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SOME FUNCTIONAL INEQUALITIES FOR THE GEOMETRIC OPERATOR MEAN

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ABSTRACT. In this paper, we give some new inequalities of functional type for the power geometric operator mean involving several arguments.

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1. INTRODUCTION

The geometric mean of two positive semi-definite operators arises in various areas and has many of the properties of the geometric mean of two positive scalars. Some of these properties are in inequalities form which have proved to be a powerful tool in several scientific problems. In one direction, many of the operator inequalities to have come under study are inequalities arising from the Löwner partial order between operators acting on a Hilbert space, [1]. A second line of research is concerned with inequalities between the norms of operators. It is in this latter direction that our present work aims.

More precisely, the fundamental goal of this paper is to give further useful inequalities, with lower and upper bounds, for the geometric operator mean, in the sense to extend that of the operator norms. Namely, if the notation $\mathcal{G}_p(A, B)$ refers to the power geometric mean of two positive operators A and B, i.e.

$$\mathcal{G}_p(A, B) := B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)^{1-p}B^{1/2} = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{p}A^{1/2},$$

with $0 \leq p \leq 1$, then we establish that (see Theorem 3.2)

$$\Phi(\mathcal{G}_p(A, B)) \leq \mathcal{G}_p(\Phi(A), \Phi(B)) := (\Phi(A))^{1-p}(\Phi(B))^p,$$

together with a lower bound, where the notation Φ stands for a real-valued map, called sub-linear functional, extending the norm map of operators. In particular, our study includes the following statements

$$\Phi(A) = Tr\ A, \text{ trace of } A,$$

$$\Phi(A) = \rho(A), \text{ spectral radius of } A,$$

$$\Phi(A) = \|A\|_S, \text{ Shur's norm of } A.$$

Since the operation mean is generally not associative, our inequalities obtained here are not obvious to write immediately for three or more operators. Our approach allows us to extend our above approach from the case of two operators to that of three or more ones.

The paper will be organized as follows: After this short section, Section 2 contains a background material that will be needed in the sequel. Section 3 displays our aim for two positive semi-definite operators. In Section 4 we discuss our goal for three or more operators.

2. BASIC NOTIONS AND PRELIMINARY RESULTS

In this section, we recall some standard notations and results that are needed throughout the paper. Let H be a real or complex Hilbert space with its inner product $\langle ., .\rangle$ and the associate norm $\|.\|$. We denote by $\mathcal{L}(H)$ the Banach space of bounded linear operators defined from H into itself equipped with the standard norm and by $\mathcal{L}^+(H) \subset \mathcal{L}(H)$ the closed convex cone of self-adjoint positive semi-definite operators. The positive semi-definiteness induces a partial ordering on the subspace of the self-adjoint operators, called Löwner order, defined as follows: for A, B in $\mathcal{L}(H)$, we write $A \leq B$ as usual if A, B are self-adjoint and $B - A$ is positive semi-definite.

Henceforth, whenever we consider an order relation $A \leq B$, it will be assumed that the operators A and B are self-adjoint. If H is a finite dimensional space, where $\mathcal{L}(H)$ can be identified to the space of square matrices, we denote by $adj\ A$ the adjugate matrix of A, i.e. the transpose of the matrix of co-factors of A. If A is an invertible matrix then $adj\ A = (det\ A).A^{-1}$.

An m-tuple of positive real numbers $p = (p_1, p_2, ..., p_m)$ summing to 1 will be called a probability vector. We now state the following lemma which will be needed in the sequel.

Lemma 2.1. *Let* $p_1, p_2, ..., p_m$ *be a probability vector and* $a_1, a_2, ..., a_m$ *be non-negative reals. Define the map* $F : (\mathbb{R}_+^*)^m \longrightarrow \mathbb{R}_+$ *by*

$$F(x_1, x_2, ..., x_m) = \frac{\sum_{j=1}^m p_j a_j x_j}{\prod_{j=1}^m x_j^{p_j}}.$$

Then there holds

$$\inf_{x_1,...,x_m>0} F(x_1, x_2, ..., x_m) = \prod_{j=1}^m a_j^{p_j}.$$

Proof. By the concavity of the real-valued map $t \longmapsto ln\ t$ on $]0, +\infty[$, it is not hard to establish the following inequality (called generalized Young's inequality):

$$\prod_{j=1}^m \gamma_j^{p_j} \leq \sum_{j=1}^m p_j \gamma_j,$$

for all positive real numbers $\gamma_1, \gamma_2, ..., \gamma_m$. Taking $\gamma_j = a_j x_j$ in this latter inequality we obtain, after a simple manipulation,

$$\prod_{j=1}^m a_j^{p_j} \prod_{j=1}^m x_j^{p_j} \leq \sum_{j=1}^m p_j a_j x_j.$$

It follows that

$$\prod_{j=1}^m a_j^{p_j} \leq F(x_1, x_2, ..., x_m),$$

for all $x_1, x_2, ..., x_m > 0$. To complete the proof, it is sufficient to ensure that the lower bound of the latter inequality is attained for some m-tuple of positive real numbers $x_1^0, x_2^0, ..., x_m^0$. Such m-tuple will be determined as the critic point, i.e. vanishing the partial derivatives, of the real function $(x_1, x_2, ..., x_m) \longmapsto F(x_1, x_2, ..., x_m)$. An elementary computation, with a routine manipulation, yields

$$\frac{\partial F}{\partial x_k}(x_1, x_2, ..., x_m) = \frac{(p_k a_k x_k)\left(\prod_{j=1}^m x_j^{p_j}\right) - p_k\left(\sum_{j=1}^m p_j a_j x_j\right)\left(\prod_{j=1}^m x_j^{p_j}\right)}{x_k\left(\prod_{j=1}^m x_j^{p_j}\right)^2},$$

for all integer $k = 1, 2, ..., m$. The researched critic point is then such that

$$x_k^0 a_k = \sum_{j=1}^m p_j a_j x_j,$$

for every $k = 1, 2, ..., m$. The second side of the last equality doesn't depend on k and so $x_k^0 a_k = x_1^0 a_1$ for all $k = 1, 2, ..., m$. Substituting this in the expression of $F(x_1, x_2, ..., x_m)$ we obtain, after a simple reduction,

$$F(x_1^0, x_2^0, ..., x_m^0) = \prod_{j=1}^m a_j^{p_j},$$

which is nothing other than the desired result. The proof of the lemma is complete. ∎

Now, let $\Phi : \mathcal{L}^+(H) \longrightarrow \mathbb{R}$ be a real-valued function (called functional below) with operator arguments. We say that Φ is monotone increasing if $A \leq B$ implies $\Phi(A) \leq \Phi(B)$. The functional Φ will be called sub-linear if the two following assertions are both satisfied

$$\Phi \text{ is sub-additive, } i.e. \quad \forall A, B \in \mathcal{L}^+(H) \quad \Phi(A+B) \leq \Phi(A) + \Phi(B),$$

$$\Phi \text{ is positively homogenous, } i.e. \quad \forall A \in \mathcal{L}^+(H), \ \forall t > 0 \quad \Phi(tA) = t\Phi(A).$$

Let us present a list of examples about monotone increasing and sub-linear functionals.

Example 2.2. Every linear functional is sub-linear. In particular, let us consider the "trace" functional $\Phi : \mathcal{L}^+(H) \longrightarrow I\!R$ defined by:

$$\forall A \in \mathcal{L}^+(H) \quad \Phi(A) = Tr\ A := \sum_{i=1}^{\infty} \langle Ae_i, e_i \rangle ,$$

where (e_i) is an orthonormal basis of H. It is easy to verify that Φ is monotone increasing and sub-linear.

Example 2.3. Every norm of $\mathcal{L}(H)$ is sub-linear. In particular, the standard norm

$$\Phi(A) = \|A\| = sup\{\|Au\|, \ \ \|u\| = 1\},$$

which satisfies

$$\forall A \in \mathcal{L}^+(H) \quad \rho(A) = \sup_{u \neq 0} \frac{\langle Au, u\rangle}{\|u\|^2},$$

where $\rho(A)$ denotes the spectral radius of A, is then sub-linear and monotone increasing.

Example 2.4. Let $u \in H$ be a fixed vector and set

$$\forall A \in \mathcal{L}^+(H) \quad \mathcal{Q}_u(A) = \langle Au, u\rangle .$$

Clearly, the functional $A \longmapsto \mathcal{Q}_u(A)$, for fixed $u \in H$, is monotone increasing and sub-linear.

Example 2.5. For a self-adjoint operator A, the Schur's norm is the functional $A \longmapsto \|A\|_S := \sqrt{Tr\ A^2}$. It is not hard to see that $A \longmapsto \|A\|_S$ is sub-linear and monotone increasing.

Example 2.6. Let $\lambda_{min}(A)$ be the smallest eigenvalue of the self-adjoint operator A. It is easy to see that the functional $A \longmapsto \lambda_{min}(A)$ is monotone increasing but not sub-linear.

Example 2.7. Let $\mathcal{C}$ be an open convex set of $\mathcal{L}(H)$ containing 0 and consider the Minkowski functional $\mu_{\mathcal{C}}$ of $\mathcal{C}$ defined by

$$\forall A \in \mathcal{L}(H) \quad \mu_{\mathcal{C}}(A) = inf\{t > 0, \ A \in t\mathcal{C}\}.$$

It is well known that $\mu_{\mathcal{C}}$ is sub-linear, but generally not monotone increasing.
However, if we take

$$\mathcal{C} = \{A \in \mathcal{L}(H), \ \langle Au, u\rangle < 1, \ \forall u \in H, \ \|u\| = 1\},$$

then it is easy to verify that $\mu_{\mathcal{C}}$ is (sub-linear and) monotone increasing.

3. GEOMETRIC MEAN OF TWO OPERATORS

Let p be a real number such that $0 \leq p \leq 1$. The power geometric mean $\mathcal{G}_p(A, B)$ of two positive definite operators A and B defined by

$$\mathcal{G}_p(A, B) = B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)^{1-p} B^{1/2} = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{p} A^{1/2}, \tag{3.1}$$

was introduced as an extension of the standard geometric operator mean corresponding to the intermediary value $p = 1/2$. If A and B are commuting, particularly for the scalar case, then $\mathcal{G}_p(A, B) = A^{1-p}B^p$. In the case where the operators A and B are only semi-definite (i.e. not necessary invertible) then $\mathcal{G}_p(A, B)$ can be similarly defined via the continuity criterion for an operator mean by setting

$$A^{-1} = \lim_{\epsilon \downarrow 0}(A + \epsilon.I)^{-1},$$

for the sake of convenience.

The elementary properties of $\mathcal{G}_p(A,B)$, that are needed in the sequel, are summarized in the following: for all $A,B \in \mathcal{L}^+(H),\ 0 \le p \le 1$, there hold
(i) The joint homogeneity, i.e.

$$\forall t,s>0 \qquad \mathcal{G}_p(tA,sB) = t^{1-p}s^p\mathcal{G}_p(A,B).$$

(ii) The self-duality relationship, i.e.

$$\big(\mathcal{G}_p(A,B)\big)^{-1} = \mathcal{G}_p(A^{-1},B^{-1}).$$

(iii) The power arithmetic-geometric mean inequality, i.e.

$$\mathcal{G}_p(A,B) \le (1-p)A + pB.$$

(iv) The relative entropy inequality, i.e.

$$\mathcal{G}_p(A,B) \ge (1+p)A - pAB^{-1}A.$$

In this section, we will display some other inequalities, called of functional type, for the power geometric mean $\mathcal{G}_p(A,B)$. We start with the next simple result.

Proposition 3.1. *Assume here that H is a finite dimensional space. For all positive semi-definite matrices A,B and $0 \le p \le 1$ the following relationship holds*

$$det\big(\mathcal{G}_p(A,B)\big) = (det\ A)^{1-p}\,(det\ B)^p = \mathcal{G}_p(det\ A, det\ B).$$

Proof. Follows from the explicit form (3.1) of $\mathcal{G}_p(A,B)$ with the standard properties of the determinant. ∎

Now, we are in position to state the following theorem which is the first main result of this paper.

Theorem 3.2. *Let $\Phi : \mathcal{L}^+(H) \longrightarrow I\!R^+$ be a monotone increasing sub-linear functional. Then, for all $A,B \in \mathcal{L}^+(H)$ and $0 \le p \le 1$, there holds*

$$\big(\Phi(A)\big)^{1+p}\big(\Phi(AB^{-1}A)\big)^{-p} \le \Phi\big(\mathcal{G}_p(A,B)\big) \le \big(\Phi(A)\big)^{1-p}\big(\Phi(B)\big)^p. \tag{3.2}$$

Proof. Let us show the right hand of the inequality (3.2). By the power arithmetic- geometric mean inequality (iii), i.e.

$$\mathcal{G}_p(A,B) \le (1-p)A + pB,$$

we have, with Φ as in the above,

$$\Phi(\mathcal{G}_p(A,B)) \le (1-p)\Phi(A) + p\Phi(B).$$

Replacing in this latter inequality A and B by tA and sB ($t>0, s>0$) respectively, we obtain after a simple reduction

$$\Phi(\mathcal{G}_p(A,B)) \le \frac{(1-p)t\Phi(A) + ps\Phi(B)}{t^{1-p}s^p},$$

and hence

$$\Phi(\mathcal{G}_p(A,B)) \le \inf_{t>0,s>0}\left(\frac{(1-p)t\Phi(A) + ps\Phi(B)}{t^(1-p)s^p}\right).$$

Applying Lemma 2.1 with $m=2, p_1 = 1-p, p_2 = p, a_1 = \Phi(A), a_2 = \Phi(B)$ and

$$F(t,s) = \frac{(1-p)t\Phi(A) + ps\Phi(B)}{t^{1-p}s^p},$$

we deduce immediately the desired result.
Now let us prove the left side of (3.2). Writing the relative entropy inequality (iv) in the form

$$\mathcal{G}_p(A,B) + pAB^{-1}A \ge (1+p)A,$$

we deduce, with Φ as in the above,

$$\Phi\big(\mathcal{G}_p(A,B)\big) + p\Phi(AB^{-1}A) \geq (1+p)\Phi(A),$$

or again

$$\Phi\big(\mathcal{G}_p(A,B)\big) \geq (1+p)\Phi(A) - p\Phi(AB^{-1}A).$$

Similarly to the above we obtain, after all reductions

$$\Phi(\mathcal{G}_p(A,B)) \geq (1+p)t^p s^{-p}\Phi(A) - pt^{1+p}s^{-1-p}\Phi(AB^{-1}A)$$

for all reals $t, s > 0$. It follows that

$$\Phi(\mathcal{G}_p(A,B)) \geq \sup_{t>0,s>0}\Big((1+p)t^p s^{-p}\Phi(A) - pt^{1+p}s^{-1-p}\Phi(AB^{-1}A)\Big).$$

It is easy to verify that the second member of this latter inequality is equal to

$$\big(\Phi(A)\big)^{1+p}\big(\Phi(AB^{-1}A)\big)^{-p},$$

so completes the proof. ∎

Now, choosing appropriate functionals Φ in Theorem 3.2, we will deduce some corollaries stated in the following.

Corollary 3.3. *For all $A, B \in \mathcal{L}^+(H)$ and $0 \leq p \leq 1$ we have*

$$(Tr\ A)^{1+p}\Big(Tr(AB^{-1}A)\Big)^{-p} \leq Tr\big(\mathcal{G}_p(A,B)\big) \leq (Tr\ A)^{1-p}(Tr\ B)^p.$$

If moreover the space H is with finite dimensional then

$$\big(Tr(adj\ A)\big)^{1+p}\Big(Tr(adj\ AB^{-1}A)\Big)^{-p} \leq Tr\big(adj\ \mathcal{G}_p(A,B)\big) \leq \big(Tr(adj\ A)\big)^{1-p}\big(Tr(adj\ B)\big)^p.$$

Proof. The first inequality is immediate from Theorem 3.2 with Example 2.2. Let us show the second one. By an argument of continuity, it is sufficient to establish the inequality for A and B invertible. From the self-duality relationship (ii) we deduce, with the above inequality

$$Tr\ \big(\mathcal{G}_p(A,B)\big)^{-1} = Tr\ \mathcal{G}_p(A^{-1},B^{-1}) \leq (Tr\ A^{-1})^{1-p}(Tr\ B^{-1})^p.$$

This, with the fact that,

$$Tr\ (adj\ A) = (det\ A)Tr\ A^{-1},$$

and Proposition 3.1, yields

$$Tr\big(adj\ \mathcal{G}_p(A,B)\big) \leq (det\ A)^{1-p}(det\ B)^p(Tr\ A^{-1})^{1-p}(Tr\ B^{-1})^p,$$

so proving the right side of the desired inequality. Similarly, we show the left hand of the inequality and the proof is complete. ∎

Corollary 3.4. *For every $A, B \in \mathcal{L}^+(H)$ and $0 \leq p \leq 1$ the following inequality holds*

$$\big(\rho(A)\big)^{1+p}\big(\rho(AB^{-1}A)\big)^{-p} \leq \rho\big(\mathcal{G}_p(A,B)\big) \leq \big(\rho(A)\big)^{1-p}\big(\rho(B)\big)^p.$$

Proof. Combining Theorem 3.2 with Example 2.3 we obtain the desired result. ∎

Corollary 3.5. *For all $A, B \in \mathcal{L}^+(H)$, $0 \leq p \leq 1$ and $u \in H$, we have*

$$\big(\mathcal{Q}_u(A)\big)^{1+p}\big(\mathcal{Q}_u(AB^{-1}A)\big)^{-p} \leq \mathcal{Q}_u\big(\mathcal{G}_p(A,B)\big) \leq \big(\mathcal{Q}_u(A)\big)^{1-p}\big(\mathcal{Q}_u(B)\big)^p,$$

where $\mathcal{Q}_u(A) = \langle Au, u\rangle$ is the quadratic form of A at u.

Proof. Comes from Theorem 3.2 with Example 2.4. ∎

Corollary 3.6. *Let $A, B \in \mathcal{L}^+(H)$ and $0 \le p \le 1$, then one has*

$$\|A\|_S^{1+p}\|AB^{-1}A\|_S^{-p} \le \|\mathcal{G}_p(A,B)\|_S \le \|A\|_S^{1-p}\|B\|_S^p.$$

Proof. It is sufficient to combine Theorem 3.2 with Example 2.5. ∎

We notice that, for all positive invertible operators A, B and Φ as in the above, inequality (3.2) implies immediately

$$\big(\Phi(A)\big)^2 \le \Phi(B)\Phi(AB^{-1}A),$$

which is not obvious to establish directly. By choosing a convenient functional Φ, as in the previous, we deduce many related inequalities.

Remark 3.1. The above results for the geometric operator mean $\mathcal{G}_p(A,B)$ yields some related functional inequalities for the Tsallis relative entropy $T_p(A|B)$ given by

$$T_p(A|B) := \frac{\mathcal{G}_p(A,B) - A}{p}.$$

In particular, if Φ is a functional linear map, we have

$$\Phi(T_p(A|B)) \le T_p(\Phi(A)|\Phi(B)),$$

which is to compare, as result and related proof, with Theorem 2.3 of [7].
When the parameter p tends to 0, we obtain that of the relative operator entropy:

$$S(A|B) := A^{1/2}\log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

We left to the reader the routine task of formulating the relevant inequalities in the above cases. For further details about relative operator entropy, we refer to [4, 6, 7] for instance.

4. GEOMETRIC MEAN OF THREE OR MORE OPERATORS

This section is devoted to extend the results of the above section from two positive operators to three or more ones. We preserve the same notations as in the previous.

Theorem 4.1. *Let $p = (p_1, p_2, ..., p_m)$ be a probability vector and $\mathcal{G}_p(A_1, A_2, ..., A_m)$ a geometric operator mean satisfying the two following properties*

(i) Joint homogeneity, i.e.

$$\mathcal{G}_p(t_1A_1, t_2A_2, ..., t_mA_m) = \left(\prod_{i=1}^m t_i^{p_i}\right)\mathcal{G}_p(A_1, A_2, ..., A_m),\ \forall t_1, ..., t_m > 0,$$

(ii) The arithmetic-geometric mean inequality, i.e.

$$\mathcal{G}_p(A_1, A_2, ..., A_m) \le \sum_{i=1}^m p_iA_i.$$

Let $\Phi : \mathcal{L}^+(H) \longrightarrow \mathbb{R}$ be a monotone increasing sub-linear functional. Then the following inequality holds

$$\Phi\big(\mathcal{G}_p(A_1, A_2, ..., A_m)\big) \le \prod_{i=1}^m \big(\Phi(A_i)\big)^{p_i}.$$

Proof. Using Lemma 2.1, the proof is similar to that of Theorem 3.2. The details are left to the reader. ∎

Corollary 4.2. *Let $p = (p_1, p_2, ..., p_m)$ be a probability vector and $\mathcal{G}_p(A_1, A_2, ..., A_m)$ be a geometric operator mean satisfying the assertions (i) and (ii) of Theorem 4.1. Then the following inequalities are met*

$$Trace\big(\mathcal{G}_p(A_1, A_2, ..., A_m)\big) \le \prod_{i=1}^{m} (Trace(A_i))^{p_i},$$

$$\rho\big(\mathcal{G}_p(A_1, A_2, ..., A_m)\big) \le \prod_{i=1}^{m} \big(\rho(A_i)\big)^{p_i},$$

$$\forall u \in H \quad \mathcal{Q}_u\big(\mathcal{G}_p(A_1, A_2, ..., A_m)\big) \le \prod_{i=1}^{m} \big(\mathcal{Q}_u(A_i)\big)^{p_i},$$

$$\|\mathcal{G}_p(A_1, A_2, ..., A_m)\|_S \le \prod_{i=1}^{m} \big(\|A_i\|_S\big)^{p_i}.$$

Proof. Follow from Theorem 4.1 when combined, respectively, with Example 2.2, Example 2.3, Example 2.4 and Example 2.5. ∎

Example 4.3. There are many different geometric operator means involving several variables introduced in the literature that verify the above properties (i) and (ii), see [2, 8, 9] for instance. So, these operator means satisfy the conclusions of Theorem 4.1 and Corollary 4.2.

We notice that, if the geometric operator mean $\mathcal{G}_p(A_1, A_2, ..., A_m)$ verifies further the self-duality relationship, as the case for that of [8], then it satisfies an analogue inequality to the second one of Corollary 3.3.

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