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ELLIPSES INSCRIBED IN PARALLELOGRAMS

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ABSTRACT. We prove that there exists a unique ellipse of minimal eccentricity, E_I, inscribed in a parallelogram, Ð. We also prove that the smallest nonnegative angle between equal conjugate diameters of E_I equals the smallest nonnegative angle between the diagonals of Ð. We also prove that if E_M is the unique ellipse inscribed in a rectangle, $R,$ which is tangent at the midpoints of the sides of R, then E_M is the unique ellipse of minimal eccentricity, maximal area, and maximal arc length inscribed in R. Let Ð be any convex quadrilateral. In previous papers, the author proved that there is a unique ellipse of minimal eccentricity, E_I, inscribed in Ð, and a unique ellipse, E_O, of minimal eccentricity circumscribed about Ð. We defined Ð to be bielliptic if E_I and E_O have the same eccentricity. In this paper we show that a parallelogram, Ð, is bielliptic if and only if the square of the length of one of the diagonals of Ð equals twice the square of the length of one of the sides of Ð.

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1. INTRODUCTION

In [1] the author proved numerous results about ellipses inscribed in convex quadrilaterals, Đ. In particular, we proved that there exists a unique ellipse of **minimal eccentricity**, E_I, inscribed in Đ. In this paper, we discuss in detail the special case of ellipses inscribed in parallelograms. In particular, in § 2 we give a direct proof(see Proposition 2.3) that there is a unique ellipse, E_I, of minimal eccentricity *inscribed* in any given parallelogram, Đ. Our main result in this regard is to give a geometric characterization of E_I for parallelograms(see Theorem 2.4), where we prove that the smallest nonnegative angle between equal conjugate diameters of E_I equals the smallest nonnegative angle between the diagonals of Đ. Similar results are known for the unique ellipse, E_A, of **maximal area** inscribed in a parallelogram, Đ(see, for example, [5]). Then the equal conjugate diameters of E_A are *parallel* to the diagonals of Đ. It is not too hard to prove this by proving the corresponding result for the unit square and then using an affine transformation. This approach works because of the affine invariance of the ratios of corresponding areas. Since the eccentricity is not affine invariant, we cannot reduce the problem of the minimal eccentricity ellipse inscribed in a parallelogram to ellipses inscribed in squares.

In § 3 we discuss ellipses inscribed in rectangles. We prove(see Theorem 3.1) that if E_M is the unique ellipse inscribed in a rectangle, R, which is tangent at the *midpoints* of the sides of R, then E_M is the unique ellipse of minimal eccentricity, maximal area, and maximal arc length inscribed in R. While parts of Theorem 3.1 are known, this overall characterization appears to be new. Of course, it then follows by affine invariance that the unique ellipse of maximal area inscribed in a parallelogram, Đ, is tangent at the midpoints of the sides of Đ. The other parts of Theorem 3.1 do not hold in general for parallelograms, however.

In ([2], Proposition 1) the author proved that there is a unique ellipse, E_O, of minimal eccentricity circumscribed about any convex quadrilateral, Đ. Also, in [2] the author defined Đ to be bielliptic if E_I and E_O have the same eccentricity. In § 4 we show(Theorem 4.1) that a parallelogram, Đ, is bielliptic if and only if the square of the length of one of the diagonals of Đ equals twice the square of the length of one of the sides of Đ.

The following general result about ellipses is essentially what appears in [8], except that the cases with $A = B$ were added by the author.

Lemma 1.1. *Let a and b denote the lengths of the semi–major and semi–minor axes, respectively, of the ellipse, E, with equation $Ax^2 + By^2 + 2Cxy + Dx + Ey + F = 0$, with $A, B, AB - C^2, AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF > 0$. Then*

$$a^2 = \frac{AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF}{2(AB - C^2)\left(A + B - \sqrt{(B-A)^2 + 4C^2}\right)} \tag{1.1}$$

and

$$b^2 = \frac{AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF}{2(AB - C^2)\left(A + B + \sqrt{(B-A)^2 + 4C^2}\right)}. \tag{1.2}$$

2. MINIMAL ECCENTRICITY

Lemma 2.1. *Let Z be the rectangle with vertices $(0,0), (l,0), (0,k)$, and (l,k), where $l, k > 0$.*
(A) The general equation of an ellipse, Ψ, inscribed in Z is given by

$$k^2x^2 + l^2y^2 - 2l(k - 2v)xy - 2lkvx - 2l^2vy + l^2v^2 = 0, 0 < v < k. \tag{2.1}$$

The corresponding points of tangency of Ψ *are*

$$\left(\frac{lv}{k},0\right),(0,v),\left(\frac{l}{k}(k-v),k\right), \textit{and } (l,k-v). \tag{2.2}$$

(B) If a *and* b *denote the lengths of the semi–major and semi–minor axes, respectively, of* Ψ*, then*

$$\begin{aligned} a^2 &= \frac{2l^2(k-v)v}{k^2+l^2-\sqrt{(k^2+l^2)^2-16l^2(k-v)v}} \textit{ and} \\ b^2 &= \frac{2l^2(k-v)v}{k^2+l^2+\sqrt{(k^2+l^2)^2-16l^2(k-v)v}}. \end{aligned} \tag{2.3}$$

Proof. Let S be the unit square with vertices $(0,0),(0,1),(1,0)$, and $(1,1)$. The map $T(x,y)=\left(\frac{1}{l}x,\frac{1}{k}y\right)$ maps Z onto S and Ψ onto an ellipse, $T(\Psi)$. Denote the points of tangency of $T(\Psi)$ with S by $T_1=(t,0),T_2=(0,w),T_3=(s,1)$, and $T_4=(1,u)$, where $\{t,w,s,u\}\subseteq(0,1)$. We may assume that the general equation of $T(\Psi)$ has the form $Ax^2+By^2+2Cxy+Dx+Ey+F=0$ with $A,B>0$. Since $T(\Psi)$ passes thru the points of tangency, we have the equations

$$\begin{aligned} At^2+Dt+F &= 0, Bw^2+Ew+F=0 \\ As^2+B+2Cs+Ds+E+F &= 0, A+Bu^2+2Cu+D+Eu+F=0 \end{aligned} \tag{2.4}$$

Using $y'=-\dfrac{2Ax+2Cy+D}{2By+2Cx+E}$, $y'(T_1)=y'(T_3)=0$ and the fact that the tangents at T_2 and at T_4 are vertical, we also have the equations

$$\begin{aligned} 2At+D &= 0, 2Bw+E=0 \\ 2As+2C+D &= 0, 2Bu+2C+E=0. \end{aligned} \tag{2.5}$$

Solving (2.4) and (2.5) for B thru F,s,t, and u in terms of A and w yields

$$s=u=1-w, t=w, B=A, C=2Aw-A, D=-2Aw, E=-2Aw, F=Aw^2.$$

The equation of $T(\Psi)$ is then $x^2+y^2+2(2w-1)xy-2wx-2wy+w^2=0$. The corresponding points of tangency of $T(\Psi)$ are thus $(w,0),(0,w),(1-w,1),(1,1-w)$. To obtain the corresponding equation of Ψ, replace x by $\frac{1}{l}x$ and y by $\frac{1}{k}y$. After simplifying that yields $k^2x^2+l^2y^2+2kl(2w-1)xy-2k^2lwx-2kl^2wy+k^2l^2w^2=0$. The corresponding points of tangency of Ψ are $T^{-1}(w,0)=(lw,0)$, $T^{-1}(0,w)=(0,kw)$, $T^{-1}(1-w,1)=(l(1-w),k)$, and $T^{-1}(1,1-w)=(l,k(1-w))$. Now let $v=kw$ to obtain (2.1) and (2.2). (2.3) follows easily from Lemma 1.1, (1.1), and (1.2). ∎

We now prove a version of Lemma 2.1 for parallelograms.

Proposition 2.2. *Let Đ be the parallelogram with vertices* $O=(0,0),P=(l,0),Q=(d,k)$*, and* $R=(l+d,k)$*, where* $l,k>0,d\geq 0$*.*

(A) The general equation of an ellipse, Ψ*, inscribed in Đ is given by*

$$\begin{aligned} &k^3x^2+\left(k(d+l)^2-4dlv\right)y^2-2k(kd-2lv+kl)xy \\ &-2k^2lvx+2klv(d-l)y+kl^2v^2=0, 0<v<k. \end{aligned} \tag{2.6}$$

(B) If a *and* b *denote the lengths of the semi–major and semi–minor axes, respectively, of* Ψ*, then*

$$\frac{b^2}{a^2}=1+\frac{m(v)+\left[4dlv-k\left((d+l)^2+k^2\right)\right]\sqrt{m(v)}}{8k^2l^2(k-v)v}, \tag{2.7}$$

where

$$m(v) = 16l^2\left(d^2+k^2\right)v^2 - 8lk\left(dk^2+d^3+2ld^2+l^2d+2k^2l\right)v + k^2\left((d+l)^2+k^2\right)^2. \tag{2.8}$$

Remark 2.1. To be more precise, (A) means that any ellipse *inscribed* in Đ has an equation of the form (2.6) for some $0 < v < k$, and that any conic with an equation of the form (2.6) for some $0 < v < k$ defines an ellipse *inscribed* in Đ.

Proof. Let Z be the rectangle with vertices $(0,0),(0,k),(l,0)$, and (l,k). The map $T(x,y) = \left(x - \frac{d}{k}y, y\right)$ maps Đ onto Z. By Lemma 2.1, the general equation of $T(\Psi)$ is given by (2.1), with x replaced by $x - \frac{d}{k}y$ and y remaining the same. That yields $k^2\left(x-\frac{d}{k}y\right)^2 + l^2y^2 - 2l(k-2v)\left(x-\frac{d}{k}y\right)y - 2lkv\left(x-\frac{d}{k}y\right) - 2l^2vy + l^2v^2 = 0$, and some simplification gives (2.6). To prove (B), by Lemma 1.1, (1.1) and (1.2), $\frac{b^2}{a^2} = \frac{(A+B)-\sqrt{(B-A)^2+4C^2}}{(A+B)+\sqrt{(B-A)^2+4C^2}} = \frac{\left[(A+B)-\sqrt{(B-A)^2+4C^2}\right]^2}{(A+B)^2-((B-A)^2+4C^2)}$, or

$$\frac{b^2}{a^2} = \frac{(A+B)^2+(B-A)^2+4C^2-2(A+B)\sqrt{(B-A)^2+4C^2}}{4(AB-C^2)}. \tag{2.9}$$

Let

$$\begin{aligned} A &= k^3, B = k(d+l)^2 - 4dlv, C = -k\left(kd-2lv+kl\right), \\ D &= -2k^2lv, E = 2klv\left(d-l\right), \text{ and } F = kl^2v^2. \end{aligned} \tag{2.10}$$

(2.9) and (2.10) then yield (2.7). ■

Proposition 2.3. *Let Đ be a parallelogram in the xy plane. Then there is a unique ellipse, E_I, of minimal eccentricity inscribed in Đ.*

Proof. By using an isometry of the plane, we may assume that the vertices of Đ are $O = (0,0), P = (l,0), Q = (d,k)$, and $R = (l+d,k)$, where $l,k > 0, d \geq 0$. Let E denote any ellipse inscribed in Đ and let a and b denote the lengths of the semi–major and semi–minor axes, respectively, of E. Let

$$\begin{aligned} g(v) &= \frac{m(v) + \left[4dlv - k\left((d+l)^2+k^2\right)\right]\sqrt{m(v)}}{(k-v)\,v}, \\ h(v) &= 1 + \frac{1}{8k^2l^2}g(v). \end{aligned} \tag{2.11}$$

By (2.7) of Proposition 2.2, $h(v) = \frac{b^2}{a^2}$. We shall now minimize the eccentricity by maximizing $\frac{b^2}{a^2}$, or equivalently by maximizing $g(v)$. Now $g'(v) = 0 \iff$

$$(k-v)\,v\left[m'(v) + \left[4dlv - k\left((d+l)^2+k^2\right)\right]\frac{m'(v)}{2\sqrt{m(v)}} + 4dl\sqrt{m(v)}\right] - \left[m(v) + \left[4dlv - k\left((d+l)^2+k^2\right)\right]\sqrt{m(v)}\right](k-2v) = 0$$

$\iff$

$$(k-v)\,v\left[2\sqrt{m(v)}m'(v)+\left[4dlv-k\left((d+l)^2+k^2\right)\right]m'(v)+8dlm(v)\right]-$$
$$2(k-2v)\sqrt{m(v)}\left[m(v)+\left[4dlv-k\left((d+l)^2+k^2\right)\right]\sqrt{m(v)}\right]=0$$

$\iff$

$$2\,(k-v)\,v\sqrt{m(v)}m'(v)+(k-v)\,v\left(\left[4dlv-k\left((d+l)^2+k^2\right)\right]m'(v)+8dlm(v)\right)$$
$$-2(k-2v)\sqrt{m(v)}m(v)-2(k-2v)m(v)\left[4dlv-k\left((d+l)^2+k^2\right)\right]=0$$

$\iff$

$$(k-v)\,v\left(\left[4dlv-k\left((d+l)^2+k^2\right)\right]m'(v)+8dlm(v)\right)-$$
$$2(k-2v)m(v)\left(4dlv-k\left((d+l)^2+k^2\right)\right)=$$
$$\left[2(k-2v)m(v)-2v\,(k-v)\,m'(v)\right]\sqrt{m(v)}$$

$\iff$

$$-\left[2\left(k^2+l^2+d^2\right)v-k\left(2dl+k^2+l^2+d^2\right)\right]n(v)=$$
$$\left[4dlv-k\left((d+l)^2+k^2\right)\right]\left[2\left(k^2+l^2+d^2\right)v-k\left(2dl+k^2+l^2+d^2\right)\right]\sqrt{m(v)}, \tag{2.12}$$

where $n(v)=m(v)+8k^2l^2\,(k-v)\,v$. If

$$2\left(k^2+l^2+d^2\right)v-k\left(2dl+k^2+l^2+d^2\right)\neq 0,$$

then by (2.12), $g'(v)=0\Rightarrow$

$$\sqrt{m(v)}=\frac{-n(v)}{4dlv-k\left((d+l)^2+k^2\right)}$$

$\Rightarrow$

$$\left(4dlv-k\left((d+l)^2+k^2\right)\right)^2m(v)-n^2(v)=0$$

$\Rightarrow$

$$-64l^4v^2k^4\,(v-k)^2=0,$$

which implies that $v=0$ or $v=k$. Since $0<v<k$ by assumption, that yields no solution. Thus $g'(v)=0$, and hence $h'(v)=0$, if and only if $2\left(k^2+l^2+d^2\right)v-k\left(2dl+k^2+l^2+d^2\right)=0\iff v=v_\epsilon$, where

$$v_\epsilon=\frac{1}{2}k\frac{(d+l)^2+k^2}{k^2+l^2+d^2}. \tag{2.13}$$

It follows easily from l'Hospital's Rule that $\lim\limits_{v\to 0^+}g(v)=\lim\limits_{v\to k^-}g(v)=-8l^2k^2$, which implies that $\lim\limits_{v\to 0^+}h(v)=\lim\limits_{v\to k^-}h(v)=0$. Since $h(v)\geq 0$ for $0<v<k$, h attains its' global maximum at v_ϵ and the eccentricity is minimized when $v=v_\epsilon$. ■

Theorem 2.4. *: Let E_I denote the unique ellipse of minimal eccentricity inscribed in a parallelogram, Đ, in the xy plane. Then the smallest nonnegative angle between equal conjugate diameters of E_I equals the smallest nonnegative angle between the diagonals of Đ.*

Proof. As in the proof of Proposition 2.3, by using an isometry of the plane, we may assume that the vertices of Đ are $O = (0,0), P = (l,0), Q = (d,k)$, and $R = (l+d,k)$, where $l,k > 0, d \geq 0$. The diagonals of Đ are $D_1 = \overline{OR}$ and $D_2 = \overline{PQ}$. We find it convenient to define the following variables:

$$\begin{aligned} G &= (d+l)^2 + k^2, H = (d-l)^2 + k^2, \\ J &= k^2 + l^2 + d^2, I = l^2 - d^2 - k^2. \end{aligned} \tag{2.14}$$

There are three cases to consider: $I > 0, I = 0$(which implies that Đ is a rhombus), and $I < 0$. Assume first that $I > 0$. Then $d^2 + k^2 < l^2$, which implies that $d < l$ as well, and the lines containing D_1 and D_2 have equations $y = \frac{k}{l+d}x$ and $y = \frac{k}{d-l}(x-l)$, respectively. Let ϕ denote the smallest nonnegative angle between D_1 and D_2. We use the formula $\tan\phi = \frac{m_2 - m_1}{1 + m_1 m_2}$, where $m_1 = \frac{k}{d-l} < m_2 = \frac{k}{l+d}$. Some simplification gives

$$\tan\phi = \frac{2kl}{I}. \tag{2.15}$$

Let E_I denote the the unique ellipse from Proposition 2.3 of minimal eccentricity inscribed in Đ, and let L and L' denote a pair of *equal* conjugate diameters of E_I . Let a and b denote the lengths of the semi–major and semi–minor axes, respectively, of E_I. It is known(see, for example, [7]) that L and L' make equal acute angles, on opposite sides, with the major axis of E_I. Let θ denote the acute angle going counterclockwise from the major axis of E_I to one of the equal conjugate diameters, which implies that $\tan\theta = \frac{b}{a}$. We shall show that $\tan^2 2\theta = \tan^2\phi$, which will then easily yield $2\theta = \phi$. By (2.7) of Proposition 2.2, $\frac{b^2}{a^2} = h(v_\epsilon)$, where $h(v)$ is given by (2.11) and v_ϵ is given by (2.13). Thus

$$\tan\theta = \sqrt{h\left(v_\epsilon\right)}.$$

By (2.13),

$$4dlv_\epsilon - k\left((d+l)^2 + k^2\right) = 4dl\frac{1}{2}k\frac{2dl + l^2 + d^2 + k^2}{k^2 + l^2 + d^2} - k\left((d+l)^2 + k^2\right) =$$
$$-\frac{k\left((d+l)^2 + k^2\right)\left((d-l)^2 + k^2\right)}{k^2 + l^2 + d^2} = -\frac{kGH}{J},$$

$$v_\epsilon\left(k - v_\epsilon\right) = \frac{1}{2}k\frac{(d+l)^2 + k^2}{k^2 + l^2 + d^2}\left(k - \frac{1}{2}k\frac{(d+l)^2 + k^2}{k^2 + l^2 + d^2}\right) =$$
$$\frac{1}{4}k^2\left((d+l)^2 + k^2\right)\frac{(d-l)^2 + k^2}{\left(k^2 + l^2 + d^2\right)^2} = \frac{1}{4}\frac{k^2\left((d+l)^2 + k^2\right)\left((d-l)^2 + k^2\right)}{\left(k^2 + l^2 + d^2\right)^2} =$$
$$\frac{1}{4}\frac{k^2 GH}{J^2},$$

and by (2.8), after some simplification,

$$m\left(v_\epsilon\right) = m\left(\frac{1}{2}k\frac{(d+l)^2 + k^2}{k^2 + l^2 + d^2}\right) =$$
$$\frac{k^2\left((d+l)^2 + k^2\right)\left((d-l)^2 + k^2\right)\left(k^2 - l^2 + d^2\right)^2}{\left(k^2 + l^2 + d^2\right)^2} = \frac{k^2 GHI^2}{J^2}.$$

Hence $g(v_\epsilon) = \left(\frac{k^2GHI^2}{J^2} - \frac{kGH}{J} \frac{k\sqrt{GH}I}{J} \right) \frac{4J^2}{k^2GH} = 4I\left(I - \sqrt{G}\sqrt{H}\right)$, which implies, by (2.11), that

$$h(v_\epsilon) = 1 + \frac{I\left(I - \sqrt{G}\sqrt{H}\right)}{2k^2l^2} \tag{2.16}$$

Since $I^2 - GH = (l^2 - d^2 - k^2)^2 - ((d+l)^2 + k^2)((d-l)^2 + k^2) = -4l^2k^2 < 0$, we have $I^2 < GH$, which implies that $I - \sqrt{G}\sqrt{H} < 0$ and thus

$$h(v_\epsilon) < 1. \tag{2.17}$$

Now $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\sqrt{h(v_\epsilon)}}{1-h(v_\epsilon)}$, which implies that $\tan^2 2\theta = \frac{4h(v_\epsilon)}{(1-h(v_\epsilon))^2} = \frac{2k^2l^2 + I\left(I - \sqrt{G}\sqrt{H}\right)}{k^2l^2} \frac{8k^4l^4}{I^2\left(I - \sqrt{G}\sqrt{H}\right)^2} =$

$8k^2l^2 \frac{2k^2l^2 + I\left(I - \sqrt{G}\sqrt{H}\right)}{I^2\left(I - \sqrt{G}\sqrt{H}\right)^2}$. By (2.15), $\tan^2 2\theta = \tan^2\phi \iff$

$$8k^2l^2 \frac{2k^2l^2 + I\left(I - \sqrt{G}\sqrt{H}\right)}{I^2\left(I - \sqrt{G}\sqrt{H}\right)^2} = \frac{4k^2l^2}{I^2}$$

$\iff$

$$4k^2l^2 + 2I\left(I - \sqrt{G}\sqrt{H}\right) = \left(I - \sqrt{G}\sqrt{H}\right)^2$$

$\iff$

$$4k^2l^2 + 2I^2 - 2I\sqrt{GH} = I^2 - 2I\sqrt{GH} + GH$$

$\iff 4k^2l^2 + I^2 = GH \iff 4k^2l^2 + (l^2 - d^2 - k^2)^2 = ((d+l)^2 + k^2)((d-l)^2 + k^2)$, which holds for all $d, k, l \in \Re$. Thus $\tan^2 2\theta = \tan^2\phi$, and since $\tan 2\theta > 0$ and $\tan\phi > 0$ by (2.15) and (2.17), it follows that $\tan 2\theta = \tan\phi$. Now suppose that $I = 0$. One still has $d - l < 0$, but now $\phi = \frac{\pi}{2}$. One can let $I = 0$ in (2.16) above by using a limiting argument. Thus $h(v_\epsilon) = 1$, which gives $2\theta = \frac{\pi}{2}$. We omit the proof in the case when $I < 0$. ∎

Example 2.1. *Let $d = 2$, $l = 5$, and $k = 4$, so that Đ is the parallelogram with vertices $(0,0)$, $(2,4)$, $(7,4)$, and $(5,0)$. The minimal eccentricity of all ellipses inscribed in Đ is $\frac{65-\sqrt{65}}{65+\sqrt{65}} \approx 0.78$ and is attained with $v = \frac{50}{9}$. The equation of E_I is $1296x^2 - 531y^2 + 4464xy - 18000x - 13500y + 62500 = 0$. The common value of 2θ and ϕ equals $\tan^{-1} 8 \approx 82.9°$.*

Remark 2.2. Theorem 2.4 does not extend in general to any convex quadrilateral in the xy plane. For example, consider the convex quadrilateral with vertices $(0,0), (1,0), (0,1)$, and $(4,2)$. Using the formulas from [1], one can show that there are two ellipses *inscribed* in Đ which satisfy $2\theta = \phi$, but neither of those ellipses is the unique ellipse of minimal eccentricity *inscribed* in Đ.

Remark 2.3. A somewhat similar result for the ellipse of maximal area inscribed in a parallelogram, or the ellipse of minimal area circumscribed about a parallelogram, is proven in [6].

If Đ is a convex quadrilateral in the xy plane, the line segment, Z, thru the midpoints of the diagonals of Đ plays an important role–it is the precise locus of centers of ellipses inscribed in Đ. Let L be the line containing Z. There is strong evidence that the following is true.

Conjecture 2.5. *Theorem 2.4 holds for any convex quadrilateral, Đ, with the property that one of the diagonals of Đ is identical with L.*

The details of a proof of this conjecture along the lines of the proof of Theorem 2.4 look messy. It is also possible that there is a similar characterization for E_I for any convex quadrilateral in the xy plane. Such a characterization would perhaps involve the angles between each diagonal of Đ and between L and each diagonal of Đ. However, we have not found such a result which works with any examples.

3. RECTANGLES

The results in this paper have focused on ellipses of minimal eccentricity inscribed in a parallelogram. We now discuss ellipses of minimal eccentricity, maximal area, and maximal arc length inscribed in rectangles. While some of the results in the following theorem are known, the overall characterization appears to be new.

Theorem 3.1. *Let Z be a rectangle in the xy plane. Then there is a unique ellipse inscribed in Z which is tangent at the midpoints of the four sides of Z, which we call the midpoint ellipse, E_M. E_M has the following properties:*
(A) E_M is the unique ellipse of minimal eccentricity inscribed in Z.
(B) E_M is the unique ellipse of maximal area inscribed in Z.
(C) E_M is the unique ellipse of maximal arc length inscribed in Z.

Proof. By using a translation, we may assume that the vertices of Z are $O = (0,0), P = (l,0), Q = (0,k)$, and $R = (l,k)$, where $l, k > 0$. Letting $v = \frac{1}{2}k$ in (2.2) shows the existence of an ellipse inscribed in Z which is tangent at the midpoints of the four sides of Z. The fact that such an ellipse is unique follows easily and we omit the proof. Now let E denote *any* ellipse inscribed in Z and let a and b denote the lengths of the semi–major and semi–minor axes, respectively, of E. To prove (A), as earlier we minimize the eccentricity by maximizing $\frac{b^2}{a^2}$. By (2.3), $\frac{b^2}{a^2} = \frac{k^2+l^2-\sqrt{(k^2+l^2)^2-16l^2(k-v)v}}{k^2+l^2+\sqrt{(k^2+l^2)^2-16l^2(k-v)v}} = 8l^2\tau(v), 0 < v < k$, where

$$\tau(v) = \frac{v(k-v)}{(k^2+l^2)^2 + 8l^2v(v-k) + (k^2+l^2)\sqrt{(k^2+l^2)^2+16l^2v(v-k)}}.$$

A simple computation yields

$$\tau'(v) = \frac{8\left(k^2+l^2\right)l^2(k-2v)}{\sqrt{(k^2+l^2)^2+16l^2v(v-k)}\left((k^2+l^2)^2+8l^2v(v-k)+(k^2+l^2)\sqrt{(k^2+l^2)^2+16l^2v(v-k)}\right)}.$$

Thus $\tau'(v) = 0 \iff v = \frac{1}{2}k$. Since $\tau(0) = \tau(k) = 0$ and $\tau(v) \geq 0$ for $0 < v < k$, τ attains its' global maximum at $v = \frac{1}{2}k$ and the eccentricity is minimized when $E = E_M$. That proves (A). To prove (B), we maximize the area of $E, \pi ab$, by maximizing a^2b^2. By (2.3) again,

$$a^2b^2 = -\frac{4(k-v)^2 l^4 v^2}{\left(\sqrt{l^4+2l^2k^2+k^4+16l^2v^2-16l^2vk}-(k^2+l^2)\right)\left(\sqrt{l^4+2l^2k^2+k^4+16l^2v^2-16l^2vk}+(k^2+l^2)\right)}$$
$$= \frac{4\left(k-v\right)^2 l^4 v^2}{16\left(k-v\right)l^2 v} = \frac{l^2}{4}\left(k-v\right)v.$$

It follows immediately that a^2b^2 attains its' global maximum at $v = \frac{1}{2}k$, which proves (B). To prove (C), the arc length of E is given by

$$L = 2\int_0^{\pi/2} \left[a^2 + b^2 - \left(a^2 - b^2\right)\cos 2t\right]^{1/2} dt. \tag{3.1}$$

The proof we give is very similar to the proof in ([4]) that the ellipse of maximal arc length inscribed in a square is a circle. Indeed, what makes the proof work in ([4]) is that $a^2 + b^2$ does not vary as E varies over all ellipses inscribed in a square. For the rectangle, Z,

$$a^2 + b^2 = 2l^2\left(k-v\right)v\left(\frac{1}{k^2+l^2-\sqrt{(k^2+l^2)^2-16l^2(k-v)v}} + \frac{1}{k^2+l^2+\sqrt{(k^2+l^2)^2-16l^2(k-v)v}}\right)$$
$$= 2l^2\left(k-v\right)v\frac{2\left(k^2+l^2\right)}{16l^2v(k-v)} = \frac{1}{4}\left(k^2+l^2\right),$$

which of course does not vary as E varies over all ellipses inscribed in Z. Now

$$a^2 - b^2 = \frac{4(k-v)l^2v\sqrt{l^4+2l^2k^2+k^4+16l^2v^2-16l^2vk}}{\left(k^2+l^2-\sqrt{l^4+2l^2k^2+k^4+16l^2v^2-16l^2vk}\right)\left(k^2+l^2+\sqrt{l^4+2l^2k^2+k^4+16l^2v^2-16l^2vk}\right)}$$
$$= 4\left(k-v\right)l^2v\frac{\sqrt{l^4+2l^2k^2+k^4+16l^2v^2-16l^2vk}}{16\left(k-v\right)l^2v} = \frac{1}{4}\sqrt{\beta(v)},$$

where $\beta(v) = \left(k^2+l^2\right)^2 - 16l^2v\left(k-v\right)$. Hence by (3.1),

$L = L(v) = \frac{1}{2}\int_0^{\pi/2}\left[k^2+l^2-\sqrt{\beta(v)}\cos 2t\right]^{1/2} dt$. As in ([4]), splitting the integral up and making a change of variable gives

$$L(v) = \int_0^{\pi/4}\left[\left(k^2+l^2-\sqrt{\beta(v)}\cos 2t\right) + \left(k^2+l^2+\sqrt{\beta(v)}\cos 2t\right)\right]^{1/2} dt.$$

Let $p = k^2 + l^2$ and $u(v,t) = \sqrt{\beta(v)}\cos 2t$, which gives

$$L(v) = \int_0^{\pi/4}\left[(p-u(v,t))^{1/2} + (p+u(v,t))^{1/2}\right] dt. \tag{3.2}$$

Now β attains its global minimum on $(0,k)$ when $v = \frac{1}{2}k$. Thus, for each $0 < t < \frac{\pi}{4}$, $u(v,t) \geq u\left(\frac{1}{2}k, t\right)$, with equality if and only if $v = \frac{1}{2}k$. Also, the function $f(x) = (p - x)^{1/2} + (p+x)^{1/2}$ is strictly decreasing for $0 < x < p$(see ([4])). Hence, for each $0 < t < \frac{\pi}{4}$, $(p-u(v,t))^{1/2} + (p+u(v,t))^{1/2} \leq (p - u\left(\frac{1}{2}k,t\right))^{1/2} + (p + u\left(\frac{1}{2}k,t\right))^{1/2}$, again with equality if and only if $v = \frac{1}{2}k$. Thus by (3.2), $L(v)$ attains its' unique maximum on $(0, k)$ when $v = \frac{1}{2}k$. ∎

Remark 3.1. Showing that there is a unique ellipse of maximal arc length inscribed in a general convex quadrilateral and/or characterizing such an ellipse appears to be a very nontrivial problem. Even for parallelograms it appears to be difficult since $a^2 + b^2$ does not remain constant in general as E varies over all ellipses inscribed in a given parallelogram. Numerical evidence suggests strongly that the ellipse of minimal eccentricity inscribed in a parallelogram, Ð, is **not** the ellipse of maximal arc length inscribed in Ð.

4. BIELLIPTIC PARALLELOGRAMS

Let Ð be a convex quadrilateral. In ([1], Theorem 4.4) the author proved that there is a unique ellipse, E_I, of minimal eccentricity *inscribed* in Ð. In ([2], Proposition 1) we also proved that there is a unique ellipse, E_O, of minimal eccentricity *circumscribed* about Ð. In [2] the author defined Ð to be bielliptic if E_I and E_O have the **same eccentricity**. This generalizes the notion of bicentric quadrilaterals, which are quadrilaterals which have both a circumscribed and an inscribed circle. In [2] we gave an example of a bielliptic convex quadrilateral which is not a parallelogram and which is not bicentric. Of course every square is bicentric. For parallelograms we prove the following.

Theorem 4.1. *A parallelogram, Ð, is bielliptic if and only if the square of the length of one of the diagonals of Ð equals twice the square of the length of one of the sides of Ð.*

Proof. We prove the case when Ð is **not** a rectangle, in which case the proof below can be modified to show that Ð is bielliptic if and only if it's a square, which certainly satisfies the conclusion of Theorem 4.1. Then, by using an isometry of the plane, we may assume that the vertices of Ð are $O = (0,0), P = (l,0), Q = (d,k)$, and $R = (l+d,k)$, where $d, k, l > 0$. It is not hard to show that

$$kux^2 + ky^2 - 2udxy - klux + \left[ud(l+d) - k^2\right] y = 0, 0 < u < \frac{k^2}{d^2} \tag{4.1}$$

is the general equation of an ellipse passing thru the vertices of Ð. We leave the details to the reader. By Lemma 1.1, it follows that

$$\frac{b^2}{a^2} = h(u) = \frac{k(u+1) - \sqrt{k^2(1-u)^2 + 4d^2u^2}}{k(u+1) + \sqrt{k^2(1-u)^2 + 4d^2u^2}} = \frac{\left(k(u+1) - \sqrt{k^2(1-u)^2 + 4d^2u^2}\right)^2}{4u\left(k^2 - ud^2\right)}.$$

Differentiating with respect to u, it follows that $h'(u) = 0, 0 < u < \frac{k^2}{d^2}$, if and only if $u = \frac{k^2}{k^2+2d^2}$. Substituting yields $= \frac{\left(d^2 + k^2 - d\sqrt{d^2+k^2}\right)^2}{k^2\left(d^2+k^2\right)}$, and simplifying gives

$$1 - \frac{b^2}{a^2} = 1 - h\left(\frac{k^2}{k^2+2d^2}\right) = 1 - \frac{\left(d^2 + k^2 - d\sqrt{d^2+k^2}\right)^2}{k^2\left(d^2+k^2\right)} = 2d\frac{\sqrt{d^2+k^2} - d}{k^2} \tag{4.2}$$

for the unique ellipse of minimal eccentricity, E_O, circumscribed about Đ. As in the proof of Theorem 2.4, there are three cases to consider: $I > 0, I = 0, I < 0$, where $I = l^2 - d^2 - k^2$. Assume first that $I > 0$. Then by (2.16) in the proof of Theorem 2.4(see (2.14)), $\frac{b^2}{a^2} = h(v_\epsilon) = 1 + \frac{I\left(I - \sqrt{G}\sqrt{H}\right)}{2k^2l^2}$ for the unique ellipse, E_I, of minimal eccentricity inscribed in Đ. Setting the eccentricities of E_I and E_O equal is equivalent to

$$(4.3) \qquad 2d\frac{\sqrt{d^2+k^2}-d}{k^2} = \frac{I\left(\sqrt{G}\sqrt{H} - I\right)}{2k^2l^2}.$$

Then (4.3) holds if and only if $4l^2d\left(\sqrt{d^2+k^2}-d\right) = I\left(\sqrt{G}\sqrt{H} - I\right) \iff$
$\sqrt{G}\sqrt{H}I = 4l^2d\left(\sqrt{d^2+k^2}-d\right) + I^2 \iff$
$GHI^2 - \left(4l^2d\left(\sqrt{d^2+k^2}-d\right) + I^2\right)^2 = 0 \iff$
$4l^2\left(G - 2l^2\right)\left(H - 2l^2\right)\left(2d^2 + k^2 - 2d\sqrt{d^2+k^2}\right)$
$= 0 \iff$ one of the following equations holds:

$$(4.4) \qquad -2dl - I = 0$$

$$(4.5) \qquad 2dl - I = 0$$

$$(4.6) \qquad \left(\sqrt{d^2+k^2} - d\right)^2 = 0.$$

Since $k > 0$ by assumption, (4.6) cannot hold. Also, since we have assumed that $I > 0$, (4.4) cannot hold either. Thus the eccentricities of E_I and E_O are equal if and only if (4.5) holds. Similarly, it is not hard to show that if $I < 0$, then the eccentricities of E_I and E_O are equal if and only if (4.4) holds. Now the diagonals of Đ are $D_1 = \overline{OR}$ and $D_2 = \overline{PQ}$, and thus the squares of the lengths of one of the diagonals are $|D_1|^2 = (l+d)^2 + k^2$ and $|D_2|^2 = (l-d)^2 + k^2$. The squares of the lengths of the sides are $\left|\overline{OQ}\right|^2 = d^2 + k^2$ and $\left|\overline{OP}\right|^2 = l^2$. Now $|D_1|^2 = 2\left|\overline{OQ}\right|^2 \iff (l+d)^2 + k^2 = 2d^2 + 2k^2 \iff k^2 + d^2 - 2dl - l^2 = 0$, which is (4.4). Similarly, $|D_1|^2 = 2\left|\overline{OP}\right|^2 \iff (l+d)^2 + k^2 = 2l^2 \iff d^2 + 2dl - l^2 + k^2 = 0$, which is (4.5). One can easily check that $|D_2|^2 = 2\left|\overline{OQ}\right|^2$ or $|D_2|^2 = 2\left|\overline{OQ}\right|^2$ is equivalent to (4.4) or (4.5) as well. Finally suppose that $I = 0$. Letting I approach 0 in (2.16) shows that the unique ellipse, E_I, of minimal eccentricity inscribed in Đ is a circle, which has eccentricity 0. But $1 - \frac{b^2}{a^2} = 0$ in (4.2) if and only if $d = 0$. In that case Đ is a square, which satisfies the conclusion of Theorem 4.1 since Đ is also inscribed in a circle. ■

Remark 4.1. In the proof above, $|D_1|^2 + |D_2|^2 = 2\left(d^2 + l^2 + k^2\right) = 2\left|\overline{OP}\right|^2 + 2\left|\overline{OQ}\right|^2$ for *any* parallelogram, Đ, and not just a bielliptic parallelogram. Hence if, say, $|D_1|^2 = 2\left|\overline{OP}\right|^2$, then it follows automatically that $|D_2|^2 = 2\left|\overline{OQ}\right|^2$.

Example 4.1. *Let $l = 6$, $k = 2\sqrt{2}$, and $d = 2$. Then $I = 24 > 0$, and the common eccentricity of E_I and E_O is $\sqrt{3} - 1$. The squares of the lengths of one of the diagonals are 72 and 24, and the squares of the lengths of the sides are 12 and 36. $v = \frac{3}{2}\sqrt{2}$ yields the ellipse, E_I, of minimal eccentricity inscribed in Đ, and $4\sqrt{2}x^2 + 14\sqrt{2}y^2 + 4xy - 36\sqrt{2}x - 72y + 81\sqrt{2} = 0$ is the equation of E_I. $u = \frac{1}{2}$ yields the ellipse, E_O, of minimal eccentricity circumscribed about Đ, and $\sqrt{2}x^2 + 2\sqrt{2}y^2 - 2xy - 6\sqrt{2}x = 0$ is the equation of E_O.*

Remark 4.2. Some results about areas of ellipses inscribed in parallelograms were proven by the author in [3]. In particular, we prove that if E is any ellipse inscribed in a convex quadrilateral, Đ, then $\frac{\text{Area}\,(E)}{\text{Area(Đ)}} \leq \frac{\pi}{4}$, and equality holds if and only if Đ is a parallelogram and E is tangent to the sides of Đ at the midpoints. We also prove that the foci of the unique ellipse of maximal area inscribed in a parallelogram, Đ, lie on the orthogonal least squares line for the vertices of Đ.

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