GROWTH AND PRODUCTS OF SUBHARMONIC FUNCTIONS IN THE UNIT BALL

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ABSTRACT. The purpose of this paper is to link informations on the application \( u \mapsto gu \) with some growth conditions on the functions \( u \) and \( g \) subharmonic in the unit ball of \( \mathbb{R}^N \). Two kinds of growth are considered: the Bloch–type growth and growth conditions expressed through integrals involving involutions of the unit ball.

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1. Introduction

The subharmonic functions under study in this article are defined on the Euclidean open unit ball $B_N \subset \mathbb{R}^N$ (with $N \in \mathbb{N}$, $N \geq 2$). They grow according to two different patterns. For instance, such a function $u$ is said to have a Bloch–type growth if:

\begin{equation}
\exists \lambda \in \mathbb{R} \text{ such that } M_{X_\lambda}(u) := \sup_{x \in B_N} (1 - |x|^2)^\lambda u(x) < +\infty
\end{equation}

where $| \cdot |$ denotes the Euclidean norm in $\mathbb{R}^N$ (the set $X_\lambda$ will be defined explicitly later). Another kind of growth for $u$ is described through the Bloch–type growth of the function $F_{\beta,\gamma}(u)$ defined on $B_N$ by the following formula, with $\beta$ and $\gamma$ fixed reals:

\begin{equation}
(F_{\beta,\gamma}(u))(a) = \int_{B_N} (1 - |x|^2)^\beta u(x) (1 - |\Phi_a(x)|^2)^\gamma \, dx \quad \forall a \in B_N
\end{equation}

denoting by $\Phi_a$ an involution of $B_N$ which will be exploited in the next section. The function $u$ is said to have a growth of the second kind if:

\begin{equation}
\exists \alpha \in \mathbb{R} \text{ such that } M_{Y_{\alpha,\beta,\gamma}}(u) := M_{X_\alpha}(F_{\beta,\gamma}(u)) < +\infty
\end{equation}

(see Section 2 for the precise definitions of the sets $Y_{\alpha,\beta,\gamma}$ and $X_\alpha$).

The purpose of the paper is to study the links between the growth of two such functions $u$ and $g$ and the growth of $gu$. Given sets $E$, $F$ and $G$ of the kind $X_\lambda$ or $Y_{\alpha,\beta,\gamma}$, we consider the application:

\[ E \to F \]

\[ u \mapsto gu \]

and investigate the following questions:

- If $g \in G$, does there exist then a constant $C > 0$ such that

\[ M_F(gu) \leq C M_E(u) \quad \forall u ? \]

- Does the converse hold?

- In the case $E = X_\alpha$, $F = X_{\alpha+\lambda}$ and $G = X_\lambda$, Propositions 5.1 and 5.2 provide positive answers to both above questions (see Section 5 for the exact statements).

- In the case $E = Y_{\alpha,\beta,\gamma}$ and $G = Y_{\alpha,\beta,\gamma}$ too, positive answers also hold: see Propositions 5.4 and 5.5 in Section 5 for the precise assumptions.

- For the case $E = Y_{\alpha,\beta,\gamma}$ and $F = X_{\alpha+\beta+\lambda+N}$, see Section 4.

Theorem 4.1 studies the situation where $G = X_{\lambda+N/2}$ and the parameters $\alpha$, $\beta$, $\gamma$ fulfill:

\begin{equation}
-\beta - \frac{N+1}{2} < \alpha \leq \beta + \frac{N+1}{2}, \quad \beta > -\frac{N+1}{2}, \quad \gamma \geq -\alpha, \quad \gamma > \max(\alpha, -1-\beta).
\end{equation}

Theorem 4.2 studies the situation where $G = X_{\lambda+N/2-1+\alpha}$ and

\begin{equation}
\alpha \geq \frac{1}{2}, \quad \beta \geq -1 - \frac{N}{2}, \quad \gamma > \frac{N}{2}.
\end{equation}

Theorem 4.3 studies the situation where $G = X_{\lambda+N/2+\alpha}$ and

\begin{equation}
\alpha \geq 0, \quad \beta \geq -\frac{N+1}{2}, \quad \gamma > \frac{N-1}{2}.
\end{equation}

- In the case $E = Y_{\alpha,\beta,\gamma}$ and $F = Y_{\alpha,\lambda+\beta,\gamma}$, Proposition 3.1 brings an affirmative answer to the first question, with $g \in X_\lambda$. Theorems 3.2, 3.3 and 3.4 provide situations where the converse of Proposition 3.1 partly holds: given a function $g$, defined on $B_N$, for which there exists some

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constant $C$ such that $M_{Y_{\alpha,\beta,\gamma}}(gu) \leq C M_{Y_{\alpha,\beta,\gamma}}(u)$ for all $u$, Theorems 3.2, 3.3 and 3.4 obtain some $\lambda \geq \mu$ such that $g \in \mathcal{L}_\lambda$. For positive functions $g$ moreover exceeding $(1-|x|^2)^{-\lambda''}$ on some sector issued from the origin, Theorem 3.5 brings a restriction on $\lambda''$:

- in situation (1.4) : $\lambda = \frac{N-1}{2}$, $\lambda'' < \lambda + \beta + \gamma + 1$
- in situation (1.5) : $\lambda = \frac{N}{2} - 1 + \alpha$, $\lambda'' < \lambda + \gamma - \frac{N}{2}$
- in situation (1.6) : $\lambda = \frac{N-1}{2} + \alpha$, $\lambda'' < \lambda + \gamma - \frac{N-1}{2}$

(see Section 3 for all details).

Since the sets $\mathcal{L}_\lambda$ and $\mathcal{L}_{\alpha,\beta,\gamma}$ do not even have the structure of vector space, it is not surprising that the situation studied here is more delicate than the similar questions involving spaces of holomorphic functions in the unit disk $D \subset \mathbb{C}$, such as the Bloch space $\mathcal{B}_r$, the Dirichlet space $\mathcal{D}_r$ (a special case of Bergman space when $r > 1$) and the space $BMOA_\tau$ for instance. Using the above notations (1.1) and (1.3), with $N = 2$ and $x = z \in D = B_N$, the classical norms on these spaces are denoted as follows:

$$f \in \mathcal{B}_r \iff ||f||_{\mathcal{B}_r} := |f(0)| + M_{\mathcal{X}_\lambda}(|f'|) < +\infty \quad (\tau > 0)$$

$$f \in \mathcal{D}_r \iff ||f||_{\mathcal{D}_r} := M_{\mathcal{Y}_{h,\tau-2,0}}(|f|^2) < +\infty \quad (\tau > 1)$$

$$f \in BMOA_\tau \iff ||f||_{BMOA_\tau} := |f(0)|^2 + M_{\mathcal{Y}_{h,2\tau-2,1}}(|f'|^2) < +\infty \quad (\tau > 0)$$

modifying here the notations (1.2) and (1.3) with $\Phi_a(x)$ now replaced by $\varphi_a(z) = \frac{a-z}{1-\tau z}$.

Given $h$ an analytic function in $D$, such linear applications as $f \mapsto hf$ between these spaces have been studied by various authors: see for instance [8], [4] p. 197, [10].

A related question is the continuity (with respect to the above norms) of the operator

$$I_h : f \mapsto I_h(f)$$

defined by:

$$(I_h(f))(z) = \int_0^z h(\zeta) f'(\zeta) \, d\zeta \quad \forall z \in D.$$  

Some examples of known results:

- Given $\tau \geq 1$, the continuity of $I_h : \mathcal{B}_r \to \mathcal{B}_\tau$ is equivalent to the boundedness of $h$ on $D$, in other words: $h \in H^\infty$ (proved in [9] p.138).

- If $1 < \mu < \lambda$, then the operator $I_h : BMOA_\mu \to \mathcal{B}_\lambda$ is bounded if and only if $h \in \mathcal{B}_{\lambda-\mu+1}$ (see [8] p.1050).

- Given $1 < \mu < \lambda$, it is proved in [8] p.1059–1060 that $I_h : \mathcal{D}_\mu \to \mathcal{D}_\lambda$ is bounded if and only if $h \in \mathcal{B}_{1+\frac{1}{2}(\lambda-\mu)}$.

Since $|f'|$, $|f|^2$ and $|f'|^2$ are subharmonic functions on $B_2$ (see [2] p.46), the question naturally occurred whether the preceding results had some kind of analog for subharmonic functions on $B_N$ ($N \geq 2$) with a growth described by (1.1) or (1.3).
2. Various relations between the sets $X_\lambda$ and $Y_{\alpha, \beta, \gamma}$.

**Definition 2.1.** Given $\lambda \in \mathbb{R}$, let $X_\lambda$ be the set of all functions $u : B_N \to [-\infty, +\infty]$ satisfying

$$M_{X_\lambda}(u) := \sup_{x \in B_N} (1 - |x|^2)^\lambda u(x) < +\infty.$$  

Let $X_\lambda^+ = \{u \in X_\lambda : u(B_N) \subset [0, +\infty]\}$. Let $S X_\lambda$ denote the subset of all subharmonic $u \in X_\lambda$. Finally, let $S X_\lambda^+ = S X_\lambda \cap X_\lambda^+$.

**Remark 2.1.** Obviously $X_\lambda \subset X_\mu$ for $\lambda < \mu$, with $M_{X_\mu}(u) \leq M_{X_\lambda}(u)$ $\forall u \in X_\lambda$. If $\lambda < 0$ then $S X_\lambda^+ = \{0\}$ (see Proposition 6.2 of [6]).

**Definition 2.2.** Given $\alpha, \beta \in \mathbb{R}, \gamma \geq 0$, let $Y_{\alpha, \beta, \gamma}$ denote the set of all measurable functions $u : B_N \to [-\infty, +\infty]$ satisfying:

$$M_{Y_{\alpha, \beta, \gamma}}(u) := \sup_{a \in B_N} (1 - |a|^2)^\alpha \int_{B_N} (1 - |x|^2)^\beta u(x) (1 - |\Phi_a(x)|^2)^\gamma dx < +\infty$$

with $\Phi_a : B_N \to B_N$ the involution defined by:

$$\Phi_a(x) = a - P_a(x) - \sqrt{1 - |a|^2} \Phi_a(x)$$

where:

$$\langle x, a \rangle = \sum_{j=1}^N x_j a_j, \quad P_a(x) = \frac{\langle x, a \rangle}{|a|^2} \quad \text{and} \quad Q_a(x) = x - P_a(x)$$

for every $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ and $a = (a_1, a_2, \ldots, a_N) \in \mathbb{R}^N$, with $P_a(x) = 0$ when $a = 0$. As above, we similarly define $Y_{\alpha, \beta, \gamma}^+$ (resp. $S Y_{\alpha, \beta, \gamma}$) the subset of all non-negative (resp. subharmonic) functions $u \in Y_{\alpha, \beta, \gamma}$ and finally $S Y_{\alpha, \beta, \gamma}^+ = S Y_{\alpha, \beta, \gamma} \cap Y_{\alpha, \beta, \gamma}^+$.

**Remark 2.2.** In Proposition 3.1 of [6], it was proved that $S Y_{\alpha, \beta, \gamma}^+ \subset S X_\lambda^+ \cap Y_{\alpha, \beta, \lambda}^+$. Let $\alpha' \geq \alpha, \beta' \geq \beta$ and $\gamma' \geq \gamma \geq 0$, then $Y_{\alpha', \beta', \gamma'} \subset Y_{\alpha, \beta, \lambda}^+$ with $M_{Y_{\alpha', \beta', \gamma'}}(u) \leq M_{Y_{\alpha, \beta, \gamma}}(u)$ for every $u \in Y_{\alpha', \beta', \gamma'}$, since

$$(1 - |a|^2)^{\alpha'} \leq (1 - |a|^2)^\alpha$$

$$(1 - |x|^2)^{\beta'} \leq (1 - |x|^2)^\beta$$

$$(1 - |\Phi_a(x)|^2)^{\gamma'} \leq (1 - |\Phi_a(x)|^2)^\gamma$$

If $\alpha + \beta < -N$ or $\alpha < -\gamma$, then $S Y_{\alpha, \beta, \gamma}^+ = \{0\}$ (see Propositions 6.3 and 6.4 of [6]).

**Proposition 2.1.** (i) Given $\beta \geq -\frac{N}{2} - 1$, $\gamma \geq 1$ such that $\beta + \gamma > -1$, let $\alpha \geq \gamma - \beta + \frac{1}{2}$. There exists $K_0 > 0$ such that:

$$M_{Y_{\alpha, \beta, \gamma}}(u) \leq K_0 M_{X_\lambda}(u) \quad \forall \lambda \in \mathbb{R} \quad \forall u \in X_\lambda.$$

(ii) Given $\beta \geq -\frac{N+1}{2}$, $\gamma \geq \frac{1}{2}$ such that $\beta + \gamma > -1$, let $\alpha \geq \gamma - \beta$. There exists $K' > 0$ such that:

$$M_{Y_{\alpha, \beta, \gamma}}(u) \leq K' M_{X_\lambda}(u) \quad \forall \lambda \in \mathbb{R} \quad \forall u \in X_\lambda.$$

Thus, in both cases: $X_\lambda \subset Y_{\alpha, \beta, \gamma}^+$.

**Remark 2.3.** The constants $K_0$ and $K'$ respectively stem from Lemmas 6.2 and 6.3 which are postponed in annex: at the end of the paper, Section [6] gathers several technical results which will be repeatedly used throughout the proofs in Sections 2, 3, 4 and 5.
Proof. In order to establish Proposition 2.1, let us consider the following, which is available for any \( a \in B_N \):

\[
(1 - |a|^2)^\alpha \int_{B_N} (1 - |x|^2)^{\lambda + \beta} u(x) (1 - |\Phi_a(x)|^2)^\gamma dx \leq
\]

\[
\leq M_{X_0}(u) (1 - |a|^2)^{\alpha + \gamma} \int_{B_N} (1 - |x|^2)^{\frac{\beta + \gamma}{2}} dx
\]


\[
1 - |\Phi_a(x)|^2 = \frac{(1 - |a|^2) (1 - |x|^2)}{(1 - \langle x, a \rangle)^2} \quad \forall x \in B_N \quad \forall a \in B_N.
\]

Proof of (i) Lemma 6.2 is applied with \( A = 2\gamma > 0 \) and \( T = \beta + \gamma > -1 \), since

\[
2 \leq A \leq 2\gamma + 2(\beta + 1) + N = N + 2(T + 1).
\]

The above integral is thus majorized by \( K_0 (1 - |a|^2)^{\gamma - \frac{1}{2}} \). The result follows since

\[
\sup_{a \in B_N} \frac{(1 - |a|^2)^{\alpha + \beta}}{(1 - |a|^2)^{\gamma + \frac{1}{2}}} = 1.
\]

Proof of (ii) Lemma 6.3 is applied with \( T = \beta + \gamma > -1 \), \( A = 2\gamma > 0 \), \( \tau > 0 \) and \( P = 0 \) which fulfill \( 1 \leq A + P \leq 2\gamma + 2\beta + N + 1 = N + 1 + 2T \). It leads to:

\[
\int_{B_N} (1 - |x|^2)^T \frac{dx}{(1 - \langle x, a \rangle)^A} \leq \frac{K'}{(1 - |a|^2)^{\tau}} = \frac{K'}{(1 - |a|^2)^{\gamma}} \quad \forall a \in B_N
\]

generating the conclusion since \( \sup_{a \in B_N} \frac{(1 - |a|^2)^{\alpha + \beta}}{(1 - |a|^2)^{\gamma}} = 1. \]

Example 2.1. Given \( \alpha \geq \frac{1}{2}, \beta \geq -1 - \frac{N}{2} \) and \( \gamma > \frac{N}{2} \), the function \( v \) defined on \( B_N \) by:

\[
v(x) = (1 - |x|^2)^{-\frac{\alpha}{2} - 1} \quad \forall x \in B_N
\]

belongs to \( SY_{\alpha, \beta, \gamma} \) with \( M_{Y_{\alpha, \beta, \gamma}}(v) = K_0 \).

The growth of \( v \) will be studied during the proof of Theorem 3.3 in Section 3.

Example 2.2. Given \( a \in B_N \) and parameters \( \alpha, \beta, \gamma \) in configuration (1.4), the function \( f_a \) defined by \( f_a(x) = (1 - \langle x, a \rangle)^{-N - 1 - 2\beta} \forall x \in B_N \) belongs to \( SY_{\alpha, \beta, \gamma}^+ \) and

\[
M_{Y_{\alpha, \beta, \gamma}}(f_a) \leq K (1 - |a|^2)^{\alpha - \frac{N + 1}{2} - \beta}
\]

with \( K \) the constant from Lemma 6.3.

This property of \( f_a \) will be established during the proof of Theorem 3.2 just below.

Example 2.3. With parameters \( \alpha, \beta, \gamma \) in configuration (1.6), the function \( u \) defined by

\[
u(x) = (1 - |x|^2)^{-\frac{\alpha + 1}{2} - \beta} \quad \forall x \in B_N
\]

belongs to \( SY_{\alpha, \beta, \gamma}^+ \) and \( M_{Y_{\alpha, \beta, \gamma}}(u) = K' \).

This will be shown in the proof of Theorem 3.4.
3. Products $gu$ when $u \in \mathcal{Y}_{\alpha, \beta, \gamma}$ and $g$ has a Bloch–type growth.

**Proposition 3.1.** Given $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma \geq 0$ and $\lambda \in \mathbb{R}$, let $g \in \mathcal{X}_\lambda$ such that $M_{\mathcal{X}_\lambda}(g) \geq 0$. Then

$$M_{\mathcal{Y}_{\alpha, \beta, \gamma}}(gu) \leq M_{\mathcal{X}_\lambda}(g) M_{\mathcal{Y}_{\alpha, \beta, \gamma}}(u) \quad \forall u \in \mathcal{Y}_{\alpha, \beta, \gamma}.$$  

**Proof.** We have $(1 - |x|^2)\lambda g(x) u(x) \leq M_{\mathcal{X}_\lambda}(g) u(x)$ since $u(x) \geq 0 \forall x \in B_N$. Thus the required majorization follows from $M_{\mathcal{X}_\lambda}(g) \geq 0$.

**Theorem 3.2.** Given $\lambda \in \mathbb{R}$, $\beta > -\frac{N+1}{2}$, $\alpha \in [-\frac{N+1}{2} - \beta, \frac{N+1}{2} + \beta]$ and $\gamma > \max(\alpha, -1 - \beta)$ such that $\gamma \geq -\alpha$, let $g$ be a non–negative subharmonic function defined on $B_N$, satisfying:

(3.1) \[ \exists C > 0 \quad M_{\mathcal{Y}_{\alpha, \beta, \gamma}}(gu) \leq C M_{\mathcal{Y}_{\alpha, \beta, \gamma}}(u) \quad \forall u \in \mathcal{Y}_{\alpha, \beta, \gamma}. \]

Then $g \in \mathcal{X}_{\lambda + \frac{N+1}{2}}$.

**Remark 3.1.** Obviously $\gamma \geq |\alpha|$ and $\gamma > \alpha$ imply $\gamma > 0$.

**Definition 3.1.** Given $R \in [0, 1]$ and $\mu \in \mathbb{R}$, let $Q_\mu = Q_\mu(R)$ be defined by: $Q_\mu = 2^{-\mu}$ if $\mu \leq 0$ and

$$Q_\mu = \left( \frac{1 + R}{1 - R} \right)^\mu \quad \text{if } \mu \geq 0.$$

**Definition 3.2.** Given $a \in B_N$, let $R_a = R \frac{1 - |a|^2}{1 + |R||a|}$. Let $V_a$ denote the volume of the open ball $B(a, R_a) = \{ x \in B_N : |x - a| < R_a \}$.

**Remark 3.2.** Thus, through Lemma 6.1 from the last section:

(3.2) \[ (1 - |a|^2)^\mu \leq Q_\mu (1 - |x|^2)^\mu \quad \forall x \in B(a, R_a) \quad \forall a \in B_N \]

**Proof.** In order to demonstrate Theorem 3.2, let be given $a \in B_N$ and $f_a$ defined by

$$f_a(x) = \frac{1}{(1 - \langle x, a \rangle)^A} \quad \forall x \in B_N,$$

with $A = N + 1 + 2\beta$. The subharmonicity of the function $f_a$ follows from Lemma 6.4 since $A \geq 0$. We next show that $f_a \in \mathcal{Y}_{\alpha, \beta, \gamma}$. For any $b \in B_N$, the following holds:

$$J_b(f_a) = (1 - |b|^2)\alpha \int_{B_N} (1 - |x|^2)^\beta f_a(x) (1 - |\Phi_b(x)|^2)^\gamma \, dx$$

$$= (1 - |b|^2)^{\alpha + \gamma} \int_{B_N} (1 - \langle x, a \rangle)^A (1 - \langle x, b \rangle)^{2\gamma} \, dx$$

Lemma 6.3 now applies with $A = N + 1 + 2\beta > 0$, $P = 2\gamma > 0$ and $T = \beta + \gamma > -1$ which verify $A + P = N + 1 + 2\beta + 2\gamma = N + 1 + 2T > N + 1 - 2 \geq 1$. The choice $\tau = \alpha + \gamma$ is allowed since $\alpha + \gamma \geq 0$, $\alpha + \gamma \leq \frac{N+1}{2} + \beta + \gamma = \frac{A + P}{2}$, $\alpha + \gamma < \gamma + \gamma = P$ and $\frac{P - A}{2} = \gamma - \frac{N+1}{2} - \beta < \gamma + \alpha$. For all $a \in B_N$ and $b \in B_N$, this leads to:

$$J_b(f_a) \leq (1 - |b|^2)^{\alpha + \tau} \frac{K}{(1 - |a|^2)^{\frac{A + P}{2} - \tau}} = \frac{K}{(1 - |a|^2)^{\frac{A + P}{2} - \tau}}$$

where the constant $K$ stems from Lemma 6.5. Thus

$$M_{\mathcal{Y}_{\alpha, \beta, \gamma}}(f_a) = \sup_{b \in B_N} J_b(f_a) \leq \frac{K}{(1 - |a|^2)^{\frac{A + P}{2} - \tau}}$$

and

$$M_{\mathcal{Y}_{\alpha, \beta, \gamma}}(gu) \leq \frac{C K}{(1 - |a|^2)^{\frac{A + P}{2} - \tau}} \quad \forall a \in B_N.$$
Given \( R \in ]0, 1[ \), the subharmonicity of \( g \) provides:

\[
V_a g(a) \leq \int_{B(a, R)} g(x) \, dx \quad \forall a \in B_N
\]

with \( V_a \) and \( B(a, R_a) \) as in Definition 3.2. Now \( g(x) \geq 0 \ \forall x \in B_N \) hence the estimation (3.2) where \( \mu = \lambda - \frac{N+1}{2} \) leads to:

\[
(1 - |a|^2)^{\lambda - \frac{N+1}{2}} g(a) \leq \frac{N (1 + R) \sigma_N R^N}{\sigma_N R} Q_{\lambda - \frac{N+1}{2}} \int_{B(a, R_a)} (1 - |x|^2)^{\lambda - \frac{N+1}{2}} g(x) \, dx.
\]

Moreover \( \gamma + \beta + \frac{N+1}{2} \geq \alpha + \gamma \geq 0 \) hence, again through Lemma 6.1.

\[
1 \leq \left( 2 \frac{1 - |a|^2}{1 - \langle x, a \rangle} \right)^{\gamma + \beta + \frac{N+1}{2}}
\]

and

\[
1 \leq \left( 2 \frac{1 + R}{1 - R} \frac{1 - |x|^2}{1 - \langle x, a \rangle} \right)^{\gamma + \beta + \frac{N+1}{2}}.
\]

Thus

\[
(1 - |a|^2)^{\lambda - \frac{N+1}{2}} g(a) \leq Q (1 - |a|^2)^{\alpha + \beta + \frac{N+1}{2} - \alpha} = \alpha + \gamma + \beta + \frac{N+1}{2} - \alpha = \alpha + \gamma + \frac{A + P}{2} - \tau,
\]

whence

\[
(1 - |a|^2)^{\lambda + \frac{N+1}{2}} g(a) \leq Q (1 - |a|^2)^{A + P - \tau} (1 - |a|^2)^{\alpha + \gamma} \int_{B(a, R_a)} \frac{(1 - |x|^2)^{\lambda + \gamma + \beta}}{(1 - \langle x, a \rangle)^{2\gamma + A}} f_a(x) \, g(x) \, dx \leq
\]

\[
Q (1 - |a|^2)^{A + P - \tau} (1 - |a|^2)^{\alpha} \int_{B_N} (1 - |x|^2)^{\lambda + \beta + \frac{N+1}{2}} f_a(x) \, g(x) \, (1 - |\Phi_a(x)|^2)^{\gamma} \, dx
\]

because of \( g \geq 0 \) on \( B_N \). Finally

\[
(1 - |a|^2)^{\lambda + \frac{N+1}{2}} g(a) \leq Q (1 - |a|^2)^{A + P - \tau} M_{\alpha, \beta, \lambda, \gamma} (g f_a) \leq Q C K
\]

for all \( a \in B_N \).

**Theorem 3.3.** Given \( \alpha \geq \frac{1}{2} \), \( \beta > -1 - \frac{N}{2} \), \( \gamma > \frac{N}{2} \) and \( \lambda \in \mathbb{R} \), let \( g : B_N \to [0, +\infty[ \) be a subharmonic function satisfying (3.1). Then \( g \in \mathcal{A}_{\lambda+\frac{N}{2}-1+\alpha} \).

**Remark 3.3.** Theorem 3.2 did not include the case where \( \beta \in [-1 - \frac{N}{2}, -\frac{N+1}{2}] \). Even when \( \beta \in ]-\frac{N+1}{2}, -\frac{N}{2}[ \), the interval \( ]-\frac{N+1}{2}, -\beta, \frac{N+1}{2} + \beta \) did not contain the value \( \alpha = \frac{1}{2} \).

**Proof.** In order to establish Theorem 3.3, let \( H = \frac{N}{2} + 1 + \beta \) and let \( u \) be defined by

\[
u(x) = \frac{1}{(1 - |x|^2)^H} \quad \forall x \in B_N.
\]

This function \( u \) is subharmonic in \( B_N \) since \( H \geq 0 \), according to Lemma 6.4. It moreover belongs to \( \mathcal{V}_{\alpha, \beta, \gamma} \) since:

\[
(1 - |a|^2)^{\alpha} \int_{B_N} (1 - |x|^2)^{\beta} u(x) (1 - |\Phi_a(x)|^2)^{\gamma} \, dx = (1 - |a|^2)^{\alpha + \gamma} \int_{B_N} (1 - |x|^2)^{\beta - H + \gamma} \, dx
\]
and this integral is equal to $K_\alpha (1 - |a|^2)^{-\gamma - \frac{1}{2}}$ for all $a \in B_N$ according to Lemma 6.2 applied with $A = 2\gamma > N \geq 2$ and $T = \beta - H + \gamma = -\frac{N}{2} - 1 + \gamma > -1$ which fulfill: $N + 2(T + 1) = A$. Finally

$$M_{Y_{\alpha,\beta,\gamma}}(u) = K_0 \sup_{a \in B_N} (1 - |a|^2)^{\alpha - \frac{1}{2}} = K_0.$$  

Whence $M_{Y_{\alpha,\beta,\lambda,\gamma}}(gu) \leq C K_0$.

Let $R \in ]0,1]$ be fixed. For any $a \in B_N$, it follows from the estimation (3.2), together with the subharmonicity and positivity of $g$, that:

$$(1 - |a|^2)^{\lambda - \frac{N}{2} - 1} V_a g(a) \leq Q \lambda N - \frac{N}{2} - 1 \int_{B(a,Ra)} (1 - |x|^2)^{\lambda - \frac{N}{2} - 1} g(x) \, dx$$

$$(1 - |a|^2)^{\lambda - \frac{N}{2} - 1 + N} g(a) \leq \frac{N (1 + R)^N}{\sigma_N R^N} Q \lambda N - \frac{N}{2} - 1 \int_{B(a,Ra)} (1 - |x|^2)^{\lambda + \beta} u(x) g(x) \, dx$$

Again Lemma 6.1 provides:

$$(1 - |a|^2)^{\lambda + \frac{N}{2} - 1} g(a) \leq Q' \int_{B(a,Ra)} \left( 1 - |a|^2 \right)^{\gamma} \left( \frac{1 - |x|^2}{1 - \langle x, a \rangle} \right)^\gamma (1 - |x|^2)^{\lambda + \beta} u(x) g(x) \, dx$$

with a constant $Q' > 0$ independent from $a$ and $x$. In other words:

$$(1 - |a|^2)^{\lambda + \frac{N}{2} - 1 + \alpha} g(a) \leq Q'(1 - |a|^2)^{\alpha} \int_{B(a,Ra)} (1 - |x|^2)^{\lambda + \beta} u(x) g(x) (1 - |\Phi_a(x)|^2)^{\gamma} \, dx$$

As $u(x) g(x) \geq 0 \forall x \in B_N$, the above $\int_{B(a,Ra)} \ldots$ is majorized by $\int_{B_N} \ldots$, so that:

$$(1 - |a|^2)^{\lambda + \frac{N}{2} - 1 + \alpha} g(a) \leq Q' M_{Y_{\alpha,\beta,\lambda,\gamma}}(gu) \leq Q' C K_0$$

for each $a \in B_N$. \hfill \blacksquare

**Theorem 3.4.** Given $\alpha \geq 0$, $\beta \geq -\frac{N+1}{2}$, $\gamma > \frac{N-1}{2}$ and $\lambda \in \mathbb{R}$, let $g : B_N \to [0, +\infty]$ be a subharmonic function satisfying (3.1). Then $g \in \mathcal{X}_{\lambda + \frac{N}{2} + 1 + \alpha}$.

**Proof.** Given $J = \frac{N+1}{2} + \beta$, let $u$ denote the function defined by:

$$u(x) = (1 - |x|^2)^{-J} \quad \forall x \in B_N.$$  

Lemma 6.4 implies the subharmonicity of $u$ since $J \geq 0$. Besides that:

$$(1 - |a|^2)^{\alpha} \int_{B_N} (1 - |x|^2)^{\beta} u(x) (1 - |\Phi_a(x)|^2)^{\gamma} \, dx = (1 - |a|^2)^{\alpha + \gamma} \int_{B_N} \frac{(1 - |x|^2)^{-\frac{N+1}{2}}}{(1 - \langle x, a \rangle)^{2\gamma}} \, dx$$

$$= (1 - |a|^2)^{\alpha + \gamma} \frac{K'}{(1 - |a|^2)^{\gamma}}$$

$$= K' (1 - |a|^2)^{\alpha} \quad \forall a \in B_N$$

according to Lemma 6.3 applied here with $P = \tau = 0$, $A = 2\gamma > 0$ and $T = \gamma - \frac{N+1}{2} > -1$ (the condition $1 \leq A = N + 1 + 2T$ is fulfilled by these parameters, because of $\gamma > 1/2$). Thus $M_{Y_{\alpha,\beta,\gamma}}(u) = K'$ since $\alpha \geq 0$.

With $R \in ]0,1]$ and $a \in B_N$ fixed, it now follows from (3.3) and Lemma 6.1 that:

$$(1 - |a|^2)^{\lambda + \frac{N-1}{2}} g(a) \leq Q'' \int_{B(a,Ra)} \left( 1 - |a|^2 \right)^{\gamma} \left( \frac{1 - |x|^2}{1 - \langle x, a \rangle} \right)^\gamma (1 - |x|^2)^{\lambda + \beta} u(x) g(x) \, dx$$

$$\leq Q'' \int_{B_N} (1 - |x|^2)^{\lambda + \beta} u(x) g(x) (1 - |\Phi_a(x)|^2)^{\gamma} \, dx$$

AJMAA, Vol. 9, No. 1, Art. 9, pp. 1-16, 2012
since \( g \geq 0 \) on \( B_N \). Here \( Q'' \) denotes a constant depending only on \( R, N, \lambda \) and \( \gamma \). Finally
\[
(1 - |a|^2)^{\lambda + \frac{N-1}{2} + \alpha} g(a) \leq Q'' M_{\gamma, \alpha + \beta, \gamma} (gu) \leq Q'' C K'
\]
for all \( a \in B_N \). \( \blacksquare \)

**Definition 3.3.** Let \( S_N \) denote the unit sphere of \( \mathbb{R}^N \) and \( d\sigma \) the area element on \( S_N \). Let \( \sigma_N \) denote the area of \( S_N \), for information: \( \sigma_N = \frac{2\pi^{N/2}}{\Gamma(N/2)} \) (see [2] p.29).

**Theorem 3.5.** Let \( \varepsilon \in [0, \sigma_N] \) and \( E \) a measurable subset of \( B_N \) with \( \sigma(E) \geq \varepsilon \forall r \in [0, 1] \), where \( E_r = \{ \eta \in S_N : r \eta \in E \} \). Let \( g : B_N \to [0, +\infty] \) be a measurable function.

(i) Given parameters \( \alpha \geq \frac{1}{2}, \beta \geq -1 - \frac{N}{2}, \gamma > \frac{N}{2} \) and \( \lambda \in \mathbb{R} \), if the function \( g \) satisfies
\[
g(x) \geq (1 - |x|^2)^{-\lambda - \gamma + \frac{N}{2}} \quad \forall x \in E,
\]
then there does NOT exist any constant \( C > 0 \) such that
\[
M_{\gamma, \alpha + \beta, \gamma} (gu) \leq C M_{\gamma, \alpha, \beta, \gamma} (u) \quad \forall u \in S\mathcal{Y}^+_{\alpha, \beta, \gamma}.
\]

(ii) The same conclusion holds when \( g(x) \geq (1 - |x|^2)^{-\lambda - \gamma - 1} \forall x \in E \), with \( \alpha, \beta, \gamma \) and \( \lambda \) as in Theorem 3.2.

(iii) The same conclusion holds when \( g(x) \geq (1 - |x|^2)^{-\lambda - \gamma + \frac{N}{2} - \frac{1}{2}} \forall x \in E \), with parameters fulfilling: \( \alpha \geq 0, \beta \geq -\frac{N+1}{2}, \gamma > \frac{N-1}{2} \) and \( \lambda \in \mathbb{R} \).

**Proof.** Each of these three results is to be established ab absurdo: let us suppose on the contrary that (3.1) holds.

**Proof of (i)** Let \( H = \frac{N}{2} + \beta + 1 \geq 0 \) and \( v \) the function defined on \( B_N \) by \( v(x) = (1 - |x|^2)^{-H} \). Thus \( v \in S\mathcal{Y}^+_{\alpha, \beta, \gamma} \) as it was shown in the proof of Theorem 3.3. If there existed some \( C > 0 \) such that \( M_{\gamma, \alpha + \beta, \gamma} (gu) \leq C M_{\gamma, \alpha, \beta, \gamma} (u) \) for all functions \( u \in S\mathcal{Y}^+_{\alpha, \beta, \gamma} \), it would apply in particular to the function \( v \) and we should have \( M_{\gamma, \alpha + \beta, \gamma} (gv) < +\infty \).

Having fixed some \( a \in B_N \), the following integrals should then be finite too:
\[
\int_{B_N} (1 - |x|^2)^{\beta + \lambda} g(x) (1 - |\Phi_a(x)|^2)^\gamma \, dx \geq (1 - |a|^2)^{\gamma} \int_E (1 - |x|^2)^{\beta + \lambda - (\lambda + \gamma - \frac{N}{2}) - H + \gamma} \, dx
\]
\[
= (1 - |a|^2)^{\gamma} \int_E (1 - |x|^2)^{\beta + \frac{N}{2} - H} \, dx
\]
\[
> \frac{(1 - |a|^2)^{\gamma}}{4^\gamma} \int_E (1 - |x|^2)^{-1} \, dx
\]
since \( 0 < 1 - (x, a) < 2 \) for all \( a \in B_N \) and \( x \in B_N \). But the last integral diverges since
\[
\int_E \frac{dx}{1 - |x|^2} \geq \varepsilon \int_0^1 \frac{r^{N-1} \, dr}{1 - r^2}
\]
\[
> \frac{\varepsilon}{2} \int_0^1 r^{N-1} \, dr = +\infty
\]
with \( r = |x| \) and \( dx = r^{N-1} \, dr \, d\sigma \). Now the contradiction follows.

**Proof of (ii)** If, for some constant \( C \), the estimation \( M_{\gamma, \alpha + \beta, \gamma} (gu) \leq C M_{\gamma, \alpha, \beta, \gamma} (u) \) was valid for all \( u \in S\mathcal{Y}^+_{\alpha, \beta, \gamma} \), then it would hold in particular with \( u = f_a \in S\mathcal{Y}^+_{\alpha, \beta, \gamma} \) defined as in the
proof of Theorem 3.2. \( f_a(x) = (1 - \langle x, a \rangle)^{-N - 1 - 2\beta} \forall x \in B_N \) (with \( a \) fixed in \( B_N \)). Thus we should have \( M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(g f_a) < +\infty \) and the following integrals should be finite too:

\[
\int_{B_N} (1 - |x|^2)^{\beta + \lambda} g(x) f_a(x) (1 - |\Phi_a(x)|^2)^\gamma \, dx \geq (1 - |a|^2)^\gamma \int_{E} \frac{(1 - |x|^2)^{\beta + \lambda - (\lambda + \beta + \gamma + 1) + \gamma}}{(1 - \langle x, a \rangle)^{N + 1 + 2\beta + 2\gamma}} \, dx \geq \frac{(1 - |a|^2)^\gamma}{2^{N + 1 + 2\beta + 2\gamma}} \int_{E} \frac{dx}{1 - |x|^2}
\]

The last integral diverges, hence a contradiction.

Proof of (iii). The function \( v \) defined by \( v(x) = (1 - |x|^2)^{-J} \forall x \in B_N \), with \( J = \frac{N + 1}{2} + \beta \), belongs to \( \mathcal{S} \mathcal{Y}^{+}_{\alpha,\beta,\gamma} \) (see the proof of Theorem 3.4). Reasoning ab absurdo as in both previous cases, we should have \( M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(g v) < +\infty \), hence the finiteness of the following integral (with \( a \in B_N \) fixed):

\[
\int_{E} \frac{(1 - |x|^2)^{\beta + \lambda - (\lambda + \gamma - \frac{N}{2} + \frac{1}{2}) - J + \gamma}}{(1 - \langle x, a \rangle)^{2\gamma}} \, dx = \int_{E} \frac{(1 - |x|^2)^{\beta + \frac{N}{4} - \frac{1}{2} - J}}{(1 - \langle x, a \rangle)^{2\gamma}} \, dx \geq \frac{1}{4^\gamma} \int_{E} \frac{dx}{1 - |x|^2}
\]

which diverges, thus a contradiction arises.

4. A situation where the products \( gu \) have a Bloch–type growth.

Given \( R \in ]0, 1[ \) and \( g \in \mathcal{X} \), with \( M_{\mathcal{X}}(g) \geq 0 \), we already know from Proposition 3.1 of [6] and Corollary 3.2 of [6] that:

\[
M_{\mathcal{X}_{\alpha,\beta,\gamma} + \alpha + \beta + N}(gu) \leq \frac{N}{\sigma_N} \frac{(1 + R)^{N - \gamma}}{R^N (1 - R)^{2\gamma}} Q_\beta M_{\mathcal{X}_\lambda}(g) M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \quad \forall u \in \mathcal{S} \mathcal{Y}^{+}_{\alpha,\beta,\gamma}
\]

(for \( \sigma_N \) and \( Q_\beta \), see the notations of Definitions 3.2 and 3.3).

**Theorem 4.1.** With \( \alpha, \beta, \gamma \) and \( \lambda \) as in Theorem 3.2 let \( g \) be a non-negative subharmonic function defined on \( B_N \), satisfying:

\[
\exists C' > 0 \quad M_{\mathcal{X}_{\alpha,\beta,\gamma} + \alpha + \beta + N}(gu) \leq C' M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \quad \forall u \in \mathcal{S} \mathcal{Y}^{+}_{\alpha,\beta,\gamma}.
\]

Then \( g \in \mathcal{X}_{\lambda,\frac{N}{2} - 1} \).

**Remark 4.1.** This case cannot be deduced directly from Theorem 3.2 of the previous section and Proposition 3.1 of [6] which asserts: \( \mathcal{S} \mathcal{Y}^{+}_{\alpha,\beta,\gamma} \subset \mathcal{S} \mathcal{X}^{+}_{\alpha,\beta,\gamma} \) together with

\[
M_{\mathcal{X}_{\alpha,\beta,\gamma} + \alpha + \beta + N}(v) \leq C'' M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(v)
\]

for some constant \( C'' \) independent from \( v \in \mathcal{S} \mathcal{Y}^{+}_{\alpha,\beta,\gamma} \). This result cannot be used here (with \( v = gu \) and \( \beta \) replaced by \( \beta + \lambda \)) because the subharmonicity of \( g \) and \( u \) does not compulsorily imply the subharmonicity of \( gu \), as the following counterexample points out: with \( g \) and \( u \) defined by \( g(x) = 1 + x_1 \geq 0 \) and \( u(x) = 1 - x_1 \geq 0 \forall x = (x_1, x_2, \ldots, x_N) \in B_N \), we have

\[
(\Delta u)(x) = (\Delta g)(x) = 0 \geq 0
\]

but

\[
\Delta (gu)(x) = \frac{\partial^2}{\partial x_1^2} (1 - x_1^2) = -2 < 0 \quad \forall x \in B_N.
\]

However this function \( u \) belongs to \( \mathcal{Y}_{\alpha,\beta,\gamma} \) since

\[
u(x) \leq 2 \leq 2^{N + 2 + 2\beta} f_a(x) \quad \forall x \in B_N
\]

with \( f_a \in \mathcal{Y}_{\alpha,\beta,\gamma} \) as in the proof of Theorem 3.2. The previous majoration merely follows from \( 0 < 1 - \langle x, a \rangle < 2 \), hence \( \frac{1}{2} < \frac{1}{1 - \langle x, a \rangle} \), thus \( \left( \frac{1}{2} \right)^{N + 1 + 2\beta} < f_a(x) \quad \forall x \in B_N \).
Proof. Having fixed $R \in [0, 1]$, the proof of Theorem 4.1 will make use of the same notations $(V_a, B(a, R), f_a, Q_a, A, P$ and $\tau)$ as the proof of Theorem 3.2. We have for any $a \in B_N$:

$$
(1 - |a|^2)^{\lambda + \frac{N-1}{2} + \alpha} V_a g(a) \leq Q_{\lambda + \frac{N-1}{2} + \alpha} \int_{B(a, R)} (1 - |x|^2)^{\lambda + \frac{N-1}{2} + \alpha} \ g(x) \ dx
$$

$$
\leq L (1 - |a|^2)^{\beta + \frac{N+1}{2}} \int_{B(a, R)} \frac{(1 - |x|^2)^{\lambda + \frac{N-1}{2} + \alpha + \beta + \frac{N+1}{2}}}{(1 - \langle x, a \rangle)^{N+2+2\beta}} g(x) \ dx
$$

where the constant $L$ depends only on $N$, $R$, $\beta$ and $\lambda + \alpha$. The above integral is merely

$$
\int_{B(a, R)} (1 - |x|^2)^{\lambda + \alpha + \beta + N} f_a(x) \ g(x) \ dx \leq M_{\lambda + \alpha + \beta + N} (f_a \ g) V_a
$$

$$
\leq C' M_{\lambda, \alpha, \beta, \gamma} (f_a) V_a
$$

$$
\leq C' K' V_a
$$

$$
\leq \left(1 - |a|^2\right)^{\beta + \frac{N+1}{2} - \alpha}
$$

with $K$ from Lemma 6.3. Finally: $(1 - |a|^2)^{\lambda + \frac{N-1}{2}} g(a) \leq L C' K$ for every $a \in B_N$.

**Theorem 4.2.** Given $\alpha \geq \frac{1}{2}$, $\beta \geq -1 - \frac{N-1}{2}$, $\gamma > \frac{N}{2}$ and $\lambda \in \mathbb{R}$, let $g$ be a subharmonic function (on $B_N$) satisfying (4.1). Then $g \in \mathcal{X}_{\lambda + \frac{N-1}{2} + \alpha}$.

Proof. Let $u \in \mathcal{S}Y_{\alpha, \beta, \gamma}^+$ be defined by $u(x) = (1 - |x|^2)^{-\frac{N}{2} - 1 - \beta}$ as in the proof of Theorem 3.3. With the same notations as in the proof of Theorem 3.2, the subharmonicity of $g$ leads to:

$$
V_a g(a) \leq \int_{B(a, R)} g(x) \ dx = \int_{B(a, R)} (1 - |x|^2)^{\frac{N}{2} + 1 + \beta} g(x) \ u(x) \ dx
$$

$$
= \int_{B(a, R)} (1 - |x|^2)^{\lambda + \alpha + \beta + N} g(x) \ u(x) (1 - |x|^2)^{-\lambda - \alpha - \frac{N}{2} + 1} \ dx
$$

$$
\leq M_{\lambda + \alpha + \beta + N} (u) \int_{B(a, R)} (1 - |x|^2)^{-\lambda - \alpha - \frac{N}{2} + 1} \ dx
$$

$$
\leq C' M_{\lambda, \alpha, \beta, \gamma} (u) \int_{B(a, R)} Q_{\lambda + \alpha + \frac{N}{2} - 1} (1 - |a|^2)^{-\lambda - \alpha - \frac{N}{2} + 1} \ dx
$$

$$
= Q_{\lambda + \frac{N-1}{2} - 1 + \alpha} K' \ g(1 - |a|^2)^{-\lambda - \alpha - \frac{N}{2} + 1} V_a \ \forall a \in B_N.
$$

Thus $M_{\lambda + \frac{N-1}{2} - 1 + \alpha} (g) \leq Q_{\lambda + \frac{N-1}{2} - 1 + \alpha} C' K' \ g$ with $K_0$ from Lemma 6.2.

**Theorem 4.3.** Given $\alpha \geq 0$, $\beta \geq -\frac{N+1}{2}$, $\gamma > \frac{N-1}{2}$ and $\lambda \in \mathbb{R}$, let $g$ be a subharmonic function (on $B_N$) satisfying (4.1). Then $g \in \mathcal{X}_{\lambda + \frac{N-1}{2} + \alpha}$.

Proof. With this choice of parameters, the function $u : x \mapsto (1 - |x|^2)^{-\frac{N+1}{2} - \beta}$ belongs to the set $\mathcal{S}Y_{\alpha, \beta, \gamma}^+$ (see the proof of Theorem 3.4). Now $g$ is subharmonic hence:

$$
V_a g(a) \leq \int_{B(a, R)} (1 - |x|^2)^{\lambda + \alpha + \beta + N} g(x) \ u(x) (1 - |x|^2)^{-\lambda - \alpha - \frac{N}{2} + 1} \ dx
$$

$$
\leq C' M_{\lambda + \alpha + \beta + N} (u) \ g(1 - |a|^2)^{-\lambda - \alpha - \frac{N}{2} + 1} V_a \ \forall a \in B_N.
$$

Finally $M_{\lambda + \frac{N-1}{2} + \alpha} (g) \leq C' K' Q_{\lambda + \frac{N-1}{2} + \alpha}$ with $K'$ from Lemma 6.3.
5. Products $gu$ When $u$ Has a Bloch-Type Growth.

**Proposition 5.1.** Let $\lambda \in \mathbb{R}$, $g \in X_{\lambda}$ and two reals $\alpha \leq \beta$. If $M_{X_{\lambda}}(g) \geq 0$ then

$$M_{X_{\lambda}+\beta}(gu) \leq M_{X_{\lambda}}(g) M_{X_{\alpha}}(u) \quad \forall u \in X_{\alpha}^+.$$  

**Proof.** For any $x \in B_N$, the following holds: $(1 - |x|^2)^\beta \leq (1 - |x|^2)^\alpha$ since $1 - |x|^2 \in [0, 1]$. Now $(1 - |x|^2)^\lambda g(x) u(x) \leq M_{X_{\lambda}}(g) u(x)$ because of $u(x) \geq 0$. This leads to:

$$(1 - |x|^2)^{\beta+\lambda} g(x) u(x) \leq M_{X_{\lambda}}(g) (1 - |x|^2)^\alpha u(x)$$

since $u(x) M_{X_{\lambda}}(g) \geq 0$. The required majorization follows from $M_{X_{\lambda}}(g) \geq 0$. 

**Proposition 5.2.** Given two reals $\alpha \geq \beta \geq 0$, let $g$ denote a function defined on $B_N$ and satisfying:

$$\exists C > 0 \quad \exists \lambda \in \mathbb{R} \quad \text{such that} \quad M_{X_{\beta+\lambda}}(gu) \leq C M_{X_{\alpha}}(u) \quad \forall u \in SX_{\alpha}^+.$$  

Then $g \in X_{\lambda}$ with $M_{X_{\lambda}}(g) \leq C$.

**Proof.** For any $x \in B_N$, we have: $g(x) = (1 - |x|^2)^\beta g(x) u(x)$ where $u(x) = (1 - |x|^2)^{-\beta}$. Obviously $u \in X_{\alpha}$ with $M_{X_{\alpha}}(u) \leq 1$. Moreover $u$ is subharmonic in $B_N$ since $\beta \geq 0$, according to Lemma [6.4] in the next section. Thus:

$$(1 - |x|^2)^\beta g(x) \leq M_{X_{\beta+\lambda}}(gu) \leq C$$

for each $x \in B_N$. 

**Corollary 5.3.** Given $\lambda \in \mathbb{R}$ and $\alpha \geq 0$, a function $g : B_N \to [0, +\infty[$ belongs to $X_{\lambda}$ if and only if:

$$\exists C > 0 \quad \text{such that} \quad M_{X_{\beta+\lambda}}(gu) \leq C M_{X_{\alpha}}(u) \quad \forall u \in SX_{\alpha}^+.$$  

In particular $g$ is majorized on $B_N$ if and only if:

$$\exists C > 0 \quad \text{such that} \quad M_{X_{\beta}}(g) \leq C M_{X_{\alpha}}(u) \quad \forall u \in SX_{\alpha}^+.$$  

**Proposition 5.4.** Given $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma \geq 0$ and $\lambda \in \mathbb{R}$, let $g \in \mathcal{Y}_{\alpha,\beta,\gamma}$. Then

$$M_{Y_{\alpha,\beta+\lambda\gamma}}(gu) \leq M_{Y_{\alpha,\beta+\lambda\gamma}}(g) M_{X_{\lambda}}(u) \quad \forall u \in X_{\lambda} \quad \text{such that} \quad M_{X_{\lambda}}(u) \geq 0.$$  

**Proof.** The hypothesis $g(x) \geq 0$ implies:

$$(1 - |x|^2)^{\beta+\lambda} u(x) g(x) \leq M_{X_{\lambda}}(u) (1 - |x|^2)^\beta g(x) \quad \forall x \in B_N.$$  

Multiplying by $(1 - |x|^2)^{\alpha}(1 - |\Phi_{\alpha}(x)|^2)^\gamma \geq 0$ and integrating over $B_N$ do not modify the inequality. But evaluating the sup bound requires that $M_{X_{\lambda}}(u) \geq 0$. 

**Proposition 5.5.** Given $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma \geq 0$ and $\lambda \geq 0$, let $g$ be a function defined on $B_N$ satisfying:

$$\exists C > 0 \quad \text{such that} \quad M_{Y_{\alpha,\beta+\lambda\gamma}}(gu) \leq C M_{X_{\lambda}}(u) \quad \forall u \in SX_{\lambda}^+.$$  

Then $g \in \mathcal{Y}_{\alpha,\beta,\gamma}$. 

**Proof.** Let $u \in X_{\lambda}$ be given by: $u(x) = (1 - |x|^2)^{-\lambda} \forall x \in B_N$. Its subharmonicity follows from $\lambda \geq 0$ (see Lemma [6.4]). Then:

$$(1 - |x|^2)^\beta g(x) = (1 - |x|^2)^{\beta+\lambda} g(x) u(x) \quad \forall x \in B_N.$$  

Hence $M_{Y_{\alpha,\beta\gamma}}(g) = M_{Y_{\alpha,\beta+\lambda\gamma}}(gu) \leq C$ since $M_{X_{\lambda}}(u) = 1$.
6. ANNEX: SOME TECHNICAL AUXILIARY RESULTS.

Lemma 6.1. Given $a \in B_N$ and $R \in [0, 1]$, let $R_a$ and $B(a, R_a)$ be as in Definition 3.2. Then the following holds for any $x \in B(a, R_a)$:

$$
\frac{1}{2} < \frac{1 - \langle x, a \rangle}{1 - |x|^2} < 2 \frac{1 + R}{1 - R}
$$

and

$$
\frac{1}{2} < \frac{1 - \langle x, a \rangle}{1 - |a|^2} < 2
$$

Moreover the volume $V_a$ of $B(a, R_a)$ satisfies:

$$
V_a = \frac{\sigma_N}{N} R_a^N \geq \frac{\sigma_N}{N} \left( \frac{R}{1 + R} \right)^N (1 - |a|^2)^N
$$

with $\sigma_N$ as in Definition 3.3.

Proof. Cauchy–Schwarz inequality leads to:

$$
1 - \langle x, a \rangle \geq 1 - |x| = \frac{1 - |x|^2}{1 + |x|} > \frac{1 - |x|^2}{2}.
$$

See [6] and [7] for the other results.

Lemma 6.2. Given $A > 0$ and $T > -1$, let

$$
I_{A,T}(a) = \int_{B_N} \frac{(1 - |x|^2)^T}{(1 - \langle x, a \rangle)^A} dx \quad \forall a \in B_N.
$$

If $2 \leq A \leq N + 2(T + 1)$ then

$$
I_{A,T}(a) \leq K_0 \left( \frac{1}{1 - |a|^2} \right)^{\frac{A+1}{2}} \quad \forall a \in B_N
$$

where

$$
K_0 = \frac{\Gamma(T + 1)}{\Gamma(\frac{A}{2})} \pi^{N/2}.
$$

If $A = N + 2(T + 1)$, then equality holds in the above formula.

Proof. Without any restriction, it may be assumed that $a = (|a|, 0, 0, \ldots, 0)$. Polar coordinates in $\mathbb{R}^N$ provide $x_1 = r \cos \theta_1$ with $r = |x|, \theta_1 \in [0, \pi]$ if $N \geq 3$ and $\theta_1 \in [0, 2\pi]$ if $N = 2$. Let $d\sigma^{(N)}$ be the area element on the unit sphere $S_N$ of $\mathbb{R}^N$. Now $d\sigma^{(N)} = (\sin \theta_1)^{N-2} d\theta_1 d\sigma^{(N-1)}$ (polar formulas are more detailed in [11] p.15), thus

$$
I_{A,T}(a) = \int_{B_N} \frac{(1 - r^2)^T r^{N-1} dr d\sigma^{(N)}}{(1 - |a| r \cos \theta_1)^A}
$$

(6.1)

$$
= \sigma_{N-1} \int_0^1 \int_0^\pi \left( 1 - r^2 \right)^T r^{N-1} (\sin \theta_1)^{N-2} d\theta_1 dr
$$

when $N \geq 3$ (the case $N = 2$ will be studied later). For $\sigma_N$, see Definition 3.3. Let $s = r \cos \theta_1$. From the known expansion

$$
\frac{1}{(1 - |a| s)^A} = \sum_{\ell \in \mathbb{N}} \frac{\Gamma(\ell + A)}{\ell! \Gamma(A)} (|a| s)^\ell
$$
we obtain:
\[ I_{A,T}(a) = \sigma_{N-1} \sum_{k \in \mathbb{N}} \frac{\Gamma(2k + A)}{\Gamma(2k + 1) \Gamma(A)} |a|^2 |\Gamma(k + \frac{1}{2}) \Gamma(k + \frac{A+1}{2})| a |^{2k} \frac{\Gamma(N+1)}{2 \Gamma(N+1+k+T+1)}. \]

With \( t = r \sin \theta_1 \), this double integral turns into \( \int_H s^t t^{N-2} (1 - t^2) \, ds \, dt \) where
\[ H = \{(s, t) \in \mathbb{R}^2 : t \geq 0, s^2 + t^2 < 1 \}. \]

This integral has been computed in [6], whence
\[ I_{A,T}(a) = \sigma_{N-1} \sum_{k \in \mathbb{N}} \frac{2^{A-1} \Gamma(\frac{N-1}{2}) \Gamma(T+1) \Gamma(\frac{A+1}{2})}{2 \Gamma(A)} |a|^{2k}. \]

The function \( \Gamma \) is increasing on \([1, +\infty[ \) and \( 1 \leq k + A \leq \frac{N}{2} + k + T + 1 \) thus
\[ \Gamma(k + A) \leq \Gamma(\frac{N}{2} + k + T + 1), \]

with equality in the case \( A = N + 2(T + 1) \). It follows that:
\[ I_{A,T}(a) \leq \sigma_{N-1} \frac{2^{A-1} \Gamma(\frac{A+1}{2})}{2 \Gamma(A)} \sum_{k \in \mathbb{N}} \frac{\Gamma(k + \frac{A+1}{2})}{\Gamma(\frac{N}{2} + k + T + 1)}. \]

Now \( \frac{1}{2} \sigma_{N-1} \Gamma(\frac{N-1}{2}) = \pi^{(N-1)/2} \) and
\[ \frac{2^{A-1} \Gamma(\frac{A+1}{2})}{\Gamma(A)} = \sqrt{\frac{\pi}{2}} \]

through duplication formula applied with \( 2z = A \). Hence the conclusion follows in the case \( N \geq 3 \). When \( N = 2 \), we have
\[ I_{A,T}(a) = \int_0^1 \int_0^{2\pi} \frac{(1 - r^2)^T r \, d\theta_1 \, dr}{(1 - |a| r \cos \theta_1)^A} \]

but the inner integral is equal to
\[ 2 \int_0^\pi \frac{(1 - r^2)^T r \, d\theta_1}{(1 - |a| r \cos \theta_1)^A}, \]

so that the above formulas, from [6,1] on, all still hold, since \( \sigma_1 = 2 \). 

**Lemma 6.3.** Given \( A > 0, P > 0 \) and \( T > -1 \) satisfying \( 1 \leq A + P \leq N + 1 + 2T \), let
\[ I_{A,P,T}(a, b) = \int_{B_N} \frac{(1 - |x|^2)^T}{(1 - \langle x, a \rangle)^A (1 - \langle x, b \rangle)^P} \, dx \quad \forall a \in B_N \quad \forall b \in B_N \]

and \( \tau \) a number satisfying both
\[ \frac{P - A}{2} < \tau < P \quad \text{and} \quad 0 \leq \tau \leq \frac{A + P}{2}. \]

Then there exists \( K > 0 \) such that:
\[ I_{A,P,T}(a, b) \leq \frac{K}{(1 - |a|^2)^{\frac{A+P}{2}-\tau} (1 - |b|^2)^\tau} \quad \forall a \in B_N \quad \forall b \in B_N. \]
This result remains valid with $P = 0$ and $\tau = 0$, replacing $K$ by

$$K' = \frac{\Gamma(T + 1)}{\Gamma\left(\frac{A+1}{2}\right)} \frac{\pi^N}{\pi^2}.$$ 

With $P = \tau = 0$, equality holds in the above majoration when moreover $1 \leq A = N + 1 + 2T$.

**Proof.** See Theorem 4.1 of [6] and Proposition 5.1 of [6].

For information $K = 2^{P+A-1} \Gamma(T+1)\Gamma(\frac{P-\tau}{\Gamma(P)}\Gamma(\frac{A-P}{2}+\tau)\pi^{\frac{N-1}{2}}.$

**Lemma 6.4.** Given $A \geq 0$, $P \geq 0$ and $a \in B_N$, let $f$ be the function defined on $B_N$ by

$$f(x) = \frac{1}{(1 - \langle x, a \rangle)^A(1 - |x|^2)^P} \quad \forall x \in B_N.$$

Then $f$ is subharmonic in $B_N$.

**Proof.** For any $j \in \{1, 2, ..., N\}$, the following holds $\forall x \in B_N$:

$$\frac{\partial f_a}{\partial x_j}(x) = a_j A (1 - \langle x, a \rangle)^{-A-1}(1 - |x|^2)^{-P} + 2P x_j (1 - \langle x, a \rangle)^{-A}(1 - |x|^2)^{-P-1}$$

$$\frac{\partial^2 f_a}{\partial x^2_j}(x) = A(A+1)a_j^2(1 - \langle x, a \rangle)^{-A-2}(1 - |x|^2)^{-P} +$$

$$4AP |x| a_j (1 - \langle x, a \rangle)^{-A-1}(1 - |x|^2)^{-P-1} + 2P (1 - \langle x, a \rangle)^{-A}(1 - |x|^2)^{-P-1} +$$

$$4P(P+1) a_j^2 (1 - \langle x, a \rangle)^{-A}(1 - |x|^2)^{-P-2}.$$

Hence

$$(\Delta f)(x) = \frac{|a|^2 A(A+1)}{(1 - \langle x, a \rangle)^A(1 - |x|^2)^P} + \frac{4AP(x, a)}{(1 - \langle x, a \rangle)^{A+1}(1 - |x|^2)^{P+1}} +$$

$$+ \frac{2PN}{(1 - \langle x, a \rangle)^A(1 - |x|^2)^{P+1}} + \frac{4P^2 |x|^2}{(1 - \langle x, a \rangle)^A(1 - |x|^2)^{P+2}}$$

$$(\Delta f)(x) \geq (1 - \langle x, a \rangle)^{-A}(1 - |x|^2)^{-P} \left[ \frac{A|a|^2}{(1 - \langle x, a \rangle)^2} - \frac{4AP|a||x|}{(1 - \langle x, a \rangle)(1 - |x|^2)^2} + \frac{4P^2 |x|^2}{(1 - |x|^2)^2} \right]$$

$$= (1 - \langle x, a \rangle)^{-A}(1 - |x|^2)^{-P} \left( \frac{A|a|}{1 - \langle x, a \rangle} - \frac{2P|x|}{1 - |x|^2} \right)^2 \geq 0.$$

The subharmonicity of $f$ follows. □

**REFERENCES**


