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GROWTH AND PRODUCTS OF SUBHARMONIC FUNCTIONS IN THE UNIT BALL

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ABSTRACT. The purpose of this paper is to link informations on the application $u \mapsto gu$ with some growth conditions on the functions u and g subharmonic in the unit ball of $\mathbb{R}^N$. Two kinds of growth are considered: the Bloch–type growth and growth conditions expressed through integrals involving involutions of the unit ball.

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1. INTRODUCTION

The subharmonic functions under study in this article are defined on the Euclidean open unit ball $B_N \subset \mathbb{R}^N$ (with $N \in \mathbb{N}$, $N \geq 2$). They grow according to two different patterns. For instance, such a function u is said to have a Bloch–type growth if:

$$\exists \lambda \in \mathbb{R} \text{ such that } M_{\mathcal{X}_\lambda}(u) := \sup_{x \in B_N} (1-|x|^2)^\lambda \, u(x) < +\infty \tag{1.1}$$

where $|\,.\,|$ denotes the Euclidean norm in $\mathbb{R}^N$ (the set $\mathcal{X}_\lambda$ will be defined explicitely later). Another kind of growth for u is described through the Bloch–type growth of the function $F_{\beta,\gamma}(u)$ defined on B_N by the following formula, with β and γ fixed reals:

$$(F_{\beta,\gamma}(u))\,(a) = \int_{B_N} (1-|x|^2)^\beta \, u(x)\, (1-|\Phi_a(x)|^2)^\gamma \, dx \qquad \forall a \in B_N \tag{1.2}$$

denoting by Φ_a an involution of B_N which will be explicited in the next section. The function u is said to have a growth of the second kind if:

$$\exists \alpha \in \mathbb{R} \text{ such that } M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) := M_{\mathcal{X}_\alpha}(F_{\beta,\gamma}(u)) < +\infty \tag{1.3}$$

(see Section 2 for the precise definitions of the sets $\mathcal{Y}_{\alpha,\beta,\gamma}$ and $\mathcal{X}_\alpha$).

The purpose of the paper is to study the links between the growth of two such functions u and g and the growth of gu. Given sets $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ of the kind $\mathcal{X}_\lambda$ or $\mathcal{Y}_{\alpha,\beta,\gamma}$, we consider the application:

$$\begin{aligned} \mathcal{E} &\to \mathcal{F} \\ u &\mapsto gu \end{aligned}$$

and investigate the following questions:

– If $g \in \mathcal{G}$, does there exist then a constant $C > 0$ such that

$$M_{\mathcal{F}}(gu) \leq C\, M_{\mathcal{E}}(u) \qquad \forall u \text{ ?}$$

– Does the converse hold?

• In the case $\mathcal{E} = \mathcal{X}_\alpha$, $\mathcal{F} = \mathcal{X}_{\alpha+\lambda}$ and $\mathcal{G} = \mathcal{X}_\lambda$, Propositions 5.1 and 5.2 provide positive answers to both above questions (see Section 5 for the exact statements).

• In the case $\mathcal{E} = \mathcal{X}_\lambda$, $\mathcal{F} = \mathcal{Y}_{\alpha,\beta+\lambda,\gamma}$ and $\mathcal{G} = \mathcal{Y}_{\alpha,\beta,\gamma}$ too, positive answers also hold: see Propositions 5.4 and 5.5 in Section 5 for the precise assumptions.

• For the case $\mathcal{E} = \mathcal{Y}_{\alpha,\beta,\gamma}$ and $\mathcal{F} = \mathcal{X}_{\alpha+\beta+\lambda+N}$, see Section 4:
Theorem 4.1 studies the situation where $\mathcal{G} = \mathcal{X}_{\lambda+\frac{N-1}{2}}$ and the parameters α, β, γ fulfill:

$$-\beta-\frac{N+1}{2} < \alpha \leq \beta+\frac{N+1}{2}, \quad \beta > -\frac{N+1}{2}, \quad \gamma \geq -\alpha, \quad \gamma > \max(\alpha, -1-\beta). \tag{1.4}$$

Theorem 4.2 studies the situation where $\mathcal{G} = \mathcal{X}_{\lambda+\frac{N}{2}-1+\alpha}$ and

$$\alpha \geq \frac{1}{2}, \qquad \beta \geq -1-\frac{N}{2}, \qquad \gamma > \frac{N}{2}\,. \tag{1.5}$$

Theorem 4.3 studies the situation where $\mathcal{G} = \mathcal{X}_{\lambda+\frac{N-1}{2}+\alpha}$ and

$$\alpha \geq 0, \qquad \beta \geq -\frac{N+1}{2}, \qquad \gamma > \frac{N-1}{2}\,. \tag{1.6}$$

• In the case $\mathcal{E} = \mathcal{Y}_{\alpha,\beta,\gamma}$ and $\mathcal{F} = \mathcal{Y}_{\alpha,\lambda+\beta,\gamma}$, Proposition 3.1 brings an affirmative answer to the first question, with $g \in \mathcal{X}_\lambda$. Theorems 3.2, 3.3 and 3.4 provide situations where the converse of Proposition 3.1 partly holds: given a function g, defined on B_N, for which there exists some constant C such that $M_{\mathcal{Y}_{\alpha,\lambda+\beta,\gamma}}(gu) \leq C\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u)\ \forall u$, Theorems 3.2, 3.3 and 3.4 obtain some $\lambda' \geq \lambda$ such that $g \in \mathcal{X}_{\lambda'}$. For positive functions g moreover exceeding $(1-|x|^2)^{-\lambda''}$ on some sector issued from the origin, Theorem 3.5 brings a restriction on λ'':

$$\begin{array}{lll} \text{in situation (1.4) :} & \lambda' = \lambda + \dfrac{N-1}{2} & \lambda'' < \lambda+\beta+\gamma+1 \\ \text{in situation (1.5) :} & \lambda' = \lambda + \dfrac{N}{2} - 1 + \alpha & \lambda'' < \lambda+\gamma-\dfrac{N}{2} \\ \text{in situation (1.6) :} & \lambda' = \lambda + \dfrac{N-1}{2} + \alpha & \lambda'' < \lambda+\gamma-\dfrac{N-1}{2} \end{array}$$

(see Section 3 for all details).

Since the sets $\mathcal{X}_\lambda$ and $\mathcal{Y}_{\alpha,\beta,\gamma}$ do not even have the structure of vector space, it is not surprising that the situation studied here is more delicate than the similar questions involving spaces of holomorphic functions in the unit disk $D \subset \mathbb{C}$, such as the Bloch space $\mathcal{B}_\tau$, the Dirichlet space $\mathcal{D}_\tau$ (a special case of Bergman space when $\tau > 1$) and the space $BMOA_\tau$ for instance. Using the above notations (1.1) and (1.3), with $N = 2$ and $x = z \in D = B_N$, the classical norms on these spaces are denoted as follows:

$$\begin{array}{rcll} f \in \mathcal{B}_\tau & \Longleftrightarrow & ||f||_{\mathcal{B}_\tau} := |f(0)| + M_{\mathcal{X}_\lambda}(|f'|) < +\infty & (\tau > 0) \\ f \in \mathcal{D}_\tau & \Longleftrightarrow & ||f||^2_{\mathcal{D}_\tau} := M_{\mathcal{Y}_{0,\tau-2,0}}(|f|^2) < +\infty & (\tau > 1) \\ f \in BMOA_\tau & \Longleftrightarrow & ||f||^2_{BMOA_\tau} := |f(0)|^2 + M_{\mathcal{Y}_{0,2\tau-2,1}}(|f'|^2) < +\infty & (\tau > 0) \end{array}$$

modifying here the notations (1.2) and (1.3) with $\Phi_a(x)$ now replaced by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$.

Given h an analytic function in D, such linear applications as $f \mapsto hf$ between these spaces have been studied by various authors: see for instance [8], [4] p. 197, [10].

A related question is the continuity (with respect to the above norms) of the operator

$$I_h : f \mapsto I_h(f)$$

defined by:

$$(I_h(f))(z) = \int_0^z h(\zeta)\, f'(\zeta)\, d\zeta \qquad \forall z \in D.$$

Some examples of known results:

• Given $\tau \geq 1$, the continuity of $I_h : \mathcal{B}_\tau \to \mathcal{B}_\tau$ is equivalent to the boundedness of h on D, in other words: $h \in H^\infty$ (proved in [9] p.138).

• If $1 < \mu < \lambda$, then the operator $I_h : BMOA_\mu \to \mathcal{B}_\lambda$ is bounded if and only if $h \in \mathcal{B}_{\lambda-\mu+1}$ (see [8] p.1050).

• Given $1 < \mu < \lambda$, it is proved in [8] p.1059–1060 that $I_h : \mathcal{D}_\mu \to \mathcal{D}_\lambda$ is bounded if and only if $h \in \mathcal{B}_{1+\frac{1}{2}(\lambda-\mu)}$.

Since $|f'|$, $|f|^2$ and $|f'|^2$ are subharmonic functions on B_2 (see [2] p.46), the question naturally occured whether the preceeding results had some kind of an analog for subharmonic functions on B_N ($N \geq 2$) with a growth described by (1.1) or (1.3).

2. VARIOUS RELATIONS BETWEEN THE SETS $\mathcal{X}_\lambda$ AND $\mathcal{Y}_{\alpha,\beta,\gamma}$.

Definition 2.1. Given $\lambda \in \mathbb{R}$, let $\mathcal{X}_\lambda$ be the set of all functions $u : B_N \to [-\infty, +\infty[$ satisfying

$$M_{\mathcal{X}_\lambda}(u) := \sup_{x \in B_N} (1 - |x|^2)^\lambda \, u(x) < +\infty.$$

Let $\mathcal{X}^+_\lambda = \{u \in \mathcal{X}_\lambda \, : \, u(B_N) \subset [0, +\infty[\}$. Let $\mathcal{SX}_\lambda$ denote the subset of all subharmonic $u \in \mathcal{X}_\lambda$. Finally, let $\mathcal{SX}^+_\lambda = \mathcal{SX}_\lambda \cap \mathcal{X}^+_\lambda$.

Remark 2.1. Obviously $\mathcal{X}_\lambda \subset \mathcal{X}_\mu$ for $\lambda < \mu$, with $M_{\mathcal{X}_\mu}(u) \le M_{\mathcal{X}_\lambda}(u) \; \forall u \in \mathcal{X}_\lambda$.
If $\lambda < 0$ then $\mathcal{SX}^+_\lambda = \{0\}$ (see Proposition 6.2 of [6]).

Definition 2.2. Given $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma \ge 0$, let $\mathcal{Y}_{\alpha,\beta,\gamma}$ denote the set of all measurable functions $u : B_N \to [-\infty, +\infty[$ satisfying:

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) := \sup_{a \in B_N} (1 - |a|^2)^\alpha \int_{B_N} (1 - |x|^2)^\beta \, u(x) \, (1 - |\Phi_a(x)|^2)^\gamma \, dx < +\infty$$

with $\Phi_a : B_N \to B_N$ the involution defined by:

$$\Phi_a(x) = \frac{a - P_a(x) - \sqrt{1 - |a|^2} \; Q_a(x)}{1 - \langle x, a \rangle} \qquad \forall x \in B_N,$$

where:

$$\langle x, a \rangle = \sum_{j=1}^{N} x_j \, a_j \, , \qquad P_a(x) = \frac{\langle x, a \rangle}{|a|^2} \qquad \text{and} \qquad Q_a(x) = x - P_a(x)$$

for every $x = (x_1, x_2, \dots, x_N) \in I\!\!R^N$ and $a = (a_1, a_2, \dots, a_N) \in I\!\!R^N$, with $P_a(x) = 0$ when $a = 0$. As above, we similarly define $\mathcal{Y}^+_{\alpha,\beta,\gamma}$ (resp. $\mathcal{SY}_{\alpha,\beta,\gamma}$) the subset of all non-negative (resp. subharmonic) functions $u \in \mathcal{Y}_{\alpha,\beta,\gamma}$ and finally $\mathcal{SY}^+_{\alpha,\beta,\gamma} = \mathcal{SY}_{\alpha,\beta,\gamma} \cap \mathcal{Y}^+_{\alpha,\beta,\gamma}$.

Remark 2.2. In Proposition 3.1 of [6], it was proved that $\mathcal{SY}^+_{\alpha,\beta,\gamma} \subset \mathcal{SX}^+_{\alpha+\beta+N}$.
Let $\alpha' \ge \alpha$, $\beta' \ge \beta$ and $\gamma' \ge \gamma \ge 0$, then $\mathcal{Y}^+_{\alpha,\beta,\gamma} \subset \mathcal{Y}^+_{\alpha',\beta',\gamma'}$ with $M_{\mathcal{Y}_{\alpha',\beta',\gamma'}}(u) \le M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u)$ for every $u \in \mathcal{Y}^+_{\alpha,\beta,\gamma}$, since

$$\begin{aligned}
(1 - |a|^2)^{\alpha'} &\le \; (1 - |a|^2)^\alpha \\
(1 - |x|^2)^{\beta'} &\le \; (1 - |x|^2)^\beta \\
(1 - |\Phi_a(x)|^2)^{\gamma'} &\le \; (1 - |\Phi_a(x)|^2)^\gamma
\end{aligned}$$

If $\alpha + \beta < -N$ or $\alpha < -\gamma$, then $\mathcal{SY}^+_{\alpha,\beta,\gamma} = \{0\}$ (see Propositions 6.3 and 6.4 of [6]).

Proposition 2.1. *(i) Given $\beta \ge -\frac{N}{2} - 1$, $\gamma \ge 1$ such that $\beta + \gamma > -1$, let $\alpha \ge \gamma - \beta + \frac{1}{2}$. There exists $K_0 > 0$ such that:*

$$M_{\mathcal{Y}_{\alpha,\lambda+\beta,\gamma}}(u) \le K_0 \, M_{\mathcal{X}_\lambda}(u) \qquad \forall \lambda \in \mathbb{R} \qquad \forall u \in \mathcal{X}_\lambda.$$

(ii) Given $\beta \ge -\frac{N+1}{2}$, $\gamma \ge \frac{1}{2}$ such that $\beta + \gamma > -1$, let $\alpha \ge \gamma - \beta$. There exists $K' > 0$ such that:

$$M_{\mathcal{Y}_{\alpha,\lambda+\beta,\gamma}}(u) \le K' \, M_{\mathcal{X}_\lambda}(u) \qquad \forall \lambda \in \mathbb{R} \qquad \forall u \in \mathcal{X}_\lambda.$$

Thus, in both cases: $\mathcal{X}_\lambda \subset \mathcal{Y}_{\alpha,\lambda+\beta,\gamma}$.

Remark 2.3. The constants K_0 and K' respectively stem from Lemmas 6.2 and 6.3 which are postponed in annex: at the end of the paper, Section 6 gathers several technical results which will be repeatedly used throughout the proofs in Sections 2, 3, 4 and 5.

Proof. In order to establish Proposition 2.1, let us consider the following, which is available for any $a \in B_N$:

$$(1-|a|^2)^\alpha \int_{B_N} (1-|x|^2)^{\lambda+\beta} u(x)\,(1-|\Phi_a(x)|^2)^\gamma dx \leq$$

$$\leq M_{\mathcal{X}_\lambda}(u)\,(1-|a|^2)^{\alpha+\gamma} \int_{B_N} \frac{(1-|x|^2)^{\beta+\gamma}}{(1-\langle x,a\rangle)^{2\gamma}}\,dx$$

according to the next formula (see [5] pp.25–26 or [1] p.115):

$$1-|\Phi_a(x)|^2 = \frac{(1-|a|^2)\,(1-|x|^2)}{(1-\langle x,a\rangle)^2} \qquad \forall x \in B_N \qquad \forall a \in B_N.$$

Proof of (i) Lemma 6.2 is applied with $A = 2\gamma > 0$ and $T = \beta+\gamma > -1$, since

$$2 \leq A \leq 2\gamma + 2(\beta+1) + N = N + 2(T+1).$$

The above integral is thus majorized by $K_0\,(1-|a|^2)^{-\gamma-\frac{1}{2}}$. The result follows since

$$\sup_{a\in B_N} \frac{(1-|a|^2)^{\alpha+\beta}}{(1-|a|^2)^{\gamma+\frac{1}{2}}} = 1.$$

Proof of (ii) Lemma 6.3 is applied with $T = \beta+\gamma > -1$, $A = 2\gamma > 0$, $\tau > 0$ and $P = 0$ which fulfill $1 \leq A + P \leq 2\gamma + 2\beta + N + 1 = N + 1 + 2T$. It leads to:

$$\int_{B_N} \frac{(1-|x|^2)^T}{(1-\langle x,a\rangle)^A}\,dx \leq \frac{K'}{(1-|a|^2)^{\frac{A}{2}}} = \frac{K'}{(1-|a|^2)^\gamma} \qquad \forall a \in B_N$$

hence the conclusion since $\sup_{a\in B_N} \dfrac{(1-|a|^2)^{\alpha+\beta}}{(1-|a|^2)^\gamma} = 1.$ ∎

Example 2.1. *Given $\alpha \geq \frac{1}{2}$, $\beta \geq -1-\frac{N}{2}$ and $\gamma > \frac{N}{2}$, the function v defined on B_N by:*

$$v(x) = (1-|x|^2)^{-\frac{N}{2}-\beta-1} \qquad \forall x \in B_N$$

belongs to $\mathcal{SY}^+_{\alpha,\beta,\gamma}$ with $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(v) = K_0$.

The growth of v will be studied during the proof of Theorem 3.3 in Section 3.

Example 2.2. *Given $a \in B_N$ and parameters α, β, γ in configuration (1.4), the function f_a defined by $f_a(x) = (1-\langle x,a\rangle)^{-N-1-2\beta}$ $\forall x \in B_N$ belongs to $\mathcal{SY}^+_{\alpha,\beta,\gamma}$ and*

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(f_a) \leq K(1-|a|^2)^{\alpha-\frac{N+1}{2}-\beta}$$

with K the constant from Lemma 6.3.

This property of f_a will be established during the proof of Theorem 3.2 just below.

Example 2.3. *With parameters α, β, γ in configuration (1.6), the function u defined by*

$$u(x) = (1-|x|^2)^{-\frac{N+1}{2}-\beta} \qquad \forall x \in B_N$$

belongs to $\mathcal{SY}^+_{\alpha,\beta,\gamma}$ and $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) = K'$.

This will be shown in the proof of Theorem 3.4.

3. PRODUCTS gu WHEN $u \in \mathcal{Y}_{\alpha,\beta,\gamma}$ AND g HAS A BLOCH–TYPE GROWTH.

Proposition 3.1. *Given $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma \geq 0$ and $\lambda \in \mathbb{R}$, let $g \in \mathcal{X}_\lambda$ such that $M_{\mathcal{X}_\lambda}(g) \geq 0$. Then*

$$M_{\mathcal{Y}_{\alpha,\lambda+\beta,\gamma}}(gu) \leq M_{\mathcal{X}_\lambda}(g)\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \qquad \forall u \in \mathcal{Y}^+_{\alpha,\beta,\gamma}.$$

Proof. We have $(1-|x|^2)^\lambda g(x)\, u(x) \leq M_{\mathcal{X}_\lambda}(g)\, u(x)$ since $u(x) \geq 0\ \forall x \in B_N$. Thus the required majorization follows from $M_{\mathcal{X}_\lambda}(g) \geq 0$. ∎

Theorem 3.2. *Given $\lambda \in \mathbb{R}$, $\beta > -\frac{N+1}{2}$, $\alpha \in]-\frac{N+1}{2}-\beta, \frac{N+1}{2}+\beta]$ and $\gamma > \max(\alpha, -1-\beta)$ such that $\gamma \geq -\alpha$, let g be a non–negative subharmonic function defined on B_N, satisfying:*

$$\exists C > 0 \qquad M_{\mathcal{Y}_{\alpha,\lambda+\beta,\gamma}}(gu) \leq C\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \qquad \forall u \in \mathcal{S}\mathcal{Y}^+_{\alpha,\beta,\gamma}. \tag{3.1}$$

Then $g \in \mathcal{X}_{\lambda+\frac{N-1}{2}}$.

Remark 3.1. Obviously $\gamma \geq |\alpha|$ and $\gamma > \alpha$ imply $\gamma > 0$.

Definition 3.1. Given $R \in]0,1[$ and $\mu \in \mathbb{R}$, let $Q_\mu = Q_\mu(R)$ be defined by: $Q_\mu = 2^{-\mu}$ if $\mu \leq 0$ and

$$Q_\mu = \left(\frac{1+R}{1-R}\right)^\mu \qquad \text{if } \mu \geq 0.$$

Definition 3.2. Given $a \in B_N$, let $R_a = R\,\dfrac{1-|a|^2}{1+R|a|}$. Let V_a denote the volume of the open ball $B(a,R_a) = \{x \in B_N\ :\ |x-a| < R_a\}$.

Remark 3.2. Thus, through Lemma 6.1 from the last section:

$$(1-|a|^2)^\mu \leq Q_\mu\,(1-|x|^2)^\mu \qquad \forall x \in B(a,R_a) \quad \forall a \in B_N \tag{3.2}$$

Proof. In order to demonstrate Theorem 3.2, let be given $a \in B_N$ and f_a defined by

$$f_a(x) = \frac{1}{(1-\langle x,a\rangle)^A} \qquad \forall x \in B_N,$$

with $A = N+1+2\beta$. The subharmonicity of the function f_a follows from Lemma 6.4 since $A \geq 0$. We next show that $f_a \in \mathcal{Y}_{\alpha,\beta,\gamma}$. For any $b \in B_N$, the following holds:

$$\begin{aligned} J_b(f_a) :&= (1-|b|^2)^\alpha \int_{B_N} (1-|x|^2)^\beta\, f_a(x)\,(1-|\Phi_b(x)|^2)^\gamma\, dx \\ &= (1-|b|^2)^{\alpha+\gamma} \int_{B_N} \frac{(1-|x|^2)^{\beta+\gamma}}{(1-\langle x,a\rangle)^A\,(1-\langle x,b\rangle)^{2\gamma}}\, dx \end{aligned}$$

Lemma 6.3 now applies with $A = N+1+2\beta > 0$, $P = 2\gamma > 0$ and $T = \beta+\gamma > -1$ which verify $A+P = N+1+2\beta+2\gamma = N+1+2T > N+1-2 \geq 1$. The choice $\tau = \alpha+\gamma$ is allowed since $\alpha+\gamma \geq 0$, $\alpha+\gamma \leq \frac{N+1}{2}+\beta+\gamma = \frac{A+P}{2}$, $\alpha+\gamma < \gamma+\gamma = P$ and $\frac{P-A}{2} = \gamma - \frac{N+1}{2} - \beta < \gamma+\alpha$. For all $a \in B_N$ and $b \in B_N$, this leads to:

$$J_b(f_a) \leq (1-|b|^2)^{\alpha+\gamma} \frac{K}{(1-|a|^2)^{\frac{A+P}{2}-\tau}(1-|b|^2)^\tau} = \frac{K}{(1-|a|^2)^{\frac{A+P}{2}-\tau}}$$

where the constant K stems from Lemma 6.3. Thus

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(f_a) = \sup_{b \in B_N} J_b(f_a) \leq \frac{K}{(1-|a|^2)^{\frac{A+P}{2}-\tau}}$$

and

$$M_{\mathcal{Y}_{\alpha,\beta+\lambda,\gamma}}(g\, f_a) \leq \frac{C\,K}{(1-|a|^2)^{\frac{A+P}{2}-\tau}} \qquad \forall a \in B_N.$$

Given $R \in]0,1[$, the subharmonicity of g provides:

$$V_a\, g(a) \leq \int_{B(a,R_a)} g(x)\, dx \qquad \forall a \in B_N$$

with V_a and $B(a,R_a)$ as in Definition 3.2. Now $g(x) \geq 0\ \forall x \in B_N$ hence the estimation (3.2) where $\mu = \lambda - \frac{N+1}{2}$ leads to:

$$(3.3) \qquad (1-|a|^2)^{\lambda - \frac{N+1}{2} + N}\, g(a) \leq \frac{N\,(1+R)^N}{\sigma_N\, R^N}\, Q_{\lambda - \frac{N+1}{2}} \int_{B(a,R_a)} (1-|x|^2)^{\lambda - \frac{N+1}{2}}\, g(x)\, dx.$$

Moreover $\gamma + \beta + \frac{N+1}{2} \geq \alpha + \gamma \geq 0$ hence, again through Lemma 6.1:

$$1 \leq \left(2\, \frac{1-|a|^2}{1-\langle x,a\rangle} \right)^{\gamma+\beta+\frac{N+1}{2}}$$

and

$$1 \leq \left(2\, \frac{1+R}{1-R}\, \frac{1-|x|^2}{1-\langle x,a\rangle} \right)^{\gamma+\beta+\frac{N+1}{2}}.$$

Thus

$$(1-|a|^2)^{\lambda+\frac{N-1}{2}}\, g(a) \leq Q\,(1-|a|^2)^{\gamma+\beta+\frac{N+1}{2}} \int_{B(a,R_a)} \frac{(1-|x|^2)^{\lambda+\gamma+\beta}}{(1-\langle x,a\rangle)^{2\gamma+A}}\, g(x)\, dx,$$

where the constant Q is independant from a and x. Now

$$\gamma + \beta + \frac{N+1}{2} \;=\; \alpha + \gamma + \beta + \frac{N+1}{2} - \alpha \;=\; \alpha + \gamma + \frac{A+P}{2} - \tau,$$

whence

$$\begin{aligned} &(1-|a|^2)^{\lambda+\frac{N-1}{2}}\, g(a) \leq \\ &\leq Q\,(1-|a|^2)^{\frac{A+P}{2}-\tau}\,(1-|a|^2)^{\alpha+\gamma} \int_{B(a,R_a)} \frac{(1-|x|^2)^{\lambda+\gamma+\beta}}{(1-\langle x,a\rangle)^{2\gamma}}\, f_a(x)\ g(x)\, dx \leq \\ &\leq Q\,(1-|a|^2)^{\frac{A+P}{2}-\tau}\,(1-|a|^2)^{\alpha} \int_{B_N} (1-|x|^2)^{\lambda+\beta}\, f_a(x)\ g(x)\,(1-|\Phi_a(x)|^2)^{\gamma}\, dx \end{aligned}$$

because of $g \geq 0$ on B_N. Finally

$$(1-|a|^2)^{\lambda+\frac{N-1}{2}}\, g(a) \leq Q\,(1-|a|^2)^{\frac{A+P}{2}-\tau}\, M_{\mathcal{Y}_{\alpha,\beta+\lambda,\gamma}}(g\, f_a) \leq Q\, C\, K$$

for all $a \in B_N$. ∎

Theorem 3.3. *Given* $\alpha \geq \frac{1}{2}$, $\beta \geq -1 - \frac{N}{2}$, $\gamma > \frac{N}{2}$ *and* $\lambda \in \mathbb{R}$, *let* $g : B_N \to [0,+\infty[$ *be a subharmonic function satisfying* (3.1). *Then* $g \in \mathcal{X}_{\lambda+\frac{N}{2}-1+\alpha}$.

Remark 3.3. Theorem 3.2 did not include the case where $\beta \in [-1-\frac{N}{2}, -\frac{N+1}{2}]$. Even when $\beta \in]-\frac{N+1}{2}, -\frac{N}{2}[$, the interval $]-\frac{N+1}{2}-\beta, \frac{N+1}{2}+\beta]$ did not contain the value $\alpha = \frac{1}{2}$.

Proof. In order to establish Theorem 3.3, let $H = \frac{N}{2} + 1 + \beta$ and let u be defined by

$$u(x) = \frac{1}{(1-|x|^2)^H} \qquad \forall x \in B_N.$$

This function u is subharmonic in B_N since $H \geq 0$, according to Lemma 6.4. It moreover belongs to $\mathcal{Y}_{\alpha,\beta,\gamma}$ since:

$$(1-|a|^2)^{\alpha} \int_{B_N} (1-|x|^2)^{\beta} u(x)\,(1-|\Phi_a(x)|^2)^{\gamma}\, dx = (1-|a|^2)^{\alpha+\gamma} \int_{B_N} \frac{(1-|x|^2)^{\beta-H+\gamma}}{(1-\langle x,a\rangle)^{2\gamma}}\, dx$$

and this integral is equal to $K_0\,(1-|a|^2)^{-\gamma-\frac{1}{2}}$ for all $a \in B_N$ according to Lemma 6.2 applied with $A = 2\gamma > N \geq 2$ and $T = \beta - H + \gamma = -\frac{N}{2} - 1 + \gamma > -1$ which fulfill: $N + 2(T+1) = A$. Finally

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) = K_0 \sup_{a\in B_N} (1-|a|^2)^{\alpha-\frac{1}{2}} = K_0.$$

Whence $M_{\mathcal{Y}_{\alpha,\beta+\lambda,\gamma}}(gu) \leq C\,K_0$.

Let $R \in]0,1[$ be fixed. For any $a \in B_N$, it follows from the estimation (3.2), together with the subharmonicity and positivity of g, that:

$$(1-|a|^2)^{\lambda-\frac{N}{2}-1} V_a\, g(a) \leq Q_{\lambda-\frac{N}{2}-1} \int_{B(a,R_a)} (1-|x|^2)^{\lambda-\frac{N}{2}-1}\, g(x)\,dx$$

$$(1-|a|^2)^{\lambda-\frac{N}{2}-1+N}\, g(a) \leq \frac{N\,(1+R)^N}{\sigma_N\, R^N}\, Q_{\lambda-\frac{N}{2}-1} \int_{B(a,R_a)} (1-|x|^2)^{\lambda+\beta}\, u(x)\, g(x)\,dx$$

Again Lemma 6.1 provides:

$$(1-|a|^2)^{\lambda+\frac{N}{2}-1} g(a) \leq Q' \int_{B(a,R_a)} \left(\frac{1-|a|^2}{1-\langle x,a\rangle}\right)^{\gamma} \left(\frac{1-|x|^2}{1-\langle x,a\rangle}\right)^{\gamma} (1-|x|^2)^{\lambda+\beta}\, u(x)\, g(x)\,dx$$

with a constant $Q' > 0$ independant from a and x. In other words:

$$(1-|a|^2)^{\lambda+\frac{N}{2}-1+\alpha}\, g(a) \leq Q'(1-|a|^2)^{\alpha} \int_{B(a,R_a)} (1-|x|^2)^{\lambda+\beta}\, u(x)\, g(x)\, (1-|\Phi_a(x)|^2)^{\gamma}\,dx$$

As $u(x)\,g(x) \geq 0\ \forall x \in B_N$, the above $\int_{B(a,R_a)} \ldots$ is majorized by $\int_{B_N} \ldots$, so that:

$$(1-|a|^2)^{\lambda+\frac{N}{2}-1+\alpha}\, g(a) \leq Q'\, M_{\mathcal{Y}_{\alpha,\beta+\lambda,\gamma}}(gu) \leq Q'\,C\,K_0$$

for each $a \in B_N$. ∎

Theorem 3.4. *Given* $\alpha \geq 0$, $\beta \geq -\frac{N+1}{2}$, $\gamma > \frac{N-1}{2}$ *and* $\lambda \in \mathbb{R}$, *let* $g : B_N \to [0,+\infty[$ *be a subharmonic function satisfying* (3.1). *Then* $g \in \mathcal{X}_{\lambda+\frac{N-1}{2}+\alpha}$.

Proof. Given $J = \frac{N+1}{2} + \beta$, let u denote the function defined by:

$$u(x) = (1-|x|^2)^{-J} \qquad \forall x \in B_N.$$

Lemma 6.4 implies the subharmonicity of u since $J \geq 0$. Besides that:

$$\begin{aligned}
(1-|a|^2)^{\alpha} \int_{B_N} (1-|x|^2)^{\beta} u(x)\,(1-|\Phi_a(x)|^2)^{\gamma}\,dx &= (1-|a|^2)^{\alpha+\gamma} \int_{B_N} \frac{(1-|x|^2)^{\gamma-\frac{N+1}{2}}}{(1-\langle x,a\rangle)^{2\gamma}}\,dx \\
&= (1-|a|^2)^{\alpha+\gamma}\, \frac{K'}{(1-|a|^2)^{\gamma}} \\
&= K'\,(1-|a|^2)^{\alpha} \qquad \forall a \in B_N
\end{aligned}$$

according to Lemma 6.3 applied here with $P = \tau = 0$, $A = 2\gamma > 0$ and $T = \gamma - \frac{N+1}{2} > -1$ (the condition $1 \leq A = N+1+2T$ is fulfilled by these parameters, because of $\gamma > 1/2$). Thus $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) = K'$ since $\alpha \geq 0$.

With $R \in]0,1[$ and $a \in B_N$ fixed, it now follows from (3.3) and Lemma 6.1 that:

$$\begin{aligned}
(1-|a|^2)^{\lambda+\frac{N-1}{2}}\, g(a) &\leq Q'' \int_{B(a,R_a)} \left(\frac{1-|a|^2}{1-\langle x,a\rangle}\right)^{\gamma} \left(\frac{1-|x|^2}{1-\langle x,a\rangle}\right)^{\gamma} (1-|x|^2)^{\lambda+\beta}\, u(x)\, g(x)\,dx \\
&\leq Q'' \int_{B_N} (1-|x|^2)^{\lambda+\beta}\, u(x)\, g(x)\, (1-|\Phi_a(x)|^2)^{\gamma}\,dx
\end{aligned}$$

since $g \geq 0$ on B_N. Here Q'' denotes a constant depending only on R, N, λ and γ. Finally

$$(1-|a|^2)^{\lambda+\frac{N-1}{2}+\alpha}\, g(a) \leq Q'' M_{\mathcal{Y}_{\alpha,\lambda+\beta,\gamma}}(gu) \leq Q''\, C\, K'$$

for all $a \in B_N$. ∎

Definition 3.3. Let S_N denote the unit sphere of $\mathbb{R}^N$ and $d\sigma$ the area element on S_N. Let σ_N denote the area of S_N, for information: $\sigma_N = \frac{2\,\pi^{N/2}}{\Gamma(N/2)}$ (see [2] p.29).

Theorem 3.5. *Let $\varepsilon \in]0, \sigma_N[$ and E a measurable subset of B_N with $\sigma(E_r) \geq \varepsilon\ \forall r \in [0,1[$, where $E_r = \{\eta \in S_N\ :\ r\eta \in E\}$. Let $g : B_N \to [0,+\infty[$ be a measurable function.*
(i) Given parameters $\alpha \geq \frac{1}{2}$, $\beta \geq -1-\frac{N}{2}$, $\gamma > \frac{N}{2}$ and $\lambda \in \mathbb{R}$, if the function g satisfies

$$g(x) \geq (1-|x|^2)^{-\lambda-\gamma+\frac{N}{2}} \qquad \forall x \in E,$$

then there does NOT exist any constant $C > 0$ such that

$$M_{\mathcal{Y}_{\alpha,\lambda+\beta,\gamma}}(gu) \leq C\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \qquad \forall u \in \mathcal{SY}^+_{\alpha,\beta,\gamma}.$$

(ii) The same conclusion holds when $g(x) \geq (1-|x|^2)^{-\lambda-\beta-\gamma-1}\ \forall x \in E$, with α, β, γ and λ as in Theorem 3.2.
(iii) The same conclusion holds when $g(x) \geq (1-|x|^2)^{-\lambda-\gamma+\frac{N}{2}-\frac{1}{2}}\ \forall x \in E$, with parameters fulfilling: $\alpha \geq 0$, $\beta \geq -\frac{N+1}{2}$, $\gamma > \frac{N-1}{2}$ and $\lambda \in \mathbb{R}$.

Proof. Each of these three results is to be established ab absurdo: let us suppose on the contrary that (3.1) holds.

Proof of (i) Let $H = \frac{N}{2}+\beta+1 \geq 0$ and v the function defined on B_N by $v(x) = (1-|x|^2)^{-H}$. Thus $v \in \mathcal{SY}^+_{\alpha,\beta,\gamma}$ as it was shown in the proof of Theorem 3.3. If there existed some $C > 0$ such that $M_{\mathcal{Y}_{\alpha,\lambda+\beta,\gamma}}(gu) \leq C\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u)$ for all functions $u \in \mathcal{SY}^+_{\alpha,\beta,\gamma}$, it would apply in particular to the function v and we should have $M_{\mathcal{Y}_{\alpha,\lambda+\beta,\gamma}}(gv) < +\infty$.

Having fixed some $a \in B_N$, the following integrals should then be finite too:

$$\begin{aligned}\int_{B_N}(1-|x|^2)^{\beta+\lambda} g(x)\, v(x)\, (1-|\Phi_a(x)|^2)^{\gamma}\, dx &\geq (1-|a|^2)^{\gamma}\int_E \frac{(1-|x|^2)^{\beta+\lambda-(\lambda+\gamma-\frac{N}{2})-H+\gamma}}{(1-\langle x,a\rangle)^{2\gamma}}\, dx\\ &= (1-|a|^2)^{\gamma}\int_E \frac{(1-|x|^2)^{\beta+\frac{N}{2}-H}}{(1-\langle x,a\rangle)^{2\gamma}}\, dx\\ &> \frac{(1-|a|^2)^{\gamma}}{4^{\gamma}}\int_E (1-|x|^2)^{-1} dx\end{aligned}$$

since $0 < 1-\langle x,a\rangle < 2$ for all $a \in B_N$ and $x \in B_N$. But the last integral diverges since

$$\begin{aligned}\int_E \frac{dx}{1-|x|^2} &\geq \varepsilon \int_0^1 \frac{r^{N-1}\, dr}{1-r^2}\\ &> \frac{\varepsilon}{2}\int_0^1 \frac{r^{N-1}\, dr}{1-r} = +\infty\end{aligned}$$

with $r = |x|$ and $dx = r^{N-1}\, dr\, d\sigma$. Now the contradiction follows.

Proof of (ii) If, for some constant C, the estimation $M_{\mathcal{Y}_{\alpha,\lambda+\beta,\gamma}}(gu) \leq C\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u)$ was valid for all $u \in \mathcal{SY}^+_{\alpha,\beta,\gamma}$, then it would hold in particular with $u = f_a \in \mathcal{SY}^+_{\alpha,\beta,\gamma}$ defined as in the proof of Theorem 3.2: $f_a(x) = (1 - \langle x, a\rangle)^{-N-1-2\beta}$ $\forall x \in B_N$ (with a fixed in B_N). Thus we should have $M_{\mathcal{Y}_{\alpha,\beta+\lambda,\gamma}}(g\, f_a) < +\infty$ and the following integrals should be finite too:

$$\int_{B_N} (1-|x|^2)^{\beta+\lambda} g(x)\, f_a(x)\,(1-|\Phi_a(x)|^2)^\gamma\, dx \geq (1-|a|^2)^\gamma \int_E \frac{(1-|x|^2)^{\beta+\lambda-(\lambda+\beta+\gamma+1)+\gamma}}{(1-\langle x,a\rangle)^{N+1+2\beta+2\gamma}}\, dx$$
$$\geq \frac{(1-|a|^2)^\gamma}{2^{N+1+2\beta+2\gamma}} \int_E \frac{dx}{1-|x|^2}$$

The last integral diverges, hence a contradiction.

Proof of (iii). The function v defined by $v(x) = (1-|x|^2)^{-J}$ $\forall x \in B_N$, with $J = \frac{N+1}{2} + \beta$, belongs to $\mathcal{SY}^+_{\alpha,\beta,\gamma}$ (see the proof of Theorem 3.4). Reasoning ab absurdo as in both previous cases, we should have $M_{\mathcal{Y}_{\alpha,\beta+\lambda,\gamma}}(g\, v) < +\infty$, hence the finiteness of the following integral (with $a \in B_N$ fixed):

$$\int_E \frac{(1-|x|^2)^{\beta+\lambda-(\lambda+\gamma-\frac{N}{2}+\frac{1}{2})-J+\gamma}}{(1-\langle x,a\rangle)^{2\gamma}}\, dx = \int_E \frac{(1-|x|^2)^{\beta+\frac{N}{2}-\frac{1}{2}-J}}{(1-\langle x,a\rangle)^{2\gamma}}\, dx$$
$$\geq \frac{1}{4^\gamma} \int_E \frac{dx}{1-|x|^2}$$

which diverges, thus a contradiction arises. ∎

4. A SITUATION WHERE THE PRODUCTS gu HAVE A BLOCH–TYPE GROWTH.

Given $R \in]0,1[$ and $g \in \mathcal{X}_\lambda$ with $M_{\mathcal{X}_\lambda}(g) \geq 0$, we already know from Proposition 3.1 of [6] and Corollary 3.2 of [6] that:

$$M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(gu) \leq \frac{N}{\sigma_N}\, \frac{(1+R)^{N-\gamma}}{R^N\,(1-R)^\gamma}\, Q_\beta\, M_{\mathcal{X}_\lambda}(g)\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \qquad \forall u \in \mathcal{SY}^+_{\alpha,\beta,\gamma}$$

(for σ_N and Q_β, see the notations of Definitions 3.2 and 3.3).

Theorem 4.1. *With α, β, γ and λ as in Theorem 3.2, let g be a non–negative subharmonic function defined on B_N, satisfying:*

$$\text{(4.1)} \qquad \exists C' > 0 \qquad M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(gu) \leq C'\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \qquad \forall u \in \mathcal{SY}^+_{\alpha,\beta,\gamma}.$$

Then $g \in \mathcal{X}_{\lambda+\frac{N-1}{2}}$.

Remark 4.1. This conclusion cannot be deduced directly from Theorem 3.2 of the previous section and Proposition 3.1 of [6] which asserts: $\mathcal{SY}^+_{\alpha,\beta,\gamma} \subset \mathcal{SX}^+_{\alpha+\beta+N}$ together with

$$M_{\mathcal{X}_{\alpha+\beta+N}}(v) \leq C''\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(v)$$

for some constant C'' independant from $v \in \mathcal{SY}^+_{\alpha,\beta,\gamma}$. This result cannot be used here (with $v = gu$ and β replaced by $\beta+\lambda$) because the subharmonicity of g and u does not compulsorily imply the subharmonicity of gu, as the following counterexample points out: with g and u defined by $g(x) = 1 + x_1 \geq 0$ and $u(x) = 1 - x_1 \geq 0$ $\forall x = (x_1, x_2, ..., x_N) \in B_N$, we have $(\Delta u)(x) = (\Delta g)(x) = 0 \geq 0$ but

$$\Delta(gu)(x) = \tfrac{\partial^2}{\partial x_1^2}(1-x_1^2) = -2 < 0 \qquad \forall x \in B_N.$$

However this function u belongs to $\mathcal{Y}_{\alpha,\beta,\gamma}$ since

$$u(x) \leq 2 \leq 2^{N+2+2\beta} f_a(x) \qquad \forall x \in B_N$$

with $f_a \in \mathcal{Y}_{\alpha,\beta,\gamma}$ as in the proof of Theorem 3.2. The previous majoration merely follows from $0 < 1 - \langle x,a\rangle < 2$, hence $\frac{1}{2} < \frac{1}{1-\langle x,a\rangle}$, thus $\left(\frac{1}{2}\right)^{N+1+2\beta} < f_a(x)$ $\forall x \in B_N$.

Proof. Having fixed $R \in]0,1[$, the proof of Theorem 4.1 will make use of the same notations (V_a, $B(a,R_a)$, f_a, Q_μ, A, P and τ) as the proof of Theorem 3.2. We have for any $a \in B_N$:

$$\begin{aligned}(1-|a|^2)^{\lambda+\frac{N-1}{2}+\alpha} V_a\, g(a) &\le Q_{\lambda+\frac{N-1}{2}+\alpha} \int_{B(a,R_a)} (1-|x|^2)^{\lambda+\frac{N-1}{2}+\alpha} g(x)\, dx \\ &\le L\,(1-|a|^2)^{\beta+\frac{N+1}{2}} \int_{B(a,R_a)} \frac{(1-|x|^2)^{\lambda+\frac{N-1}{2}+\alpha+\beta+\frac{N+1}{2}}}{(1-\langle x,a\rangle)^{N+1+2\beta}}\, g(x)\, dx\end{aligned}$$

where the constant L depends only on N, R, β and $\lambda+\alpha$. The above integral is merely

$$\begin{aligned}\int_{B(a,R_a)} (1-|x|^2)^{\lambda+\alpha+\beta+N} f_a(x)\, g(x)\, dx &\le M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(f_a\, g)\, V_a \\ &\le C'\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(f_a)\, V_a \\ &\le \frac{C'\, K\, V_a}{(1-|a|^2)^{\frac{A+P}{2}-\tau}} \\ &\le \frac{C'\, K\, V_a}{(1-|a|^2)^{\beta+\frac{N+1}{2}-\alpha}}\end{aligned}$$

with K from Lemma 6.3. Finally: $(1-|a|^2)^{\lambda+\frac{N-1}{2}}\, g(a) \le L\, C'\, K$ for every $a \in B_N$. ∎

Theorem 4.2. *Given* $\alpha \ge \frac{1}{2}$, $\beta \ge -1-\frac{N}{2}$, $\gamma > \frac{N}{2}$ *and* $\lambda \in \mathbb{R}$, *let g be a subharmonic function (on B_N) satisfying* (4.1). *Then* $g \in \mathcal{X}_{\lambda+\frac{N}{2}-1+\alpha}$.

Proof. Let $u \in \mathcal{SY}^+_{\alpha,\beta,\gamma}$ be defined by $u(x) = (1-|x|^2)^{-\frac{N}{2}-1-\beta}$ as in the proof of Theorem 3.3. With the same notations as in the proof of Theorem 3.2, the subharmonicity of g leads to:

$$\begin{aligned}V_a\, g(a) &\le \int_{B(a,R_a)} g(x)\, dx = \int_{B(a,R_a)} (1-|x|^2)^{\frac{N}{2}+1+\beta} g(x)\, u(x)\, dx \\ &= \int_{B(a,R_a)} (1-|x|^2)^{\lambda+\alpha+\beta+N} g(x)\, u(x)\, (1-|x|^2)^{-\lambda-\alpha-\frac{N}{2}+1} dx \\ &\le M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(gu) \int_{B(a,R_a)} (1-|x|^2)^{-\lambda-\alpha-\frac{N}{2}+1} dx \\ &\le C'\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \int_{B(a,R_a)} Q_{\lambda+\alpha+\frac{N}{2}-1}\, (1-|a|^2)^{-\lambda-\alpha-\frac{N}{2}+1} dx \\ &= Q_{\lambda+\frac{N}{2}-1+\alpha}\, C'\, K_0\, (1-|a|^2)^{-\lambda-\alpha-\frac{N}{2}+1} V_a \qquad \forall a \in B_N.\end{aligned}$$

Thus $M_{\mathcal{X}_{\lambda+\frac{N}{2}-1+\alpha}}(g) \le Q_{\lambda+\frac{N}{2}-1+\alpha}\, C'\, K_0$ with K_0 from Lemma 6.2. ∎

Theorem 4.3. *Given* $\alpha \ge 0$, $\beta \ge -\frac{N+1}{2}$, $\gamma > \frac{N-1}{2}$ *and* $\lambda \in \mathbb{R}$, *let g be a subharmonic function (on B_N) satisfying* (4.1). *Then* $g \in \mathcal{X}_{\lambda+\frac{N-1}{2}+\alpha}$.

Proof. With this choice of parameters, the function $u : x \mapsto (1-|x|^2)^{-\frac{N+1}{2}-\beta}$ belongs to the set $\mathcal{SY}^+_{\alpha,\beta,\gamma}$ (see the proof of Theorem 3.4). Now g is subharmonic hence:

$$\begin{aligned}V_a\, g(a) &\le \int_{B(a,R_a)} (1-|x|^2)^{\lambda+\alpha+\beta+N} g(x)\, u(x)\, (1-|x|^2)^{-\lambda-\alpha-\frac{N-1}{2}} dx \\ &\le C'\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u)\, Q_{\lambda+\alpha+\frac{N-1}{2}}\, (1-|a|^2)^{-\lambda-\alpha-\frac{N-1}{2}}\, V_a \qquad \forall a \in B_N.\end{aligned}$$

Finally $M_{\mathcal{X}_{\lambda+\frac{N-1}{2}+\alpha}}(g) \le C'\, K'\, Q_{\lambda+\frac{N-1}{2}+\alpha}$ with K' from Lemma 6.3. ∎

5. PRODUCTS gu WHEN u HAS A BLOCH–TYPE GROWTH.

Proposition 5.1. *Let $\lambda \in \mathbb{R}$, $g \in \mathcal{X}_\lambda$ and two reals $\alpha \leq \beta$. If $M_{\mathcal{X}_\lambda}(g) \geq 0$ then*

$$M_{\mathcal{X}_{\lambda+\beta}}(gu) \leq M_{\mathcal{X}_\lambda}(g)\, M_{\mathcal{X}_\alpha}(u) \qquad \forall u \in \mathcal{X}_\alpha^+.$$

Proof. For any $x \in B_N$, the following holds: $(1-|x|^2)^\beta \leq (1-|x|^2)^\alpha$ since $1-|x|^2 \in [0,1[$. Now $(1-|x|^2)^\lambda\, g(x)\, u(x) \leq M_{\mathcal{X}_\lambda}(g)\, u(x)$ because of $u(x) \geq 0$. This leads to:

$$(1-|x|^2)^{\beta+\lambda}\, g(x)\, u(x) \leq M_{\mathcal{X}_\lambda}(g)\, (1-|x|^2)^\alpha u(x)$$

since $u(x)\, M_{\mathcal{X}_\lambda}(g) \geq 0$. The required majorization follows from $M_{\mathcal{X}_\lambda}(g) \geq 0$. ∎

Proposition 5.2. *Given two reals $\alpha \geq \beta \geq 0$, let g denote a function defined on B_N and satisfying:*

$$\exists C > 0 \quad \exists \lambda \in \mathbb{R} \quad \textit{such that} \quad M_{\mathcal{X}_{\beta+\lambda}}(gu) \leq C\, M_{\mathcal{X}_\alpha}(u) \quad \forall u \in \mathcal{S}\mathcal{X}_\alpha^+.$$

Then $g \in \mathcal{X}_\lambda$ with $M_{\mathcal{X}_\lambda}(g) \leq C$.

Proof. For any $x \in B_N$, we have: $g(x) = (1-|x|^2)^\beta\, g(x)\, u(x)$ where $u(x) = (1-|x|^2)^{-\beta}$. Obviously $u \in \mathcal{X}_\alpha$ with $M_{\mathcal{X}_\alpha}(u) \leq 1$. Moreover u is subharmonic in B_N since $\beta \geq 0$, according to Lemma 6.4 in the next section. Thus

$$(1-|x|^2)^\lambda g(x) \leq M_{\mathcal{X}_{\beta+\lambda}}(gu) \leq C$$

for each $x \in B_N$. ∎

Corollary 5.3. *Given $\lambda \in \mathbb{R}$ and $\alpha \geq 0$, a function $g : B_N \to [0,+\infty[$ belongs to $\mathcal{X}_\lambda$ if and only if:*

$$\exists C > 0 \qquad \textit{such that} \qquad M_{\mathcal{X}_{\alpha+\lambda}}(gu) \leq C\, M_{\mathcal{X}_\alpha}(u) \qquad \forall u \in \mathcal{S}\mathcal{X}_\alpha^+.$$

In particular g is majorized on B_N if and only if:

$$\exists C > 0 \qquad \textit{such that} \qquad M_{\mathcal{X}_\alpha}(gu) \leq C\, M_{\mathcal{X}_\alpha}(u) \qquad \forall u \in \mathcal{S}\mathcal{X}_\alpha^+.$$

Proposition 5.4. *Given $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma \geq 0$ and $\lambda \in \mathbb{R}$, let $g \in \mathcal{Y}^+_{\alpha,\beta,\gamma}$. Then*

$$M_{\mathcal{Y}_{\alpha,\beta+\lambda,\gamma}}(gu) \leq M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(g)\, M_{\mathcal{X}_\lambda}(u) \qquad \forall u \in \mathcal{X}_\lambda \quad \textit{such that } M_{\mathcal{X}_\lambda}(u) \geq 0.$$

Proof. The hypothesis $g(x) \geq 0$ implies:

$$(1-|x|^2)^{\beta+\lambda}\, u(x)\, g(x) \leq M_{\mathcal{X}_\lambda}(u)\, (1-|x|^2)^\beta\, g(x) \qquad \forall x \in B_N.$$

Multiplying by $(1-|a|^2)^\alpha (1-|\Phi_a(x)|^2)^\gamma \geq 0$ and integrating over B_N do not modify the inequality. But evaluating the $\sup\limits_{a\in B_N}$–bound requires that $M_{\mathcal{X}_\lambda}(u) \geq 0$. ∎

Proposition 5.5. *Given $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma \geq 0$ and $\lambda \geq 0$, let g be a function defined on B_N, satisfying:*

$$\exists C > 0 \qquad \textit{such that} \qquad M_{\mathcal{Y}_{\alpha,\lambda+\beta,\gamma}}(gu) \leq C\, M_{\mathcal{X}_\lambda}(u) \qquad \forall u \in \mathcal{S}\mathcal{X}_\lambda^+.$$

Then $g \in \mathcal{Y}_{\alpha,\beta,\gamma}$.

Proof. Let $u \in \mathcal{X}_\lambda$ be given by: $u(x) = (1-|x|^2)^{-\lambda}$ $\forall x \in B_N$. Its subharmonicity follows from $\lambda \geq 0$ (see Lemma 6.4). Then

$$(1-|x|^2)^\beta g(x) = (1-|x|^2)^{\beta+\lambda} g(x)\, u(x) \qquad \forall x \in B_N.$$

Hence $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(g) = M_{\mathcal{Y}_{\alpha,\lambda+\beta,\gamma}}(gu) \leq C$ since $M_{\mathcal{X}_\lambda}(u) = 1$. ∎

6. ANNEX: SOME TECHNICAL AUXILIARY RESULTS.

Lemma 6.1. *Given $a \in B_N$ and $R \in]0,1[$, let R_a and $B(a,R_a)$ be as in Definition 3.2. Then the following holds for any $x \in B(a,R_a)$:*

$$\frac{1}{2} < \frac{1-\langle x,a\rangle}{1-|x|^2} < 2\,\frac{1+R}{1-R}$$

$$\frac{1}{2} < \frac{1-\langle x,a\rangle}{1-|a|^2} < 2$$

and

$$\frac{1-R}{1+R} \leq \frac{1-|x|^2}{1-|a|^2} \leq 2.$$

Moreover the volume V_a of $B(a,R_a)$ satisfies:

$$V_a = \frac{\sigma_N}{N} R_a^N \geq \frac{\sigma_N}{N}\left(\frac{R}{1+R}\right)^N (1-|a|^2)^N$$

with σ_N as in Definition 3.3.

Proof. Cauchy–Schwarz inequality leads to:

$$1-\langle x,a\rangle \geq 1-|x| = \frac{1-|x|^2}{1+|x|} > \frac{1-|x|^2}{2}.$$

See [6] and [7] for the other results. ∎

Lemma 6.2. *Given $A>0$ and $T>-1$, let*

$$I_{A,T}(a) = \int_{B_N} \frac{(1-|x|^2)^T}{(1-\langle x,a\rangle)^A}\,dx \qquad \forall a \in B_N.$$

If $2 \leq A \leq N+2(T+1)$ then

$$I_{A,T}(a) \leq K_0 \left(\frac{1}{1-|a|^2}\right)^{\frac{A+1}{2}} \qquad \forall a \in B_N$$

where

$$K_0 = \frac{\Gamma(T+1)}{\Gamma(\frac{A}{2})}\,\pi^{N/2}.$$

If $A = N+2(T+1)$, then equality holds in the above formula.

Proof. Without any restriction, it may be assumed that $a = (|a|,0,0,...,0)$. Polar coordinates in $I\!R^N$ provide $x_1 = r\,\cos\theta_1$ with $r=|x|$, $\theta_1 \in]0,\pi[$ if $N \geq 3$ and $\theta_1 \in]0,2\pi[$ if $N=2$. Let $d\sigma^{(N)}$ be the area element on the unit sphere S_N of $\mathbb{R}^N$. Now $d\sigma^{(N)} = (\sin\theta_1)^{N-2} d\theta_1\, d\sigma^{(N-1)}$ (polar formulas are more detailed in [11] p.15), thus

$$\begin{aligned} I_{A,T}(a) &= \int_{B_N} \frac{(1-r^2)^T\, r^{N-1}\, dr\, d\sigma^{(N)}}{(1-|a|\,r\,\cos\theta_1)^A} \\ &= \sigma_{N-1} \int_0^1 \int_0^\pi \frac{(1-r^2)^T\, r^{N-1}\,(\sin\theta_1)^{N-2} d\theta_1\, dr}{(1-|a|\,r\,\cos\theta_1)^A} \qquad (6.1) \end{aligned}$$

when $N \geq 3$ (the case $N=2$ will be studied later). For σ_N, see Definition 3.3. Let $s = r\,\cos\theta_1$. From the known expansion

$$\frac{1}{(1-|a|\,s)^A} = \sum_{\ell \in \mathbb{N}} \frac{\Gamma(\ell+A)}{\ell!\,\Gamma(A)}\,(|a|\,s)^\ell$$

we obtain:

$$I_{A,T}(a) = \sigma_{N-1} \sum_{\ell \in \mathbb{N}} \frac{\Gamma(\ell + A)}{\ell!\, \Gamma(A)} \, |a|^\ell \int_0^1 \int_0^\pi (1 - r^2)^T \, r^{N-1} \, s^\ell \, (\sin\theta_1)^{N-2} d\theta_1 \, dr.$$

With $t = r\,\sin\theta_1$, this double integral turns into $\iint_H s^\ell \, t^{N-2} (1 - s^2 - t^2)^T ds\, dt$ where

$$H = \{(s,t) \in I\!R^2 : t \geq 0,\; s^2 + t^2 < 1\}.$$

This integral has been computed in [6], whence

$$I_{A,T}(a) = \sigma_{N-1} \sum_{k \in \mathbb{N}} \frac{\Gamma(2k + A)}{\Gamma(2k+1)\, \Gamma(A)} \, |a|^{2k} \, \frac{\Gamma(k + \frac{1}{2})\, \Gamma(\frac{N-1}{2})\, \Gamma(T+1)}{2\, \Gamma(\frac{N}{2} + k + T + 1)}.$$

The duplication formula $\sqrt{\pi}\, \Gamma(2z) = 2^{2z-1}\, \Gamma(z)\, \Gamma(z + \frac{1}{2})$ (see [3] p.3), applied successively with $2z = 2k + A$ and $2z = 2k + 1$, leads to:

$$I_{A,T}(a) = \sigma_{N-1} \sum_{k \in \mathbb{N}} \frac{2^{A-1}\, \Gamma(k + \frac{A}{2})\, \Gamma(k + \frac{A+1}{2})}{k!\, \Gamma(A)} \, |a|^{2k} \, \frac{\Gamma(\frac{N-1}{2})\, \Gamma(T+1)}{2\, \Gamma(\frac{N}{2} + k + T + 1)}.$$

The function Γ is increasing on $[1, +\infty[$ and $1 \leq k + \frac{A}{2} \leq \frac{N}{2} + k + T + 1$ thus

$$\Gamma(k + \tfrac{A}{2}) \leq \Gamma(\tfrac{N}{2} + k + T + 1),$$

with equality in the case $A = N + 2(T+1)$. It follows that:

$$I_{A,T}(a) \leq \sigma_{N-1} \frac{2^{A-1}\, \Gamma(\frac{N-1}{2})\, \Gamma(T+1)\, \Gamma(\frac{A+1}{2})}{2\, \Gamma(A)} \sum_{k \in \mathbb{N}} \frac{\Gamma(k + \frac{A+1}{2})}{k!\, \Gamma(\frac{A+1}{2})} \, |a|^{2k}.$$

Now $\frac{1}{2}\, \sigma_{N-1}\, \Gamma(\frac{N-1}{2}) = \pi^{(N-1)/2}$ and

$$\frac{2^{A-1}\, \Gamma(\frac{A+1}{2})}{\Gamma(A)} = \frac{\sqrt{\pi}}{\Gamma(\frac{A}{2})}$$

through duplication formula applied with $2z = A$. Hence the conclusion follows in the case $N \geq 3$. When $N = 2$, we have

$$I_{A,T}(a) = \int_0^1 \int_0^{2\pi} \frac{(1 - r^2)^T \, r \, d\theta_1 \, dr}{(1 - |a|\, r\, \cos\theta_1)^A}$$

but the inner integral is equal to

$$2 \int_0^\pi \frac{(1 - r^2)^T \, r \, d\theta_1}{(1 - |a|\, r\, \cos\theta_1)^A},$$

so that the above formulas, from (6.1) on, all still hold, since $\sigma_1 = 2$. ∎

Lemma 6.3. *Given $A > 0$, $P > 0$ and $T > -1$ satisfying $1 \leq A + P \leq N + 1 + 2T$, let*

$$I_{A,P,T}(a,b) = \int_{B_N} \frac{(1 - |x|^2)^T}{(1 - \langle x, a\rangle)^A \, (1 - \langle x, b\rangle)^P} \, dx \qquad \forall a \in B_N \qquad \forall b \in B_N$$

and τ a number satisfing both

$$\frac{P - A}{2} < \tau < P \qquad \textit{and} \qquad 0 \leq \tau \leq \frac{A + P}{2}.$$

Then there exists $K > 0$ such that:

$$I_{A,P,T}(a,b) \leq \frac{K}{(1 - |a|^2)^{\frac{A+P}{2} - \tau} (1 - |b|^2)^\tau} \qquad \forall a \in B_N \qquad \forall b \in B_N.$$

This result remains valid with $P = 0$ and $\tau = 0$, replacing K by

$$K' = \frac{\Gamma(T+1)}{\Gamma(\frac{A+1}{2})}\,\pi^{\frac{N}{2}}.$$

With $P = \tau = 0$, equality holds in the above majoration when moreover $1 \leq A = N + 1 + 2T$.

Proof. See Theorem 4.1 of [6] and Proposition 5.1 of [6].
For information $K = 2^{P+A-1}\,\frac{\Gamma(T+1)\,\Gamma(P-\tau)}{\Gamma(P)\,\Gamma(A)}\,\Gamma(\frac{A-P}{2}+\tau)\,\pi^{\frac{N-1}{2}}$. ∎

Lemma 6.4. *Given $A \geq 0$, $P \geq 0$ and $a \in B_N$, let f be the function defined on B_N by*

$$f(x) = \frac{1}{(1 - \langle x, a\rangle)^A (1 - |x|^2)^P} \qquad \forall x \in B_N.$$

Then f is subharmonic in B_N.

Proof. For any $j \in \{1, 2, ..., N\}$, the following holds $\forall x \in B_N$:

$$\begin{aligned}
\frac{\partial f_a}{\partial x_j}(x) =&\ a_j\, A\, (1 - \langle x, a\rangle)^{-A-1}(1 - |x|^2)^{-P} + 2P\, x_j (1 - \langle x, a\rangle)^{-A}(1 - |x|^2)^{-P-1} \\
\frac{\partial^2 f_a}{\partial x_j^2}(x) =&\ A(A+1)a_j^2(1 - \langle x, a\rangle)^{-A-2}(1 - |x|^2)^{-P} + \\
&+ 4AP\, x_j a_j (1 - \langle x, a\rangle)^{-A-1}(1 - |x|^2)^{-P-1} + 2P(1 - \langle x, a\rangle)^{-A}(1 - |x|^2)^{-P-1} + \\
&+ 4P(P+1)\, x_j^2\, (1 - \langle x, a\rangle)^{-A}(1 - |x|^2)^{-P-2}.
\end{aligned}$$

Hence

$$\begin{aligned}
(\Delta f)(x) =&\ \frac{|a|^2\, A\, (A+1)}{(1 - \langle x, a\rangle)^{A+2}(1 - |x|^2)^P} + \frac{4AP\langle x, a\rangle}{(1 - \langle x, a\rangle)^{A+1}(1 - |x|^2)^{P+1}} + \\
&+ \frac{2PN}{(1 - \langle x, a\rangle)^A (1 - |x|^2)^{P+1}} + \frac{4P(P+1)|x|^2}{(1 - \langle x, a\rangle)^A (1 - |x|^2)^{P+2}} \\
(\Delta f)(x) \geq&\ (1 - \langle x, a\rangle)^{-A}(1 - |x|^2)^{-P}\left[\tfrac{A^2\,|a|^2}{(1-\langle x,a\rangle)^2} - \tfrac{4AP|a|\,.\,|x|}{(1-\langle x,a\rangle)(1-|x|^2)} + \tfrac{4P^2\,|x|^2}{(1-|x|^2)^2}\right] \\
=&\ (1 - \langle x, a\rangle)^{-A}(1 - |x|^2)^{-P}\left(\frac{A\,|a|}{1 - \langle x, a\rangle} - \frac{2P|x|}{1 - |x|^2}\right)^2 \geq 0.
\end{aligned}$$

The subharmonicity of f follows. ∎

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