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TRAUB–POTRA–TYPE METHOD FOR SET–VALUED MAPS

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ABSTRACT. We introduce a new iterative method for approximating a locally unique solution of variational inclusions in Banach spaces by using generalized divided differences of the first order. This method extends a method considered by Traub [17] (in the scalar case) and by Potra [12] (in the Banach spaces case) for solving nonlinear equations to variational inclusions. An existence–convergence theorem and a radius of convergence are given under some conditions on divided differences operator and Lipschitz–like continuity property of set–valued mappings. The R–order of the method is equal to the unique positive root of a certain cubic equation, which is $1.839\cdots$, and as such it compares favorably to related methods such as the Secant method which is only of order $1.618\cdots$

Key words and phrases: Banach space; Divided differences operators, Generalized equation, Aubin's continuity, Radius of convergence; Fréchet derivative, Set–valued map.

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the generalized equation

$$0 \in F(x) + G(x), \tag{1.1}$$

where F is a continuous function from an open subset $\mathcal{D}$ of $\mathcal{X}$ into $\mathcal{Y}$, G is a set–valued map from $\mathcal{X}$ to the subsets of $\mathcal{Y}$ with closed graph, and $\mathcal{X}$, $\mathcal{Y}$ are Banach spaces.

A large number of problems in applied mathematics and engineering are solved by finding the solutions of generalized equation (1.1), introduced by Robinson [13], [14].

In the particular case $G = \{0\}$, (1.1) is a nonlinear equation in the form

$$F(x) = 0. \tag{1.2}$$

For example, dynamic systems are mathematically modeled by differential or difference equations, and their solutions usually represent the states of the systems, which are determined by solving equation (1.2).

Most of the numerical approximation methods require the expensive computation of the Fréchet–derivatives $F'(x)$ and $F''(x)$ of operator F at each step (Newton's method, Euler–Chebysheff's method, Halley's method, $\cdots$). A comprehensive bibliography of these methods is given in [4]. In this study, we are interested in numerical method for solving generalized equation (1.1) that avoid the expensive computation of any Fréchet–derivative. Here, we generalize Traub–Potra–type method restricted to the resolution of nonlinear operator equations [12], [17]. Recently, we have considered the problem of approximating a locally unique solution $x^\star$ of (1.1) using a third–order iterative method as follows [7]:

$$0 \in F(x_n) + [2\ x_n - x_{n-1}, x_{n-1}; F]\ (x_{n+1} - x_n) + G(x_{n+1}), \quad (x_0, x_1 \in \mathcal{D}), \quad (n \geq 1) \tag{1.3}$$

which generalizes the results obtained in [5] for nonlinear equations.

In this paper, for approximating $x^\star$, we consider Traub–Potra–type method as follows

$$0 \in F(x_n) + \mathcal{T}_n\ (x_{n+1} - x_n) + G(x_{n+1}), \quad (x_0, x_1, x_2 \in \mathcal{D}), \quad (n \geq 2) \tag{1.4}$$

where,

$$\mathcal{T}_n = [x_n, x_{n-1}; F] + [x_{n-2}, x_n; F] - [x_{n-2}, x_{n-1}; F], \tag{1.5}$$

with $[x, y; F] \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is called a divided difference of F of order one at the points x and y (to be precised in section 2).

In the case of nonlinear equations (1.2), the method (1.4) is reduced to the following algorithm (see [17], [12]):

$$x_{n+1} = x_n - \mathcal{T}_n^{-1}\ F(x_n), \quad (x_0, x_1, x_2 \in \mathcal{D}), \quad (n \geq 2). \tag{1.6}$$

Potra [12] provided a local/semilocal convergence analysis for method (1.6) for solving (1.2) using some Lipschitz conditions on the first and the second order divided differences operator (see Theorem 4.1 and Theorem 5.1 in [12]).

Here, we are motivated by the extension of convergence of method (1.6) to generalized equations and by the works in [5], [7], [10]. Using some conditions on divided differences operator introduced for nonlinear equation in [4], and under Aubin's continuity of set–valued map G^{-1} around $(-f(x^\star), x^\star)$, we provide a local convergence of method (1.4). Our approach has the following advantages: we do not need to evaluate any Fréchet derivative and we extend the results obtained by Traub [17] (in the scalar case) and by Potra [12] (in the Banach spaces case) to variational inclusions.

The structure of this paper is the following. In section 2, we collect a number of basic definitions and recall a fixed points theorem for set–valued maps. In section 3 we show an existence–convergence theorem of sequence given by (1.4). Some remarks are also presented.

2. BACKGROUND MATERIAL

In order to make the paper as self–contained as possible we reintroduce some definitions and some results on fixed point theorems [2]–[9], [10]–[16]. Let us begin with some notations that will used throughout this paper. We let $\mathcal{Z}$ be a Banach space equiped with the norm $\|\ .\ \|$. The distance from a point x to a set A in $\mathcal{Z}$ is defined by $\mathrm{dist}\,(x, A) = \inf_{y \in A} \| x - y \|$, with the convention $\mathrm{dist}\,(x, \emptyset) = +\infty$ (according to the general convention $\inf \emptyset = +\infty$). Given a subset C of $\mathcal{Z}$, we denote by $e(C, A)$ the Hausdorff–Pompeiu excess of C into A, defined by

$$e(C, A) = \sup_{x \in C} \ \mathrm{dist}\,(x, A),$$

with the conventions $e(\emptyset, A) = 0$ and $e(C, \emptyset) = +\infty$ whenever $C \neq \emptyset$. For a set–mapping $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$, we denote by $\mathrm{gph}\,\Lambda$ the set $\{(x, y) \in \mathcal{X} \times \mathcal{Y},\ y \in \Lambda(x)\}$ and $\Lambda^{-1}(y)$ the set $\{x \in \mathcal{X},\ y \in \Lambda(x)\}$. The norms in both the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ will be denoted by $\|\ .\ \|$ and the closed ball centered at x with radius r by $\mathbb{B}_r(x)$.

We also need to define the pseudo–Lipschitzian concept of set–valued maps, introduced by Aubin [8] and also known as Lipschitz–like property [11]:

Definition 2.1. A set–valued Γ is pseudo–Lipschitz around $(\overline{x}, \overline{y}) \in \mathrm{gph}\,\Gamma$ with modulus M if there exist constants a and b such that

$$\sup_{z \in \Gamma(y') \cap \mathbb{B}_a(\overline{y})} \mathrm{dist}\,(z, \Gamma(y'')) \leq M \| y' - y'' \|, \quad \text{for all } y' \text{ and } y'' \text{ in } \mathbb{B}_b(\overline{x}). \tag{2.1}$$

In the term of excess, we have an equivalent definition of pseudo–Lipschitzian property replacing the inequality (2.1) by

$$e(\Gamma(y') \cap \mathbb{B}_a(\overline{y}), (y'')) \leq M \| y' - y'' \|, \quad \text{for all } y' \text{ and } y'' \text{ in } \mathbb{B}_b(\overline{x}). \tag{2.2}$$

Pseudo–Lipschitzian property play a crutial role in many aspects of variational analysis and applications [11], [16]. Let us note that the Lipschitz–like of Γ is equivalent to the metric regularity of Γ^{-1} which is a basic well–posedness property in optimization problems. Other characterization is by Mordukhovich [11] via the concept of coderivative $\mathcal{D}^*\Gamma(x/y)$, i.e.,

$$v \in \mathcal{D}^*\Gamma(x/y)(u) \Longleftrightarrow (v, -u) \in N_{\mathrm{gph}\,\Gamma}(x, y). \tag{2.3}$$

Then the Mordukhovich criterion says that Γ is pseudo–Lipschitz around $(\overline{x}, \overline{y})$ if and only if

$$\| \mathcal{D}^*\Gamma(\overline{x}/\overline{y}) \|^+ = \sup_{u \in \mathbb{B}_1(0)} \ \sup_{v \in \mathcal{D}^*\Gamma(\overline{x}/\overline{y})(u)} \| v \| < \infty. \tag{2.4}$$

For some characterizations and applications of the Lipschitz–like property the reader could be referred to [8], [9], [11], [15], [16] and the references given there.

We need the definitions of the first and second divided difference operators [4].

Definition 2.2. An operator $[x, y; G]$ belonging to the space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is called the *first order divided difference* of the operator $G : \mathcal{X} \to \mathcal{Y}$ at the points x, y if :

(1) $[x, y; G]\ (x - y) = G(x) - G(y)$, for $x \neq y$;
(2) if g is Fréchet differentiable at $x \in X$ then $[x, x; G] = G'(x)$.

Definition 2.3. We say that an operator belonging to the space $\mathcal{L}(\mathcal{X}, \mathcal{L}(\mathcal{X}, \mathcal{Y}))$ denoted by $[x, y, z; G]$ is called the second order divided difference of the operator $G : \mathcal{X} \to \mathcal{Y}$ on the points $x, y, z \in \mathcal{X}$ if :

(1) $[x,y,z;G]\ (y-z) = [x,y;G] - [x,z;G]$ for the distinct points x,y and z;
(2) if G is twice differentiable at $x \in \mathcal{X}$ then $[x,y,z;G] = \dfrac{1}{2}\ G''(x)$.

Finally, we need also the following fixed point theorem [9].

Lemma 2.1. *Let ϕ be a set–valued map from $\mathcal{X}$ into the closed subsets of $\mathcal{X}$. We suppose that for $\eta_0 \in \mathcal{X}$, $r \geq 0$ and $0 \leq \lambda < 1$ the following properties hold*
(1) dist $(\eta_0, \phi(\eta_0)) \leq r(1-\lambda)$.
(2) $e(\phi(x_1) \cap \mathbb{B}_r(\eta_0), \phi(x_2)) \leq \lambda\ \parallel x_1 - x_2 \parallel,\ \ \forall x_1, x_2 \in \mathbb{B}_r(\eta_0)$.
Then ϕ has a fixed point in $\mathbb{B}_r(\eta_0)$. That is, there exists $x \in \mathbb{B}_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single–valued, then x is the unique fixed point of ϕ in $\mathbb{B}_r(\eta_0)$.

3. LOCAL CONVERGENCE OF METHOD (1.4)

Before presenting our main result of convergence of method (1.4), we give the local convergence theorem restricted to the resolution of nonlinear equation (1.2) (due to Potra) [12, Theorem 4.1, p. 87–88]:

Theorem 3.1. [12] *Let $F\ :\ \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be an operator such that for every distinct points x, y and z in $\mathcal{D}$, there exists a first and a second order divided differences of F $[x,y;F]$ and $[x,y,z;F]$. Suppose that (1.2) has a solution $x^\star \in \mathcal{D}$ at which the Fréchet–derivative $F'(x^\star)$ exists and is invertible. Assume:*

($\mathcal{P}$1) *There exists $p_\star > 0$, such that for all x, y, u, v in $\mathcal{D}$*
$$\parallel F'(x^\star)^{-1}\ ([x,y;F] - [u,v;F]) \parallel \leq p_\star\ (\parallel x-u \parallel + \parallel y-v \parallel);$$
($\mathcal{P}$2) *There exists $q_\star > 0$, such that for all x, y, u, v in $\mathcal{D}$*
$$\parallel F'(x^\star)^{-1}\ ([u,x,y;F] - [v,x,y;F]) \parallel \leq q_\star\ \ \parallel u-v \parallel;$$
($\mathcal{P}$3)
$$\mathbb{B}(x^\star, r_\star) \subseteq \mathcal{D},$$
where,
$$r_\star = \frac{2}{3\ p_\star + \sqrt{9\ p_\star^2 + 16\ q_\star}}.$$

Then, the sequence given by (1.6) remains in $\mathbb{B}(x^\star, r_\star)$, converges to the unique solution $x^\star$ of (1.2) and satisfies the following estimation for all $n \geq 2$:
$$\parallel x_{n+1} - x^\star \parallel \leq \rho_n\ \ \parallel x_n - x^\star \parallel,$$
where,
$$\rho_n = \frac{p_\star\ \parallel x_n - x^\star \parallel + q_\star\ (\parallel x_n - x^\star \parallel + \parallel x_{n-2} - x^\star \parallel)\ (\parallel x_n - x^\star \parallel + \parallel x_{n-1} - x^\star \parallel)}{1 - 2\ p_\star\ \ \parallel x_n - x^\star \parallel - q_\star\ (\parallel x_n - x^\star \parallel + \parallel x_{n-2} - x^\star \parallel)\ \ \parallel x_{n-1} - x^\star \parallel},$$
i.e., there exists a constant $\rho > 0$ and a integer $N \geq 2$ such that:
$$\parallel x_{n+1} - x^\star \parallel \leq \rho\ \ \parallel x_n - x^\star \parallel\ \parallel x_{n-1} - x^\star \parallel\ \parallel x_{n-2} - x^\star \parallel \qquad for\ all\ \ n \geq N.$$

We will be concerned with the existence and the convergence of the sequence defined by (1.4) to the solution $x^\star$ of (1.1). The main result of this study is as follows.

Theorem 3.2. *Let $F\ :\ \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be an operator such that for every distinct points x, y and z in $\mathcal{D}$, there exists a first order and a second order divided differences of F $[x,y;F]$ and $[x,y,z;F]$ at these points. Assume:*

($\mathcal{H}1$) *There exists $\alpha > 0$, such that for all x, y, u, v in $\mathcal{D}$*

$$\| [x,y;F] - [u,v;F] \| \leq \alpha \left(\| x-u \| + \| y-v \| \right);$$

($\mathcal{H}2$) *There exists $\beta > 0$, such that for all x, y, u, v in $\mathcal{D}$*

$$\| [u,x,y;F] - [v,x,y;F] \| \leq \beta \ \| u-v \|;$$

($\mathcal{H}3$) *The set–valued map G^{-1} is pseudo–Lipschitz around $(-F(x^\star), x^\star)$ with constants M, a and b (These constants are given in Definition 2.1);*

($\mathcal{H}4$) *There exists $\kappa > 0$ such that for all $x, y \in V$, we have*

$$\| [x,y;F] \| \leq \kappa \qquad \text{and} \qquad M\ \kappa < \frac{1}{3}.$$

Then, for every constant C such that

$$C \geq C_0 = \frac{M\ (\alpha + 4\ \beta)}{1 - 3\ M\ \kappa}, \tag{3.1}$$

exists $\delta > 0$ satisfying

$$\delta < \delta_0 = \min \left\{ a;\ \sqrt{\frac{1}{4} + \frac{1}{C}} - \frac{1}{2};\ \frac{b}{3\ \kappa};\ \delta_1;\ \delta_2;\ \delta_3;\ \delta_4 \right\} \tag{3.2}$$

where,

$$\delta_1 = \sqrt{\frac{b}{2\ (\alpha + 4\ \beta)}};\ \delta_2 = \sqrt[3]{\frac{b}{2\ (\alpha + 4\ \beta)}};\ \delta_3 = \sqrt{\frac{b}{6\ \alpha}};\ \delta_4 = \sqrt[3]{\frac{b}{24\ \beta}}$$

such that for every distinct starting points x_0, x_1 and x_2 in $\mathbb{B}_\delta(x^\star)$ (with $x_0 \neq x^\star$, $x_1 \neq x^\star$ and $x_2 \neq x^\star$), and a sequence (x_k) defined by (1.4) which is convergent to $x^\star$, and satisfies the following inequality for $k \geq 2$

$$\begin{aligned} &\| x_{k+1} - x^\star \| \leq \\ &\quad C\ \| x_k - x^\star \|\ \max \Big\{ \| x_k - x^\star \| + \| x_k - x^\star \|^2,\ \| x_k - x^\star \|\ \| x_{k-2} - x^\star \|, \\ &\quad \| x_k - x^\star \|\ \| x_{k-1} - x^\star \|,\ \| x_{k-1} - x^\star \|\ \| x_{k-2} - x^\star \| \Big\}. \end{aligned} \tag{3.3}$$

We need to introduce some notations [6]. First, define the set–valued maps $Q : \mathcal{X} \rightrightarrows \mathcal{Y}$ and $\psi_k : \mathcal{X} \rightrightarrows \mathcal{X}$ by

$$Q(x) = F(x^\star) + G(x), \qquad \psi_k(x) = Q^{-1}(Z_k(x)), \quad k \geq 2 \tag{3.4}$$

where Z_k is a mapping from $\mathcal{X}$ to $\mathcal{Y}$ defined by

$$Z_k(x) = F(x^\star) - F(x_k) - \mathcal{T}_k\ (x - x_k), \qquad k \geq 2. \tag{3.5}$$

The proof of theorem 3.2 is given by induction on k. We first state a result involving the starting points (x_0, x_1, x_2). Let us note that the point x_3 is a fixed point of ψ_2 if and only if $0 \in F(x_2) + \mathcal{T}_2\ (x_3 - x_2) + G(x_3)$.

Proposition 3.3. *Under the assumptions of Theorem 3.2, and for every distinct starting points x_0, x_1 and x_2 in $\mathbb{B}_\delta(x^\star)$ (with $x_0 \neq x^\star$, $x_1 \neq x^\star$ and $x_2 \neq x^\star$), the set–valued map ψ_2 has a* *fixed point x_3 in $\mathbb{B}_\delta(x^\star)$ satisfying*

$$\begin{aligned} &\| x_3 - x^\star \| \le \\ &C \ \| x_2 - x^\star \| \ \max\Big\{ \| x_2 - x^\star \| + \| x_2 - x^\star \|^2, \ \| x_2 - x^\star \| \ \| x_0 - x^\star \|, \\ &\| x_2 - x^\star \| \ \| x_1 - x^\star \|, \ \| x_1 - x^\star \| \ \| x_0 - x^\star \| \Big\}. \end{aligned} \tag{3.6}$$

where C and δ are given by Theorem 3.2.

Proof. By hypothesis ($\mathcal{H}2$) we have

$$e(Q^{-1}(y') \cap \mathbb{B}_a(x^\star), Q^{-1}(y'')) \le M \ \| y' - y'' \|, \ \forall y', y'' \in \mathbb{B}_b(0). \tag{3.7}$$

According to the definition of excess e, we have

$$\text{dist } (x^\star, \psi_2(x^\star)) \le e\Big(Q^{-1}(0) \cap \mathbb{B}_\delta(x^\star), \psi_2(x^\star)\Big). \tag{3.8}$$

Moreover, for all distinct points x_0, x_1 and x_2 in $\mathbb{B}_\delta(x^\star)$, and by assumptions ($\mathcal{H}1$) and ($\mathcal{H}2$) we obtain the following

(3.9)

$$\begin{aligned} \| Z_2(x^\star) \| &= \| F(x^\star) - F(x_2) - \mathcal{T}_2 \ (x^\star - x_2) \| \\ &= \| \big[([x^\star, x_2; F] - [x_2, x_2; F]) + ([x_2, x_2; F] - [x_2, x_1; F]) \\ &\quad -([x_0, x_2; F] - [x_0, x_1; F]) \big] \ (x^\star - x_2) \| \\ &= \| \big[([x^\star, x_2; F] - [x_2, x_2; F]) + \\ &\quad ([x_2, x_2, x_1; F] - [x_0, x_2, x_1; F]) \, (x_2 - x_1) \big] \ (x^\star - x_2) \| \\ &\le \Big(\alpha \ \| x_2 - x^\star \| + \beta \ \| x_2 - x_0 \| \, \| x_2 - x_1 \| \Big) \| x^\star - x_2 \| \\ &\le \Big(\alpha \ \| x_2 - x^\star \| + \beta \, (\| x_2 - x^\star \| + \| x_0 - x^\star \|) \\ &\quad (\| x_2 - x^\star \| + \| x_1 - x^\star \|) \Big) \| x^\star - x_2 \| \\ &\le (\alpha + 4\,\beta) \ \| x_2 - x^\star \| \ \max\Big\{ \| x_2 - x^\star \| + \| x_2 - x^\star \|^2, \\ &\quad \| x_2 - x^\star \| \, \| x_0 - x^\star \|, \| x_2 - x^\star \| \, \| x_1 - x^\star \|, \ \| x_1 - x^\star \| \, \| x_0 - x^\star \| \Big\}. \end{aligned}$$

By (3.2) we have $Z_2(x^\star) \in \mathbb{B}_b(0)$.

Hence from (3.7) one gets

(3.10)

$$\begin{aligned} &e\Big(Q^{-1}(0) \cap \mathbb{B}_\delta(x^\star), \psi_2(x^\star)\Big) = e\Big(Q^{-1}(0) \cap \mathbb{B}_\delta(x^\star), Q^{-1}[Z_2(x^\star)]\Big) \\ &\le M \ (\alpha + 4 \ \beta) \ \| x_2 - x^\star \| \ \max\Big\{ \| x_2 - x^\star \| + \| x_2 - x^\star \|^2, \ \| x_2 - x^\star \| \ \| x_0 - x^\star \|, \\ &\qquad \| x_2 - x^\star \| \ \| x_1 - x^\star \|, \ \| x_1 - x^\star \| \ \| x_0 - x^\star \| \Big\}. \end{aligned}$$

Using (3.8) the following inequality holds:

(3.11)

$$\begin{aligned} &\text{dist } (x^\star, \psi_2(x^\star)) \le \\ &M \ (\alpha + 4 \ \beta) \ \| x_2 - x^\star \| \ \max\Big\{ \| x_2 - x^\star \| + \| x_2 - x^\star \|^2, \ \| x_2 - x^\star \| \ \| x_0 - x^\star \|, \\ &\qquad \| x_2 - x^\star \| \ \| x_1 - x^\star \|, \ \| x_1 - x^\star \| \ \| x_0 - x^\star \| \Big\}. \end{aligned}$$

Since $C\ (1-3\ M\ \kappa) > M\ (\alpha + 4\ \beta)$, there exists $\lambda \in [3\ M\ \kappa, 1[$ such that

$$C\ (1-\lambda) \geq M\ (\alpha + 4\ \beta),$$

and

(3.12)

$$\begin{aligned}&\text{dist }(x^\star, \psi_2(x^\star)) \leq \\ &C\ (1-\lambda)\ \parallel x_2 - x^\star \parallel\ \max\Big\{ \parallel x_2 - x^\star \parallel + \parallel x_2 - x^\star \parallel^2,\ \parallel x_2 - x^\star \parallel \parallel x_0 - x^\star \parallel, \\ &\qquad \parallel x_2 - x^\star \parallel \parallel x_1 - x^\star \parallel,\ \parallel x_1 - x^\star \parallel \parallel x_0 - x^\star \parallel \Big\}.\end{aligned}$$

Identifying η_0, ϕ and r in Lemma 2.1 by

$$\begin{aligned}&x^\star, \quad \psi_2 \\ &\text{and} \\ &r_2 = C\ \parallel x_2 - x^\star \parallel\ \max\Big\{ \parallel x_2 - x^\star \parallel + \parallel x_2 - x^\star \parallel^2,\ \parallel x_2 - x^\star \parallel \parallel x_0 - x^\star \parallel, \\ &\qquad \parallel x_2 - x^\star \parallel \parallel x_1 - x^\star \parallel,\ \parallel x_1 - x^\star \parallel \parallel x_0 - x^\star \parallel \Big\}\end{aligned}$$

respectively, we can deduce from the inequality (3.12) that the first assumption in Lemma 2.1 is satisfied.

We prove now that the second assumption of Lemma 2.1 is verified. Using (3.2) we have $r_2 \leq \delta \leq a$, and moreover for $x \in \mathbb{B}_\delta(x^\star)$ we get in turn

(3.13)

$$\begin{aligned}\parallel Z_2(x) \parallel &= \parallel F(x^\star) - F(x_2) - \mathcal{T}_2\,(x - x_2) \parallel \\ &= \parallel [x^\star, x_2; F]\,(x^\star - x + x - x_2) - \\ &\quad ([x_2, x_1; F] + [x_0, x_2; F] - [x_0, x_1; F])\,(x - x_2) \parallel \\ &\leq \parallel [x^\star, x_2; F] \parallel \parallel x - x^\star \parallel + \Big(\parallel ([x^\star, x_2; F] - [x_2, x_2; F]) + \\ &\quad ([x_2, x_2, x_1; F] - [x_0, x_2, x_1; F])\,(x_2 - x_1) \Big) \parallel x - x_2 \parallel .\end{aligned}$$

Using the assumptions ($\mathcal{H}1$)–($\mathcal{H}3$) we obtain

(3.14)

$$\begin{aligned}\parallel Z_2(x) \parallel &\leq \kappa\ \parallel x - x^\star \parallel + \Big(\alpha\ \parallel x_2 - x^\star \parallel + \beta\ \parallel x_2 - x_0 \parallel \parallel x_2 - x_1 \parallel \Big)\ \parallel x - x_2 \parallel \\ &\leq \kappa\ \delta + (\alpha\ \delta + 4\ \beta\ \delta^2)\ 2\ \delta = \kappa\ \delta + 2\ \alpha\ \delta^2 + 8\ \beta\ \delta^3.\end{aligned}$$

Then by (3.2), we deduce that for all $x \in \mathbb{B}_\delta(x^\star)$ we have $Z_2(x) \in \mathbb{B}_b(0)$. Then it follows that for all $x', x'' \in \mathbb{B}_{r_2}(x^\star)$ we have

$$e(\psi_2(x') \cap \mathbb{B}_{r_2}(x^\star), \psi_2(x'')) \leq e(\psi_2(x') \cap \mathbb{B}_\delta(x^\star), \psi_2(x'')),$$

which yields by (3.7)

(3.15)

$$\begin{aligned}e(\psi_2(x') \cap \mathbb{B}_{r_2}(x^\star), \psi_2(x'')) &\leq M \parallel Z_2(x') - Z_2(x'') \parallel \\ &= M \parallel ([x_2, x_1; F] + [x_0, x_2; F] - [x_0, x_1; F])\ (x'' - x') \parallel \\ &\leq 3\ M\ \kappa\ \parallel x'' - x' \parallel\end{aligned}$$

Using ($\mathcal{H}3$) and the fact that $\lambda \geq 3\ M\ \kappa$, we obtain

$$e(\psi_2(x') \cap \mathbb{B}_{r_2}(x^\star), \psi_2(x'')) \leq \lambda\ \parallel x'' - x' \parallel . \tag{3.16}$$

The second condition of Lemma 2.1 is satisfied. By Lemma 2.1 we can deduce the existence of a fixed point $x_3 \in \mathbb{B}_{r_2}(x^\star)$ for the map ψ_2.

The proof of Proposition 3.3 is complete. ∎

Proof. (Proof of Theorem 3.2) Keep $\eta_0 = x^\star$, and for $k \geq 3$ set:

$$r := r_k = C \ \parallel x^\star - x_k \parallel \ \max\Big\{ \parallel x_k - x^\star \parallel + \parallel x_k - x^\star \parallel^2, \ \parallel x_k - x^\star \parallel \parallel x_{k-2} - x^\star \parallel, \ \parallel x_k - x^\star \parallel \parallel x_{k-1} - x^\star \parallel, \ \parallel x_{k-1} - x^\star \parallel \parallel x_{k-2} - x^\star \parallel \Big\}.$$

Then, the application of Proposition 3.3 to the map ψ_k gives the desired result. ∎

Remark 3.1. We can show by using more precise estimates that under less computational cost, and weaker hypothesis than ($\mathcal{H}1$), ($\mathcal{H}2$) and ($\mathcal{H}4$): the ratio of convergence of method (1.4) is improved and the radius of convergence is enlarged. The idea is taken from the works on nonlinear equations [3]–[5]. We can show the following result for the local convergence of method (1.4).

Proposition 3.4. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be an operator such that for every distinct points x, y and z in $\mathcal{D}$, there exists a first order and a second order divided differences of F $[x, y; F]$ and $[x, y, z; F]$ at these points. We suppose that assumption ($\mathcal{H}3$) of Theorem 3.2 is satisfied, and assume:*

($\mathcal{H}1$)* *There exists $\overline{\alpha} > 0$, such that for all x, y, z in $\mathcal{D}$*

$$\parallel [x^\star, x; F] - [y, x; F] \parallel \leq \overline{\alpha} \ \parallel y - x^\star \parallel;$$

($\mathcal{H}2$)* *There exists $\overline{\beta} > 0$, such that for all x, y, z in $\mathcal{D}$*

$$\parallel [x, x, y; F] - [z, x, y; F] \parallel \leq \overline{\beta} \ \parallel x - z \parallel;$$

($\mathcal{H}4$)* *There exists $\kappa > 0$ and $\overline{\kappa} > 0$ such that for all $x, y \in V$, we have*

$$\parallel [x, y; F] \parallel \leq \kappa, \qquad \parallel [x^\star, y; F] \parallel \leq \overline{\kappa} \qquad \text{and} \qquad M \ \kappa < \frac{1}{3}.$$

Then, for every constant $\overline{C}$ such that

$$\overline{C} \geq \overline{C_0} = \frac{M \ (\overline{\alpha} + 4 \ \overline{\beta})}{1 - 3 \ M \ \kappa}, \tag{3.17}$$

exists $\overline{\delta} > 0$ satisfying

$$\overline{\delta} < \overline{\delta_0} = \min\left\{ a; \ \sqrt{\frac{1}{4} + \frac{1}{\overline{C}}} - \frac{1}{2}; \ \frac{b}{3 \ \overline{\kappa}}; \ \overline{\delta_1}; \ \overline{\delta_2}; \ \overline{\delta_3}; \ \overline{\delta_4} \right\} \tag{3.18}$$

where,

$$\overline{\delta_1} = \sqrt{\frac{b}{2 \ (\overline{\alpha} + 4 \ \overline{\beta})}}; \ \overline{\delta_2} = \sqrt[3]{\frac{b}{2 \ (\overline{\alpha} + 4 \ \overline{\beta})}}; \ \overline{\delta_3} = \sqrt{\frac{b}{6 \ \overline{\alpha}}}; \ \overline{\delta_4} = \sqrt[3]{\frac{b}{24 \ \overline{\beta}}}$$

such that for every distinct starting points x_0, x_1 and x_2 in $\mathbb{B}_\delta(x^\star)$ (with $x_0 \neq x^\star$, $x_1 \neq x^\star$ and $x_2 \neq x^\star$), and a sequence (x_k) defined by (1.4) which is convergent to $x^\star$, and satisfies the following inequality for $k \geq 2$

$$\parallel x_{k+1} - x^\star \parallel \leq \overline{C} \ \parallel x_k - x^\star \parallel \ \max\Big\{ \parallel x_k - x^\star \parallel + \parallel x_k - x^\star \parallel^2, \ \parallel x_k - x^\star \parallel \parallel x_{k-2} - x^\star \parallel, \ \parallel x_k - x^\star \parallel \parallel x_{k-1} - x^\star \parallel, \ \parallel x_{k-1} - x^\star \parallel \parallel x_{k-2} - x^\star \parallel \Big\}. \tag{3.19}$$

Proof. (Idea) The proof of Proposition 3.4 is the same one as that of the principal theorem (Theorem 3.2). It is enough to make some modifications by replacing estimates (3.9) and (3.14) by
(3.20)

$$\begin{aligned}\| Z_2(x^\star) \| \ &\leq \ (\overline{\alpha}+4\,\overline{\beta})\ \| x_2 - x^\star \| \ \max\Big\{ \| x_2 - x^\star \| + \| x_2 - x^\star \|^2, \\ &\| x_2 - x^\star \| \, \| x_0 - x^\star \|, \| x_2 - x^\star \| \, \| x_1 - x^\star \|, \| x_1 - x^\star \| \, \| x_0 - x^\star \| \Big\}\end{aligned}$$

and
(3.21)

$$\begin{aligned}\| Z_2(x) \| \ &\leq \ \overline{\kappa}\ \| x - x^\star \| + \Big(\overline{\alpha}\ \| x_2 - x^\star \| + \overline{\beta}\ \| x_2 - x_0 \| \, \| x_2 - x_1 \| \Big)\ \| x - x_2 \| \\ &\leq \ \overline{\kappa}\,\overline{\delta} + (\overline{\alpha}\,\overline{\delta} + 4\,\overline{\beta}\,\overline{\delta}^2)\ 2\,\overline{\delta} = \overline{\kappa}\,\overline{\delta} + 2\,\overline{\alpha}\,\overline{\delta}^2 + 8\,\overline{\beta}\,\overline{\delta}^3.\end{aligned}$$

∎

Remark 3.2. In general, α, β and κ given in **(**$\mathcal{H}1$**)**, **(**$\mathcal{H}2$**)** and **(**$\mathcal{H}4$**)** are not easy to compute. This is our motivation for introducing even weaker hypotheses **(**$\mathcal{H}1$**)***, **(**$\mathcal{H}2$**)*** and **(**$\mathcal{H}4$**)*** in Proposition 3.4.

We clearly have:

$$\overline{\alpha} \leq \alpha, \tag{3.22}$$

$$\overline{\beta} \leq \beta, \tag{3.23}$$

$$\overline{\kappa} \leq \kappa, \tag{3.24}$$

$$\overline{C_0} \leq C_0, \tag{3.25}$$

$$\overline{\delta_0} \geq \delta_0, \tag{3.26}$$

and $\dfrac{\overline{\beta}}{\beta}$, $\dfrac{\overline{\alpha}}{\alpha}$, $\dfrac{\overline{\kappa}}{\kappa}$ can be arbitrarily large [3]–[4].

It follows using (3.25) and (3.26) that the radius of convergence is larger and the convergence of method (1.4) is faster in Proposition 3.4 than the corresponding in Theorem 3.2.

Remark 3.3. We can enlarge the radius of convergence in Theorem 3.2 even further as follows: using inequalities (3.14), (3.9), (3.21) and (3.20), we can improve δ and $\overline{\delta}$ given by (3.2) and (3.18) by considering the constants δ' and $\overline{\delta}'$ respectively:

$$\delta' < \delta_0' = \min\left\{ a;\ \sqrt{\frac{1}{4}+\frac{1}{C}} - \frac{1}{2};\ \delta_5;\ \delta_6 \right\}$$

and

$$\overline{\delta}' < \overline{\delta_0}' = \min\left\{ a;\ \sqrt{\frac{1}{4}+\frac{1}{\overline{C}}} - \frac{1}{2};\ \delta_5';\ \delta_6' \right\}$$

where δ_5, δ_6, δ_5' and δ_6' are the constants given respectively by

$$\begin{aligned}\delta_5 &= \max\ \{\eta > 0\ :\ \eta^3 + \eta - \frac{b}{\alpha + 4\,\beta} < 0\}, \\ \delta_6 &= \max\ \{\eta > 0\ :\ \kappa\ \eta + 2\ \alpha\ \eta^2 + 8\ \beta\ \eta^3 - b < 0\}, \\ \delta_5' &= \max\ \{\eta > 0\ :\ \eta^3 + \eta - \frac{b}{\overline{\alpha} + 4\,\overline{\beta}} < 0\},\end{aligned}$$

and

$$\delta'_6 = \max \{\eta > 0 \,:\, \overline{\kappa}\,\eta + 2\,\overline{\alpha}\,\eta^2 + 8\,\overline{\beta}\,\eta^3 - b < 0\}.$$

The constants κ, $\overline{\kappa}$, α, $\overline{\alpha}$, β, $\overline{\beta}$, and b are as given in Theorem 3.2 and Proposition 3.4.

It follows by (3.3) that there exists $C > 0$, and an integer $N \geq 2$, such that for all $n \geq N$:

$$\| x_{n+1} - x^\star \| \leq C \;\; \| x_n - x^\star \| \, \| x_{n-1} - x^\star \| \, \| x_{n-2} - x^\star \| \,.$$

It then follows that the R–order of convergence as already noted in the abstract of the article is $1.839 \cdots$, which compares favorably with say the Secant method (see [4], [6]) using similar information:

$$0 \in F(x_n) + [x_{n-1}, x_n; F] \; (x_{n+1} - x_n) + G(x_{n+1}), \quad (x_0, x_1 \in \mathcal{D}), \quad (n \geq 1),$$

which is only of order $1.618 \cdots$

CONCLUSION

We provided a new iterative method to approximate solutions for generalized equations. This method extends the one related to the resolution of nonlinear equations [12], [17]. Moreover, we obtain a local convergence result (see Theorem 3.2) using Lipschitz–type assumptions on the first order and the second order divided differences operator and Lipschitz–like concept for set–valued maps.

Under some ideas given in [3], [4] for nonlinear equations, and using some observations (see Remarks 3.2 and 3.3), we provided under weaker hypotheses than used in Theorem 3.2, and less computational cost a local convergence analysis (see Proposition 3.4) with the following advantages:

(1) A larger radius of convergence which allows a larger choice of initial guesses x_0, x_1 and x_2;

(2) Finer error estimates on the distances $\| x_n - x^\star \|$ $(n \geq 1)$.

These observations are very important in computational mathematics [3], [4].

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