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A DIFFERENTIAL SANDWICH THEOREM FOR ANALYTIC FUNCTIONS DEFINED BY THE GENERALIZED SĂLĂGEAN OPERATOR

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ABSTRACT. We obtain some subordination and superordination results involving the generalized Sălăgean differential operator for certain normalized analytic functions in the open unit disk. Our results extend corresponding previously known results.

Key words and phrases: Differential subordination, Differential superordination, Generalized Sălăgean derivative.

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1. INTRODUCTION

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + ...\}.$$

We also consider the class

$$\mathcal{A} = \left\{f \in \mathcal{H} : f(z) = z + a_2 z^2 + ...\right\}.$$

We denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\right\},$$

and which are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Since we use the terms of subordination and superordination, we review here those definitions. Let $f, F \in \mathcal{H}$. The function f is said to be *subordinate* to F, or F is said to be *superordinate* to f, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Since most of the functions considered in this paper and conditions on them are defined uniformly in the unit disk U, we shall omit the requirement "$z \in U$".

Let $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let h be univalent in U and $q \in \mathcal{Q}$. In [3], the authors considered the problem of determining conditions on admissible functions ψ such that

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z) \tag{1.1}$$

implies $p(z) \prec q(z)$, for all functions $p \in \mathcal{H}[a,n]$ that satisfy the differential subordination (1.1). Moreover, they found conditions so that the function q is the "smallest" function with this property, called the best dominant of the subordination (1.1).

Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let $h \in \mathcal{H}$ and $q \in \mathcal{H}[a,n]$. Recently, in [4], the authors studied the dual problem and determined conditions on φ such that

$$h(z) \prec \varphi\left(p(z), zp'(z), z^2p''(z); z\right) \tag{1.2}$$

implies $q(z) \prec p(z)$, for all functions $p \in \mathcal{Q}$ that satisfy the above differential superordination. Moreover, they found conditions so that the function q is the "largest" function with this property, called the best subordinant of the superodination (1.2).

For two functions $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $\lambda > 0$. The generalized Sălăgean derivative of a function f is defined in [1] by

$$D_\lambda^o f(z) = f(z), D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z), D_\lambda^m f(z) = D_\lambda^1\left(D_\lambda^{m-1} f(z)\right).$$

If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, then we write the generalized Sălăgean derivative as a Hadamard product

$$D_\lambda^m f(z) = f(z) * \left\{z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^m z^n\right\} = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^m a_n z^n.$$

When $\lambda = 1$, we get the classic Sălăgean derivative [7], denoted by $D^m f(z)$.

In this paper we will determine some properties on admissible functions defined with the generalized Sălăgean derivative.

2. Preliminaries

In our present investigation we shall need the folllowing results.

Theorem 2.1 ([3], Theorem 3.4h., p.132)**.** *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z) \cdot \phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either*

(i) *h is convex or*
(ii) *Q is starlike.*
In addition, assume that
(iii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} > 0$.

If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta[p(z)] + zp'(z) \cdot \phi[p(z)] \prec \theta[q(z)] + zp'(z) \cdot \phi[q(z)] = h(z),$$

then $p \prec q$, and q is the best dominant.

By taking $\theta(w) := w$ and $\phi(w) := \gamma$ in Theorem 2.1, we get

Corollary 2.2. *Let q be univalent in U, $\gamma \in \mathbb{C}^*$ and suppose*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\operatorname{Re}\frac{1}{\gamma}\right\}.$$

If p is analytic in U, with $p(0) = q(0)$ and

$$p(z) + \gamma zp'(z) \prec q(z) + \gamma zq'(z),$$

then $p \prec q$, and q is the best dominant.

Theorem 2.3 ([5])**.** *Let θ and ϕ be analytic in a domain D and let q be univalent in U, with $q(0) = a$, $q(U) \subset D$. Set $Q(z) = zq'(z) \cdot \phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that*

(i) $\operatorname{Re}\left[\dfrac{\theta'[q(z)]}{\phi[q(z)]}\right] > 0$ *and*
(ii) *$Q(z)$ is starlike.*

If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, $p(U) \subset D$ and $\theta[p(z)] + zp'(z) \cdot \phi[p(z)]$ is univalent in U, then

$$\theta[q(z)] + zp'(z) \cdot \phi[q(z)] \prec \theta[p(z)] + zp'(z) \cdot \phi[p(z)] \Rightarrow q \prec p$$

and q is the best subordinant.

By taking $\theta(w) := w$ and $\phi(w) := \gamma$ in Theorem 2.3, we get

Corollary 2.4 ([2])**.** *Let q be convex in U, $q(0) = a$ and $\gamma \in \mathbb{C}$, $\operatorname{Re}\gamma > 0$. If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$ and $p(z) + \gamma zp'(z)$ is univalent in U, then*

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z) \Rightarrow q \prec p$$

and q is the best subordinant.

3. MAIN RESULTS

Theorem 3.1. *Let q be univalent in U with $q(0)=1$, $\alpha\in\mathbb{C}^*$, $\delta>0$ and suppose*

$$\operatorname{Re}\left[1+\frac{zq''(z)}{q'(z)}\right]>\max\left\{0,-\operatorname{Re}\frac{\delta}{\alpha}\right\}.$$

If $f\in\mathcal{A}$ satisfies the subordination

$$\left(1-\frac{\alpha}{\lambda}\right)\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta+\frac{\alpha}{\lambda}\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta\cdot\frac{D_\lambda^{m+1}f(z)}{D_\lambda^m f(z)}\prec q(z)+\frac{\alpha}{\delta}zq'(z), \tag{3.1}$$

then

$$\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta\prec q(z)$$

and q is the best dominant.

Proof. We define the function

$$p(z):=\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta.$$

By calculating the logarithmic derivative of p, we obtain

$$\frac{zp'(z)}{p(z)}=\delta\left(\frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)}-1\right) \tag{3.2}$$

Because the generalised Sălăgean derivative satisfies the identity

$$z\left(D_\lambda^m f(z)\right)'=\frac{1}{\lambda}D_\lambda^{m+1}f(z)+\left(1-\frac{1}{\lambda}\right)D_\lambda^m f(z) \tag{3.3}$$

equation (3.2) becomes

$$\frac{zp'(z)}{p(z)}=\frac{\delta}{\lambda}\left(\frac{D_\lambda^{m+1}f(z)}{D_\lambda^m f(z)}-1\right)$$

and, therefore,

$$\frac{zp'(z)}{\delta}=\frac{1}{\lambda}\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta\left(\frac{D_\lambda^{m+1}f(z)}{D_\lambda^m f(z)}-1\right).$$

The subordination (3.1) from the hypothesis becomes

$$p(z)+\frac{\alpha}{\delta}zp'(z)\prec q(z)+\frac{\alpha}{\delta}zq'(z).$$

We apply now Corrolary 2.4 with $\gamma=\dfrac{\alpha}{\delta}$ to obtain the conclusion of our theorem. □

If we consider $m=0$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let q be univalent in U with $q(0)=1$, $\alpha\in\mathbb{C}^*$, $\delta>0$ and suppose*

$$\operatorname{Re}\left[1+\frac{zq''(z)}{q'(z)}\right]>\max\left\{0,-\operatorname{Re}\frac{\delta}{\alpha}\right\}.$$

If $f\in\mathcal{A}$ satisfies the subordination

$$\left(1-\frac{\alpha}{\lambda}\right)\left(\frac{f(z)}{z}\right)^\delta+\frac{\alpha}{\lambda}\left(\frac{f(z)}{z}\right)^\delta\cdot\frac{(1-\lambda)f(z)+\lambda zf'(z)}{f(z)}\prec q(z)+\frac{\alpha}{\delta}zq'(z), \tag{3.4}$$

then

$$\left(\frac{f(z)}{z}\right)^\delta\prec q(z)$$

and q is the best dominant.

For $\lambda = 1$ in Theorem 3.1 we get the following corrolary.

Corollary 3.3. *Let q be univalent in U with $q(0) = 1$, $\alpha \in \mathbb{C}^*$, $\delta > 0$ and suppose*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\operatorname{Re}\frac{\delta}{\alpha}\right\}.$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1-\alpha)\left(\frac{D^m f(z)}{z}\right)^\delta + \alpha\left(\frac{D^m f(z)}{z}\right)^\delta \cdot \frac{D^{m+1} f(z)}{D^m f(z)} \prec q(z) + \frac{\alpha}{\delta} z q'(z),$$

then

$$\left(\frac{D^m f(z)}{z}\right)^\delta \prec q(z)$$

and q is the best dominant.

If we take $m = 0$ and $\lambda = 1$ in Theorem 3.1, then we obtain the next result.

Corollary 3.4 ([6]). *Let q be univalent in U with $q(0) = 1$, $\alpha \in \mathbb{C}^*$, $\delta > 0$ and suppose*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\operatorname{Re}\frac{\delta}{\alpha}\right\}.$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^\delta + \alpha\left(\frac{f(z)}{z}\right)^\delta \cdot \frac{zf'(z)}{f(z)} \prec q(z) + \frac{\alpha}{\delta} z q'(z),$$

then

$$\left(\frac{f(z)}{z}\right)^\delta \prec q(z)$$

and q is the best dominant.

We consider a particular convex function $q(z) = \dfrac{1+Az}{1+Bz}$ to give the following application to Theorem 3.1.

Corollary 3.5. *Let $A, B, \alpha \in \mathbb{C}$, $A \neq B$ be such that $|B| \leq 1$, $\operatorname{Re}\alpha > 0$ and let $\delta > 0$. If $f \in \mathcal{A}$ satisfies the subordination*

$$\left(1-\frac{\alpha}{\lambda}\right)\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta + \frac{\alpha}{\lambda}\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta \cdot \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{\delta}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \dfrac{1+Az}{1+Bz}$ is the best dominant.

The next theorem is a result concerning a differential superodination.

Theorem 3.6. *Let q be convex in U with $q(0) = 1$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, $\delta > 0$. If $f \in \mathcal{A}$ such that $\left(\dfrac{D_\lambda^m f(z)}{z}\right)^\delta \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $\left(1-\dfrac{\alpha}{\lambda}\right)\left(\dfrac{D_\lambda^m f(z)}{z}\right)^\delta + \dfrac{\alpha}{\lambda}\left(\dfrac{D_\lambda^m f(z)}{z}\right)^\delta \cdot \dfrac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)}$ is univalent in U and satisfies the superordination*

$$q(z) + \frac{\alpha}{\delta} z q'(z) \prec \left(1-\frac{\alpha}{\lambda}\right)\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta + \frac{\alpha}{\lambda}\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta \cdot \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)}, \tag{3.5}$$

then

$$q(z) \prec \left(\frac{D_\lambda^m f(z)}{z}\right)^\delta$$

and q is the best subordinant.

Corollary 3.7. *Let q be convex in U with $q(0)=1$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, $\delta > 0$. If $f \in \mathcal{A}$ such that $\left(\frac{f(z)}{z}\right)^\delta \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $\left(1-\frac{\alpha}{\lambda}\right)\left(\frac{f(z)}{z}\right)^\delta + \frac{\alpha}{\lambda}\left(\frac{f(z)}{z}\right)^\delta \cdot \frac{(1-\lambda)f(z)+\lambda z f'(z)}{f(z)}$ is univalent in U and satisfies the superordination*

$$q(z) + \frac{\alpha}{\delta} z q'(z) \prec \left(1-\frac{\alpha}{\lambda}\right)\left(\frac{f(z)}{z}\right)^\delta + \frac{\alpha}{\lambda}\left(\frac{f(z)}{z}\right)^\delta \cdot \frac{(1-\lambda)f(z)+\lambda z f'(z)}{f(z)},$$

then

$$q(z) \prec \left(\frac{f(z)}{z}\right)^\delta$$

and q is the best subordinant.

Corollary 3.8. *Let q be convex in U with $q(0)=1$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, $\delta > 0$. If $f \in \mathcal{A}$ such that $\left(\frac{D^m f(z)}{z}\right)^\delta \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $(1-\alpha)\left(\frac{D^m f(z)}{z}\right)^\delta + \alpha\left(\frac{D^m f(z)}{z}\right)^\delta \cdot \frac{D^{m+1} f(z)}{D^m f(z)}$ is univalent in U and satisfies the superordination*

$$q(z) + \frac{\alpha}{\delta} z q'(z) \prec (1-\alpha)\left(\frac{D^m f(z)}{z}\right)^\delta + \alpha\left(\frac{D^m f(z)}{z}\right)^\delta \cdot \frac{D^{m+1} f(z)}{D^m f(z)},$$

then

$$q(z) \prec \left(\frac{D^m f(z)}{z}\right)^\delta$$

and q is the best subordinant.

Concluding the results of differential subordination and superordination we state the following sandwich result.

Theorem 3.9. *Let q_1, q_2 be convex in U with $q_1(0) = q_2(0) = 1$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, $\delta > 0$. If $f \in \mathcal{A}$ such that $\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $\left(1-\frac{\alpha}{\lambda}\right)\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta + \frac{\alpha}{\lambda}\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta \cdot \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)}$ is univalent in U and satisfies*

$$q_1(z) + \frac{\alpha}{\delta} z q_1'(z) \prec \left(1-\frac{\alpha}{\lambda}\right)\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta + \frac{\alpha}{\lambda}\left(\frac{D_\lambda^m f(z)}{z}\right)^\delta \cdot \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} \prec q_2(z) + \frac{\alpha}{\delta} z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{D_\lambda^m f(z)}{z}\right)^\delta \prec q_2(z)$$

and q_1, q_2 are the best subordinant and the best dominant respectively.

Corollary 3.10. *Let q_1, q_2 be convex in U with $q_1(0) = q_2(0) = 1$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, $\delta > 0$. If $f \in \mathcal{A}$ such that $\left(\frac{f(z)}{z}\right)^\delta \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $\left(1-\frac{\alpha}{\lambda}\right)\left(\frac{f(z)}{z}\right)^\delta + \frac{\alpha}{\lambda}\left(\frac{f(z)}{z}\right)^\delta \cdot$*

$\frac{(1-\lambda) f(z)+\lambda z f'(z)}{f(z)}$ *is univalent in* U *and satisfies*

$$q_1(z)+\frac{\alpha}{\delta} z q_1'(z) \prec \left(1-\frac{\alpha}{\lambda}\right)\left(\frac{f(z)}{z}\right)^{\delta}+\frac{\alpha}{\lambda}\left(\frac{f(z)}{z}\right)^{\delta} \cdot \frac{(1-\lambda) f(z)+\lambda z f'(z)}{f(z)} \prec q_2(z)+\frac{\alpha}{\delta} z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{f(z)}{z}\right)^{\delta} \prec q_2(z)$$

and q_1, q_2 *are the best subordinant and the best dominant respectively.*

Corollary 3.11. *Let* q_1, q_2 *be convex in* U *with* $q_1(0)=q_2(0)=1$, $\alpha \in \mathbb{C}$ *with* $\operatorname{Re} \alpha>0$, $\delta>0$. *If* $f \in \mathcal{A}$ *such that* $\left(\frac{D^m f(z)}{z}\right)^{\delta} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $(1-\alpha)\left(\frac{D^m f(z)}{z}\right)^{\delta}+\alpha\left(\frac{D^m f(z)}{z}\right)^{\delta} \cdot \frac{D^{m+1} f(z)}{D^m f(z)}$ *is univalent in* U *and satisfies*

$$q_1(z)+\frac{\alpha}{\delta} z q_1'(z) \prec(1-\alpha)\left(\frac{D^m f(z)}{z}\right)^{\delta}+\alpha\left(\frac{D^m f(z)}{z}\right)^{\delta} \cdot \frac{D^{m+1} f(z)}{D^m f(z)} \prec q_2(z)+\frac{\alpha}{\delta} z q_2'(z),$$

then

$$q_1(z) \prec\left(\frac{D^m f(z)}{z}\right)^{\delta} \prec q_2(z)$$

and q_1, q_2 *are the best subordinant and the best dominant respectively.*

REFERENCES

[1] F.M. AL-OBOUDI, On univalent functions defined by a generalized Sălăgean operator, *International Journal of Mathematics and Mathematical Sciences*, **2004**(2004), No. 27, pp. 1429–1436.

[2] T. BULBOACĂ, Classes of first order differential superordinations, *Demonstr. Math.*, **35**(2002), No.2, pp. 287–292.

[3] S.S. MILLER and P.T. MOCANU, *Differential Subordinations. Theory and Applications*, Marcel Dekker, Inc., New York, Basel, 2000.

[4] S.S. MILLER and P.T. MOCANU, Subordinants of differential superordinations, *Complex Variables*, **48**(2003), No.10, pp. 815–826.

[5] S.S. MILLER and P.T. MOCANU, Briot-Bouquet differential superordinations and sandwich theorems, *J. Math.Anal.Appl.*, **329**(2007), No.1, pp. 327–335.

[6] G. MURUGUSUNDARAMOORTHY and N. MAGESH, Differential subordinations and superordinations for analytic functions defined by the Dziok-Srivastava linear operator, *JIPAM*, **7**(2006), No.4, Art.152. [Online: http://www.emis.de/journals/JIPAM/article771.html?sid=771].

[7] GR.ŞT. SĂLĂGEAN, Subclasses of univalent functions, Complex analysis - Proc. 5th Rom.-Finn. Semin., Bucharest 1981, Part 1, *Lect. Notes Math.*, **1013**(1983), pp. 362–372.