The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 9, Issue 2, Article 10, pp. 1-7, 2012

HARMONIC FUNCTIONS WITH POSITIVE REAL PART

SİBEL YALÇIN

Received 9 March, 2009; accepted 15 December, 2010; published 28 December, 2012.

ULUDAĞ ÜNİVERSİTESİ, FEN EDEBİYAT FAKÜLTESİ, MATEMATİK BÖLÜMÜ, 16059 BURSA, TURKEY
syalcin@uludag.edu.tr

ABSTRACT. In this paper, the class of harmonic functions $f = h + \bar{g}$ with positive real part and normalized by $f(\zeta) = 1$, $(|\zeta| < 1)$ is studied, where h and g are analytic in $U = \{z : |z| < 1\}$. Some properties of this class are searched. Sharp coefficient relations are given for functions in this class.

On the other hand, the author make use of Alexander integral transforms of certain analytic functions (which are starlike with respect to $f(\zeta)$) with a view to investigating the construction of sense preserving, univalent and close to convex harmonic functions.

Key words and phrases: Harmonic functions with positive real part, Coefficient inequalities, Alexander integral transforms, Analytic functions, Starlike functions, Close to convex functions, Convex functions.

2000 *Mathematics Subject Classification.* Primary 30C45 ; Secondary 30C50,30C20.

ISSN (electronic): 1449-5910

1. Introduction

In papers [2], [3] and [7] there were studied some classes of complex harmonic functions defined in simply connected domains. As is known, such functions are representable in a simply connected domain D in the form $f = h + \bar{g}$, where h and g are analytic in D. Moreover, for mappings f of this type, the Jacobian

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2, z \in D,$$

$J_f(z) > 0$ implies that a harmonic function f is sense preserving in D and locally 1-1.

Next we denote by S_H the class of functions $f = h + \bar{g}$ which are harmonic univalent and sense -preserving in the open unit disk $U = \{z : |z| < 1\}$with the normalization $f(0) = f_z(0) - 1 = 0$.

There has been interest [5] in studying the class P_H of all the functions of the form $f = h + \bar{g}$ that are harmonic in U and such that $z \in U$, $\mathrm{Re}\{f(z)\} > 0$, where

$$h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \ , \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in U.

We recall that with the usual normalization

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$$

is necessary and sufficient for $f(z)$ to be starlike in U. The set of all functions is denoted by ST.

Suppose that $f(z)$ is a bounded univalent function and $f(\zeta)$ lies inside $D = f(U)$. Then $w_0 = f(\zeta)$ for some unique $\zeta = \rho e^{i\theta_0}$. If $\rho < r < 1$ and $\theta_0 \le \theta \le \theta_0 + 2\pi$, then the domain $D_r = f(U_r)$ is starlike with respect to $w_0 = f(\zeta)$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z) - f(\zeta)}\right\} > 0, \quad z \in C_r : |z| = r.$$

We let $STN(\zeta)$ be the set of functions

$$f(z) = \sum_{n=1}^{\infty} b_n z^n$$

which are analytic and univalent in U for which $f'(\zeta) = 1$, and $f(U)$ is starlike with respect to $f(\zeta)$.

In the other hand, $P(\zeta)$ is the class of all functions

$$q(z) = \sum_{n=0}^{\infty} q_n z^n \equiv 1 + \sum_{n=1}^{\infty} Q_n (z - \zeta)^n, \ \ |\zeta| = \rho < 1$$

that are analytic in U, have positive real part in U and for which $q(\zeta) = 1$. This class is studied by Wald [8]. Similarly, we defined the class $P_H(\zeta)$, $|\zeta| < 1$ in this paper.

The set $STN(\zeta)$ will lead naturally to a related set, $P(\zeta)$. Following Lemma 1.1 is due to Wald [8].

Lemma 1.1. *If $f(z)$ is in $STN(\zeta)$, then*

$$\frac{f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right)-f(\zeta)}{1-\rho^2}$$

is in ST.

Also let C_H denote the subclass S_H consisting of functions f for which the image domain $f(U)$ is close to convex. A domain D is said to be close to convex if the complement of D can be expressed as the union of non-crossing half lines. The construction of harmonic close to convex functions using convex analytic functions was investigated earlier by Clunie and Sheil Small [2]. Jahangiri et al.[4] make use of the Alexander integral transforms(see [1]) of certain analytic functions (which are starlike or convex of positive order) with a view to investigating the construction of sense-preserving, univalent, and close to convex harmonic functions. Similarly, we apply the Alexander integral transforms of analytic functions which are starlike respect to $f(\zeta)$ with a view to constructing sense preserving, univalent, and close to convex harmonic functions.

The following result (due to [2]) will be required in our present investigation.

Lemma 1.2. *If $f=h+\bar{g}$ is locally univalent in U and $h+\epsilon g$ is convex for some $\epsilon(|\epsilon|<1)$, then f is univalent close to convex in U.*

2. THE CLASS $P_H(\zeta)$ AND MAIN RESULTS

Let h and g be analytic in $U=\{z:|z|<1\}$. Denote by $P_H(\zeta)$ the class of all functions of the form

$$f(z)=h(z)+\overline{g(z)}=\sum_{n=0}^{\infty}a_nz^n+\sum_{n=0}^{\infty}\overline{b_nz^n} \tag{2.1}$$

$$\equiv 1+\sum_{n=1}^{\infty}A_n(z-\zeta)^n+\sum_{n=1}^{\infty}\overline{B_n(z-\zeta)^n},\ z\in U,\ \ |\zeta|<1,$$

that are harmonic in U, have positive real part in U and for which $f(\zeta)=1$. The subclass of $P_H(\zeta)$ with $B_1=0$ will be denoted by $P_H^0(\zeta)$.

If $f\in P_H(\zeta)$, the Jacobian $J_f(\zeta)\neq 0$ and A_1, B_1 are real, then the function

$$f_0(z)=\frac{A_1f(z)-B_1\overline{f(z)}}{A_1-B_1}\in P_H^0(\zeta);$$

next, if $f_0\in P_H^0(\zeta)$, then the function f takes the form

$$f(z)=\frac{A_1f_0(z)+B_1\overline{f_0(z)}}{A_1+B_1}.$$

Theorem 2.1. *If $f=h+\overline{g}\in P_H^0(\zeta)$, then $q=h+g\in P(\zeta)$. Conversely, if h, g are analytic in U, $h(\zeta)-1=g(\zeta)=0$ and $q=h+g\in P(\zeta)$, then $f=h+\overline{g}\in P_H^0(\zeta)$.*

Proof. Let $f=h+\overline{g}\in P_H^0(\zeta)$. Since $q=h+g$ is analytic in U,

$$\mathrm{Re}\{q(z)\}=\mathrm{Re}\{h(z)+g(z)\}=\mathrm{Re}\{f(z)\}>0,$$

and

$$q(\zeta)=h(\zeta)+g(\zeta)=1,$$

$q\in P(\zeta)$.

Conversely, let $q = h + g \in P(\zeta)$, let h, g be analytic in U and $h(\zeta) - 1 = g(\zeta) = 0$. Then $f = h + \overline{g} \in P_H^0(\zeta)$, since $\mathrm{Re}\{f(z)\} = \mathrm{Re}\{q(z)\} > 0$. ∎

Proposition 2.2. *If f and F are related by*

$$f(L(z)) = F(z) \text{ or } F(z) = f(M(z)),$$

then $F(z)$ is in P_H if and only if $f(z)$ is in $P_H(\zeta)$, where

$$L(z) = \frac{z+\zeta}{1+\overline{\zeta}z} \text{ and } M(z) = \frac{z-\zeta}{1-\overline{\zeta}z}$$

are two mappings of U onto itself.

Proof. If $F(z) \in P_H$, then $f(\zeta) = F(M(\zeta)) = F(0) = 1$ and

$$\mathrm{Re}\{f(z)\} = \mathrm{Re}\{f(M(z)\} > 0, |M(z)| < 1.$$

Consequently, $f(z) \in P_H(\zeta)$.

Conversely, if $f(z) \in P_H(\zeta)$, then $F(0) = f(L(0)) = f(\zeta) = 1$ and

$$\mathrm{Re}\{F(z)\} = \mathrm{Re}\{f(L(z)\} > 0,\ \ |L(z)| < 1.$$

Hence, $F(z)$ is in P_H. ∎

Theorem 2.3. *If f is in $P_H(\zeta)$ and has the series expansions (2.1), then for all $n \geq 1$ and for $|\zeta| = \rho < 1$*

$$\begin{aligned} ||a_0| - |b_0|| &\leq \frac{1+\rho}{1-\rho}, \\ ||a_n| - |b_n|| &\leq 2\frac{1+\rho}{1-\rho} \end{aligned}$$

and

$$||A_n| - |B_n|| \leq \frac{2}{(1+\rho)(1-\rho)^n}.$$

Proof. If $f = h + \overline{g} \in P_H(\zeta)$, then by Theorem 2.1, $h + g$ is in $P(\zeta)$ and has the form

$$h(z) + g(z) = \sum_{n=0}^{\infty}(a_n + b_n)z^n = 1 + \sum_{n=1}^{\infty}(A_n + B_n)(z-\zeta)^n.$$

Hence, by [8],

$$\begin{aligned} ||a_0| - |b_0|| &\leq |a_0 + b_0| \leq \frac{1+\rho}{1-\rho}, \\ ||a_n| - |b_n|| &\leq |a_n + b_n| \leq 2\frac{1+\rho}{1-\rho} \end{aligned}$$

and

$$||A_n| - |B_n|| \leq |A_n + B_n| \leq \frac{2}{(1+\rho)(1-\rho)^n}.$$

All the inequalities are sharp. ∎

Theorem 2.4. *If $f \in P_H^0(\zeta)$ and f is sense preserving in U, then*

$$|a_n| \leq \frac{1+\rho}{1-\rho}(n+1), \quad |b_n| \leq \frac{1+\rho}{1-\rho}(n-1) \tag{2.2}$$

and

$$|A_n| \leq \frac{n+1}{(1+\rho)(1-\rho)^{n+1}}, \quad |B_n| \leq \frac{n-1+2\rho}{(1+\rho)(1-\rho)^{n+1}}. \tag{2.3}$$

All the inequalities are sharp. Equality in (2.2), (2.3) occur for the functions

$$f(z) = \frac{1+\rho}{1-\rho}\left[\frac{1}{(1-z)^2} + \overline{\left(\frac{z}{1-z}\right)^2}\right],\ z \in U$$

and

$$f(z) = \frac{1-\rho}{1+\rho}\left[\frac{1}{(1-z)^2} + \overline{\left(\frac{2(z-2)}{(1-z)^2}\right)}\right],\ z \in U$$

respectively.

Proof. If $f = h + \overline{g} \in P_H^0(\zeta)$, then by Theorem 2.1, $q = h + g \in P(\zeta)$. Since f is sense preserving in U, then $|g'(z)| < |h'(z)|$, for $z \in U$. If we say that $w(z) = g'(z)\ /\ h'(z)$, then $w(z)$ satisfy the conditions as Schwarz Lemma in U. Thus we get

$$g'(z) = \frac{w(z)}{1+w(z)}q'(z),\ h'(z) = \frac{g'(z)}{w(z)}. \tag{2.4}$$

Next, we have for $z \in U$, $|\zeta| = \rho < 1$

$$q(z) \ll \frac{1+\rho}{1-\rho}\frac{1+z}{1-z} \text{ and } \frac{w(z)}{1+w(z)} \ll \frac{z}{1-z};\ q(-\rho) = 1,$$

therefore by (2.4), we obtain the following results

$$g'(z) \ll \frac{1+\rho}{1-\rho}\frac{2z}{(1-z)^3} \text{ and } h'(z) \ll \frac{1+\rho}{1-\rho}\frac{1}{(1-z)^3}$$

where $\ll$ means that the moduli of the coefficients of the left are bounded by the corresponding coefficients of the function on the right. Using the technique of dominant power series we have

$$g(z) \ll \frac{1+\rho}{1-\rho}\frac{z^2}{(1-z)^2},\ h(z) \ll \frac{1+\rho}{1-\rho}\frac{1}{(1-z)^2}$$

and

$$g^{(n)}(z) \ll \frac{1+\rho}{1-\rho}\frac{n!(n+2z-1)}{(1-z)^{n+2}},\quad h^{(n)}(z) \ll \frac{1+\rho}{1-\rho}\frac{(n+1)!}{(1-z)^{n+2}}.$$

Furthermore, so for $n \geq 1$

$$a_n = \frac{h^{(n)}(0)}{n!} \text{ and } b_n = \frac{g^{(n)}(0)}{n!},$$

we obtain (2.2).

In the other hand, using

$$q(z) \ll \frac{1-\rho}{1+\rho}\frac{1+z}{1-z} \text{ and } \frac{w(z)}{1+w(z)} \ll \frac{z}{1-z};\ q(\rho) = 1,$$

we have from (2.4),

$$g(z) \ll 2\frac{1-\rho}{1+\rho}\frac{z-2}{(1-z)^2},\quad h(z) \ll \frac{1-\rho}{1+\rho}\frac{1}{(1-z)^2}$$

and

$$g^{(n)}(z) \ll \frac{1-\rho}{1+\rho}\frac{n!(n+2z-1)}{(1-z)^{n+2}},\quad h^{(n)}(z) \ll \frac{1-\rho}{1+\rho}\frac{(n+1)!}{(1-z)^{n+2}}.$$

Therefore, so

$$A_n = \frac{h^{(n)}(\zeta)}{n!} \text{ and } B_n = \frac{g^{(n)}(\zeta)}{n!},$$

for $|\zeta| = \rho < 1$, we obtain the inequalities in (2.3), where h and g converge for $|z-\zeta| < 1-\rho$. ∎

Theorem 2.5. *If $f = h + \overline{g} \in P_H(\zeta)$, then for $X = \{\eta : |\eta| = 1\}$ and for $z \in U$,*

$$h(z) + g(z) = \int\limits_{|\eta|=1} \frac{1 - \eta\zeta + z(\eta - \overline{\zeta})}{1 + \eta\zeta - z(\eta + \overline{\zeta})} d\mu(\eta)\ ;\ \frac{1}{2\pi} \int\limits_{|\eta|=1} d\mu(\eta) = 1,\ \ |\zeta| < 1. \tag{2.5}$$

Proof. If $f(z) \in P_H(\zeta)$, then by Proposition 2.2 and [8] for $X = \{\eta : |\eta| = 1\}$ and $z \in U$,

$$h(z) + g(z) = \int\limits_{|\eta|=1} \frac{1 + \eta(M(z))}{1 - \eta(M(z))} d\mu(\eta)\ ;\ \frac{1}{2\pi} \int\limits_{|\eta|=1} d\mu(\eta) = 1.$$

Hence, we have (2.5). ∎

Theorem 2.6. *$P_H(\zeta)$ is convex and compact.*

Proof. Let $f = \lambda f_1 + (1-\lambda) f_2$, for $f_1, f_2 \in P_H(\zeta)$, $0 < \lambda < 1$. Thus,

$$f = h + \overline{g} = \lambda h_1 + (1-\lambda) h_2 + \overline{\lambda g_1 + (1-\lambda) g_2}$$

and h and g are analytic in U. Furthermore,

$$\operatorname{Re}\{f(z)\} = \lambda \operatorname{Re}\{f_1(z)\} + (1-\lambda) \operatorname{Re}\{f_2(z)\} > 0$$

and

$$f(\zeta) = \lambda f_1(\zeta) + (1-\lambda) f_2(\zeta) = \lambda + 1 - \lambda = 1.$$

Hence, the harmonic function $f = h + \overline{g}$ belongs to the class $P_H(\zeta)$.

In the other hand, P_H is compact implies that $P_H(\zeta)$ is compact, [6]. ∎

Next theorem makes use of certain analytic functions (which are starlike with respect to $f(\zeta)$) in the construction of harmonic close to convex functions.

Theorem 2.7. *Let $\lambda \in \mathbb{C}$ and $0 < |\lambda| < \frac{1}{2}$. If f is in $STN(\zeta)$, then*

$$\begin{aligned} F(z) &= H(z) + \overline{G(z)} \\ &= \int\limits_0^z \frac{f(L(t)) - f(\zeta)}{(1-\rho^2)t} dt + \frac{\lambda}{1-\rho^2} \left[\overline{f(L(z))} - f(\zeta) \right] \in C_H. \end{aligned} \tag{2.6}$$

Proof. First of all it follows from Lemma 1.1 that

$$\frac{f\left(L(z)\right) - f(\zeta)}{1-\rho^2}$$

is starlike of positive order in U. Therefore, by a result of Alexander[1], its integral transform:

$$H(z) = \int\limits_0^z \frac{f(L(t)) - f(\zeta)}{(1-\rho^2)t} dt \tag{2.7}$$

is convex in U. Thus, by choosing H in Lemma 1.2 as in (2.7), and letting

$$G(z) = \frac{\lambda}{1-\rho^2} \left[\overline{f(L(z))} - f(\zeta) \right] \ \text{ and } \ \epsilon = 0,$$

the assertion (2.6) of Theorem 2.7 would follow if we can show that

$$|H'(z)| > |G'(z)| \quad (z \in U).$$

Since

$$|H'(z)| = \left| \frac{f(L(z)) - f(\zeta)}{(1-\rho^2)z} \right| \ \text{ and } \ |G'(z)| = |\lambda| \left| f'(L(z)) \frac{1}{(1+\bar{\zeta}z)^2} \right|,$$

it is sufficient to show that

$$\left|\frac{zf'(L(z))\frac{1}{(1+\bar{\zeta}z)^2}}{f(L(z))-f(\zeta)}\right| < \frac{1}{|\lambda|(1-\rho^2)}, \quad (z \in U).$$

But, from

$$\left|\frac{zf'(L(z))\frac{1-\rho^2}{(1+\bar{\zeta}z)^2}}{f(L(z))-f(\zeta)} - 1\right| < 1$$

and by $0 < |\lambda| < \frac{1}{2}$, we have

$$\left|\frac{zf'(L(z))\frac{1}{(1+\bar{\zeta}z)^2}}{f(L(z))-f(\zeta)}\right| < \frac{2}{1-\rho^2} < \frac{1}{|\lambda|(1-\rho^2)}, \quad (z \in U),$$

which evidently completes the proof of Theorem 2.7. ∎

Example 2.8. By setting

$$f(z) = (1-\rho^2)\left(M(z)e^{M(z)} + \zeta e^{-\zeta}\right)$$

in Theorem 2.7, we observe that the integral function:

$$F(z) = \int_0^z e^t dt + \overline{\lambda z e^z} = e^z + \bar{\lambda}\,\bar{z}e^{\bar{z}} - 1$$

is sense preserving, univalent, and close to convex harmonic in U if $0 < |\lambda| < \frac{1}{2}$.

References

[1] J. W. ALEXANDER, Functions which map the interior of the unit circle upon simple regions, *Ann. of Math.*, (Ser. 2), **17** (1915-1916), pp. 12-22.

[2] J. CLUNIE, and T. SHEIL-SMALL, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Math.* **9** (1984), pp. 3-25.

[3] W. HENGARTNER, and G., SCHOBER, *Univalent harmonic functions, Trans. Amer. Math. Soc.* **299** (1987), pp. 1-31.

[4] J. M. JAHANGIRI, Y. C. KIM, and H. M. SRIVASTAVA, Construction of a certain class of harmonic close-to-convex functions associated with the alexander integral transfrom, *Int. Trans. and Special Func.* Vol. **14**, No.3 (2003), pp. 237-242.

[5] Z. J. JAKUBOWSKI, W. MAJCHRZAK, and K. SKALSKA Harmonic mappings with a positive real part, *Materialy XIV Konferencji z Teorii Zagadnien Exremalnych,* Lodz, (1993), pp. 17-24.

[6] M. ÖZTÜRK, Harmonic functions with positive real part, and which are typically real, *Hacettepe Bull. of Nat. Sci. and Eng.* Vol. **24**, Series B, (1995), pp. 79-89.

[7] T. SHEIL-SMALL, Constants for planar harmonic mappings, *J. London Math. Soc.,* **42**, No. 2, (1990), pp. 237-248.

[8] J. K. WALD, *On Starlike Functions*, PhD Thesis, University of Delaware, Newark, Delaware, (1978).