2.9). ∎

3. APPLICATIONS FOR L_p-MEANS

Let us consider the convex mapping $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^p$, $p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ and $0<a<b$. Define the mapping

$$H_p(t):=\frac{1}{b-a}\int_a^b (tx+(1-t)A(a,b))^p\,dx,\ t\in[0,1].$$

It is obvious that $H_p(0)=A^p(a,b)$, $H_p(1)=L_p^p(a,b)$ where, we recall that $A(a,b)=\frac{a+b}{2}$,

$$L_p^p(a,b):=\frac{1}{p+1}\frac{b^{p+1}-a^{p+1}}{b-a},\ p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$$

and for $t\in(0,1)$ we have

$$\begin{aligned}(3.1)\qquad H_p(t)&=\frac{1}{[tb+(1-t)A(a,b)]-[ta+(1-t)A(a,b)]}\int_{ta+(1-t)A(a,b)}^{tb+(1-t)A(a,b)} y^p\,dy \\ &=L_p^p(ta+(1-t)A(a,b),tb+(1-t)A(a,b)).\end{aligned}$$

Now, consider the function

$$F_p(t):=\frac{1}{(b-a)^2}\int_a^b\int_a^b (tx+(1-t)y)^p\,dxdy.$$

We observe that $F_p(1)=F_p(0)=L_p^p(a,b)$ and for $t\in(0,1)$ we have

$$\begin{aligned}(3.2)\qquad F_p(t)&=\frac{1}{b-a}\int_a^b\left(\frac{1}{b-a}\int_a^b (tx+(1-t)y)^p\,dx\right)dy \\ &=\frac{1}{b-a}\int_a^b\left(\frac{1}{[tb+(1-t)y]-[ta+(1-t)y]}\int_{ta+(1-t)y}^{tb+(1-t)y} s^p\,ds\right)dy \\ &=\frac{1}{b-a}\int_a^b L_p^p(ta+(1-t)y,tb+(1-t)y)\,dy.\end{aligned}$$

We can calculate the double integral

$$\begin{aligned} F_p\left(\frac{1}{2}\right) &= \frac{1}{(b-a)^2}\int_a^b\int_a^b\left(\frac{x+y}{2}\right)^p dxdy \\ &= \begin{cases} \frac{4}{(b-a)^2(p+1)(p+2)}\left[b^{p+2}-2\left(\frac{b+a}{2}\right)^{p+2}+a^{p+2}\right] & p\neq -2, \\ \frac{8}{(b-a)^2}\ln\left(\frac{A(a,b)}{G(a,b)}\right) & p=-2 \end{cases} \end{aligned}$$

for $p\neq -1$, where $G(a,b)$ denotes the geometric mean of a,b (see [7]).

Let us consider the convex mapping $f_p:(0,\infty)\to\mathbb{R},\ f_p(x)=x^p,\ p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ and $0<a<b$. Define the quantities

$$K_p := p(p-1)\times\begin{cases} b^{p-2}, & \text{if } p\geq 2 \\ a^{p-2}, & \text{if } p\in(-\infty,0)\cup[1,2)\setminus\{-1\} \end{cases}$$

and

$$k_p := p(p-1)\times\begin{cases} a^{p-2}, & \text{if } p\geq 2 \\ b^{p-2}, & \text{if } p\in(-\infty,0)\cup[1,2)\setminus\{-1\}. \end{cases}$$

We observe that with these notations we have that

$$k_p\leq f_p''(x)\leq K_p$$

for any $p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ and $0<a\leq x\leq b$.

We can state the following result:

Proposition 3.1. *We have the following inequalities:*

$$\begin{aligned} \frac{1}{24}k_p(1-t)t(b-a)^2 &\leq tL_p^p(a,b)+(1-t)A^p(a,b)-H_p(t) \\ &\leq \frac{1}{24}K_p(1-t)t(b-a)^2, \end{aligned} \tag{3.3}$$

$$\frac{1}{12}k_p(1-t)t(b-a)^2\leq L_p^p(a,b)-F_p(t)\leq\frac{1}{12}K_p(1-t)t(b-a)^2 \tag{3.4}$$

and

$$\frac{1}{12}\left(t-\frac{1}{2}\right)^2k_p(b-a)^2\leq F_p(t)-F_p\left(\frac{1}{2}\right)\leq\frac{1}{12}\left(t-\frac{1}{2}\right)^2K_p(b-a)^2$$

for any $t\in[0,1]$.

The proof follows by Theorem 2.1 and 2.2 and the details are omitted.

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