A-NORMAL OPERATORS IN SEMI HILBERTIAN SPACES

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ABSTRACT. In this paper we study some properties and inequalities of A-normal operators in semi-Hilbertian spaces by employing some known results for vectors in inner product spaces. We generalize also most of the inequalities of \((\alpha, \beta)\)-normal operators discussed in Hilbert spaces [7].

Key words and phrases: A-adjoint, Semi-inner product, Normal operators.

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1. Introduction

Throughout this paper \( \mathcal{H} \) denotes a complex Hilbert space with inner product \( \langle \cdot , \cdot \rangle \) and norm \( \| \cdot \| \). \( \mathcal{L}(\mathcal{H}) \) stands the Banach algebra of all bounded linear operators on \( \mathcal{H} \). \( I = I_{\mathcal{H}} \) being the identity operator and if \( V \subset \mathcal{H} \) is a closed subspace, \( P_{V} \) is the orthogonal projection onto \( V \).

For \( T \in \mathcal{L}(\mathcal{H}) \) its range is denoted by \( R(T) \), its null space by \( N(T) \), its adjoint by \( T^{*} \) and its spectrum by \( \sigma(T) \). The numerical range of \( T \) is a subset of the set of complex numbers \( \mathbb{C} \) and it is defined by

\[
W(T) = \{ \langle Tx | x \rangle, \ x \in \mathcal{H}, \ |x| = 1 \}
\]

The spectral radius and the numerical radius and the minimum modulus of \( T \) will be denoted respectively by \( r(T) \) and \( w(T) \) and \( \gamma(T) \). They are defined as \( r(T) = \sup\{ |\lambda|, \ \lambda \in \sigma(T) \} \) and \( w(T) = \sup\{ |\lambda|, \ \lambda \in \mathcal{R}(T) \} \) and \( \gamma(T) = \inf\{ |T x||, \ x \in \mathcal{N}(T) \} \) and \( |x| = 1 \). It is well known that \( \gamma(T) \geq 0 \) if and only if \( R(T) \) is closed and that \( w(T) \) is a norm on the Banach algebra \( \mathcal{L}(\mathcal{H}) \) (for more detail about the concept of numerical radius, see for example [4,9]). Moreover for \( T \in \mathcal{L}(\mathcal{H}) \), we have

\[
w(T) \leq |T|| \leq 2w(T),
\]

and that for a normal operator \( T \) ([3]), one has

\[
r(T) = w(T) = |T||
\]

\( \mathcal{L}(\mathcal{H})^{+} \) is the cone of positive operators, i.e.

\[
\mathcal{L}(\mathcal{H})^{+} = \{ A \in \mathcal{L}(\mathcal{H}) : \langle Ax | x \rangle \geq 0, \ \forall x \in \mathcal{H} \}.
\]

Any positive operator \( A \in \mathcal{L}(\mathcal{H})^{+} \) defines a positive semi-definite sesquilinear form

\[
\langle \cdot , \cdot \rangle_{A} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}, \ \langle x | y \rangle_{A} = \langle Ax | y \rangle.
\]

By \( \| \cdot \|_{A} \) we denote the seminorm induced by \( \langle \cdot , \cdot \rangle_{A} \), i.e., \( \| x \|_{A} = \langle x | x \rangle_{A}^{\frac{1}{2}} \). Note that \( \| x \|_{A} = 0 \) if and only if \( x \in N(A) \). Then \( \| \cdot \|_{A} \) is a norm on \( \mathcal{H} \) if and only if \( A \) is an injective operator, and the semi-normed space \( \langle \mathcal{L}(\mathcal{H}), \| \cdot \|_{A} \rangle \) is complete if and only if \( R(A) \) is closed. Moreover \( \langle \cdot , \cdot \rangle_{A} \) induces a seminorm on the subspace \( \{ T \in \mathcal{L}(\mathcal{H}) : \exists c > 0, \ |T x|| \leq c\|x\|_{A}, \ \forall x \in \mathcal{H} \} \). For this subspace of operators it holds

\[
\| T \|_{A} = \sup_{x \in R(A), x \neq 0} \frac{\| Tx \|_{A}}{\| x \|_{A}} < \infty,
\]

Moreover

\[
\| T \|_{A} = \sup\{ |\langle Tx | y \rangle_{A}|; \ x, y \in \mathcal{H} \ \text{and} \ \| x \|_{A} \leq 1, \| y \|_{A} \leq 1 \}.
\]

For \( x, y \in \mathcal{H} \), we say that \( x \) and \( y \) are \( A \)-orthogonal if \( \langle x | y \rangle_{A} = 0 \). Note that this definition is a natural extension of the usual notion of orthogonality which represents the \( I \)-orthogonality case. For a set \( S \subset \mathcal{H} \), its \( A \)-orthogonal subspace \( S^{\perp_{A}} \) is given by

\[
S^{\perp_{A}} = \{ x \in \mathcal{H}; \langle x | y \rangle_{A} = 0, \ \forall y \in S \}.
\]

Note that \( S^{\perp_{A}} = (AS)^{\perp} = A^{-1}(S^{\perp}) \) and since \( A(A^{-1}(S) = S \cap R(A) \), then \( (S^{\perp_{A}})^{\perp_{A}} = (S^{\perp} \cap R(A))^{-1} \). The concept of \( A \)-spectral radius, \( A \)-numerical radius and \( A \)-minimum modulus of an operator are a natural generalization of the spectral radius, the numerical radius and the minimum modulus respectively. In the next, we give the following definition.

**Definition 1.1.** Let \( T \in \mathcal{L}(\mathcal{H}) \). The \( A \)-spectral radius, the \( A \)-numerical radius and the \( A \)-minimum modulus of \( T \) are denoted respectively \( r_{A}(T) \), \( w_{A}(T) \) and \( \gamma_{A}(T) \) and they are defined as

\[
r_{A}(T) = \lim_{n \rightarrow +\infty} \sup ||T^{n}||^{\frac{1}{n}}_{A}
\]
\[ w_A(T) = \sup\{ |\langle Tx | x \rangle_A| \; x \in \mathcal{H}, \| x \|_A = 1 \} \]

and
\[ \gamma_A(T) = \inf\{ \| Tx \|_A; x \in N(A^{\frac{1}{2}} T)^{\perp A} \text{ and } \| x \|_A = 1 \}. \]

For any \( T, S \in \mathcal{L}(\mathcal{H}) \), the following properties are immediate:

1. \( w_A(T) \geq 0 \) and \( w_A(T) = 0 \) if and only if \( AT = 0 \).
2. \( w_A(\lambda T) = |\lambda| w_A(T) \) for any \( \lambda \in \mathbb{C} \).
3. \( w_A(T + S) \leq w_A(T) + w_A(S) \).
4. \( \forall x \in \mathcal{H}, |\langle Tx | x \rangle_A| \leq w_A(T) \| x \|^2_A \leq \| T \|_A \| x \|^2_A \) and \( \| Tx \|_A \geq \gamma_A(T) d_A(x, N(A^{\frac{1}{2}} T)) \) where \( d_A(x, V) = \inf \{ \| x - y \|_A; y \in V \} \) for any \( V \subset \mathcal{H} \).

Note that \( w_A(.) \) is a seminorm on \( \mathcal{L}(\mathcal{H}) \) and it is a norm if \( A \) is injective. Moreover \( w_A(T) \leq \| T \|_A \) for any \( T \in \mathcal{L}(\mathcal{H}) \). The following theorem due to Douglas will be used (see [5] for its proof).

**Theorem 1.1.** Let \( T, S \in \mathcal{L}(\mathcal{H}) \). The following conditions are equivalent.

1. \( R(S) \subset R(T) \).
2. There exists a positive number \( \lambda \) such that \( SS^* \leq \lambda TT^* \).
3. There exists \( W \in \mathcal{L}(\mathcal{H}) \) such that \( TW = S \).

From now on, \( A \) denotes a positive operator on \( \mathcal{H} \) (i.e. \( A \in \mathcal{L}(\mathcal{H})^+ \)).

**Definition 1.2.** Let \( T \in \mathcal{L}(\mathcal{H}) \), an operator \( W \in \mathcal{L}(\mathcal{H}) \) is called an \( A \)-adjoint of \( T \) if
\[ \langle Tu | v \rangle_A = \langle u | Wv \rangle_A \quad \text{for every } u, v \in \mathcal{H}, \]

or equivalently
\[ AW = T^* A; \]

\( T \) is called \( A \)-selfadjoint if \( AT = T^* A \) and it is called \( A \)-positive if \( AT \) is positive.

By Douglas Theorem, an operator \( T \in \mathcal{L}(\mathcal{H}) \) admits an \( A \)-adjoint if and only if \( R(T^* A) \subset R(A) \) and if \( W \) is an \( A \)-adjoint of \( T \) and \( AZ = 0 \) for some \( Z \in \mathcal{L}(\mathcal{H}) \) then \( W + Z \) is also an \( A \)-adjoint of \( T \). Hence neither the existence nor the uniqueness of an \( A \)-adjoint operator is guaranteed. In fact an operator \( T \in \mathcal{L}(\mathcal{H}) \) may admit none, one or many \( A \)-adjoints.

From now on, \( \mathcal{L}_A(\mathcal{H}) \) denotes the set of all \( T \in \mathcal{L}(\mathcal{H}) \) which admit an \( A \)-adjoint, i.e.
\[ \mathcal{L}_A(\mathcal{H}) = \{ T \in \mathcal{L}(\mathcal{H}) : R(T^* A) \subset R(A) \}. \]

\( \mathcal{L}_A(\mathcal{H}) \) is a subalgebra of \( \mathcal{L}(\mathcal{H}) \) which is neither closed nor dense in \( \mathcal{L}(\mathcal{H}) \).

On the other hand the set of all \( A \)-bounded operators in \( \mathcal{L}(\mathcal{H}) \) (i.e. with respect the seminorm \( \| . \|_A \)) is
\[ \mathcal{L}_{A^\frac{1}{2}}(\mathcal{H}) = \{ T \in \mathcal{L}(\mathcal{H}) : T^* R(A^{\frac{1}{2}}) \subset R(A^{\frac{1}{2}}) \} = \{ T \in \mathcal{L}(\mathcal{H}) : R(A^{\frac{1}{2}} T^* A^{\frac{1}{2}}) \subset R(A) \}. \]

Note that \( \mathcal{L}_A(\mathcal{H}) \subset \mathcal{L}_{A^\frac{1}{2}}(\mathcal{H}) \), which shows that if \( T \) admits an \( A \)-adjoint then it is \( A \)-bounded. Section 2, contains some inequalities giving upper bounds of the difference between the \( A \)-norm and \( A \)-numerical radius of an \( A \)-bounded operator in semi-Hilbertian spaces and under appropriate conditions. In section 3, we introduce the notion of \( A \)-normal operators, we prove a characterization involving the \( A \)-norm, \( \| . \|_A \), we give some properties on \( A \)-normal operators, then we establish new operator norm inequalities. Our inequalities generalize the well known properties for normal operators.
2. Inequalities involving A-Numerical Radius

If \( T \in \mathcal{L}(\mathcal{H}) \) with \( R(T^*A) \subseteq R(A) \), then \( T \), admits an \( A \)-adjoint operator, Moreover there exists a distinguished \( A \)-adjoint operator of \( T \), namely, the reduced solution of the equation \( AX = T^*A \), i.e. \( T^\dagger = A^\dagger T^*A \), where \( T^\dagger \) is the Moore-Penrose inverse of \( T \). The \( A \)-adjoint operator \( T^\dagger \) verifies

\[
AT^\dagger = T^*A, \quad R(T^\dagger) \subseteq R(A) \quad \text{and} \quad N(T^\dagger) = N(T^*A).
\]

In the next we add without proof some important properties of \( T^\dagger \) (for more details we refer the reader to [1] and [2]).

**Theorem 2.1.** Let \( T \in \mathcal{L}_A(\mathcal{H}) \). Then

1. If \( AT = TA \) then \( T^\dagger = PT^* \).
2. \( T^\dagger T \) and \( TT^\dagger \) are \( A \)-selfadjoint and \( A \)-positive.
3. \( \|T\|^2_A = \|T^\dagger\|^2_A = \|T^\dagger T\| = \|TT^\dagger\| = w_A(T^\dagger T) = w_A(TT^\dagger) \).
4. \( \|S\|_A = \|T^\dagger\|_A \) for every \( S \in \mathcal{L}(\mathcal{H}) \) which is an \( A \)-adjoint of \( T \).
5. If \( S \in \mathcal{L}_A(\mathcal{H}) \) then \( ST \in \mathcal{L}_A(\mathcal{H}) \), \( (ST)^\dagger = T^\dagger S^\dagger \) and \( \|TS\|_A = \|ST\|_A \).
6. \( T^\dagger \in \mathcal{L}_A(\mathcal{H}) \), \( (T^\dagger)^\dagger = PT^P \) and \( (T^\dagger)^2 = T^\dagger \).
7. \( \|T^\dagger\| \leq \|S\| \) for every \( S \in \mathcal{L}(\mathcal{H}) \) which is an \( A \)-adjoint of \( T \). Nevertheless, \( T^\dagger \) is not in general the unique \( A \)-adjoint of \( T \) that realizes the minimal norm.

**Lemma 2.1.** Let \( T \in \mathcal{L}_A(\mathcal{H}) \). If \( M \) is an invariant subspace for \( T \) and \( T^\dagger \), then \( M^\perp_A \) is also invariant for \( T \) and \( T^\dagger \).

**Proof.** Let \( x \in M^\perp_A \), and \( y \in M \), then \( \langle Tx | y \rangle_A = \langle x | T^\dagger y \rangle_A = 0 \), since \( T^\dagger y \in M \). Thus \( Tx \in M^\perp_A \), so \( T(M^\perp_A) \subseteq M^\perp_A \). Similarly, we show that \( T^\dagger(M^\perp_A) \subseteq M^\perp_A \).

In the following, we establish various inequalities between the operator seminorm \( \|\cdot\|_A \) and the \( A \)-numerical radius \( w_A(\cdot) \) of operators in semi-Hilbertian spaces.

**Theorem 2.2.** Let \( T \in \mathcal{L}_A(\mathcal{H}) \), \( \lambda \in \mathbb{C} \) and \( \alpha \geq 0 \) are such that \( \|T - \lambda I\|_A \leq \alpha \). Then

\[
(0 \leq \lambda)(\|T\|_A - w_A(T)) \leq \frac{\alpha^2}{2}
\]

Moreover, if \( |\lambda| > \alpha \) then

\[
\sqrt{1 - \frac{\alpha^2}{|\lambda|^2} \|T\|_A} \leq w_A(T) \leq \|T\|_A
\]

**Proof.** Since \( \|T - \lambda I\|_A \leq \alpha \) then for \( x \in \mathcal{H} \) with \( \|x\|_A = 1 \), we have \( \|Tx - \lambda x\|_A \leq \alpha \), or equivalently \( \|Tx - \lambda x\|^2_A \leq \alpha^2 \), which implies that

\[
\|Tx\|^2_A + \|\lambda\|^2 \leq 2\Re(\langle Tx | x \rangle_A) + \alpha^2 \leq 2|\lambda|\|Tx | x \rangle_A| + \alpha^2
\]

By taking the supremum over \( x \in \mathcal{H} \), \( \|x\|_A = 1 \), it follows

\[
2|\lambda|\|T\|_A \leq \|T\|^2_A + \|\lambda\|^2 \leq 2|\lambda|w_A(T) + \alpha^2
\]

Hence the desired inequality (2.1) is obtained.

Now if \( |\lambda| > \alpha \), on dividing with \( |\lambda|^2 \) in (2.3) we obtain

\[
\frac{\|T\|^2_A}{|\lambda|^2} + 1 \leq 2 \frac{w_A(T)}{|\lambda|} + \frac{\alpha^2}{|\lambda|^2}
\]
then by using an elementary inequality, we deduce

$$2\sqrt{1 - \frac{\alpha^2}{|\lambda|^2}} \frac{||T||_A}{|\lambda|} \leq \frac{||T||_A^2}{|\lambda|^2} + 1 - \frac{\alpha^2}{|\lambda|^2} \leq 2 w_A(T)$$

from which the inequality (2.2) is easily holds.

**Remark 2.1.** Note that for $T \in \mathcal{L}_A(H)$, $\lambda \in \mathbb{C}$ and $|\lambda| > \alpha \geq 0$ such that $||T - \lambda I||_A \leq \alpha$, (1.1) and (2.2) lead a refinement and improve (1.1) and they provide the following inequalities

$$w_A(T) \leq ||T||_A \leq \sqrt{\frac{||T||_A^2}{|\lambda|^2 - \alpha^2} w_A(T)} \leq 2 w_A(T), \text{ if } \frac{\alpha}{|\lambda|} \leq \frac{\sqrt{3}}{2}$$

Using the fact that for $x, y, z \in \mathcal{H}$, one has

$$Re\langle y - x|x - z\rangle_A \geq 0 \iff ||x - \frac{y + z}{2}||_A \leq \frac{1}{2}||y - z||_A$$

and by applying Theorem 2.2, (2.1), the following corollary is immediately deduced.

**Corollary 2.3.** Let $T \in \mathcal{L}_A(H)$, $\lambda, \mu \in \mathbb{C}$, $\lambda \neq \mu$. If $Re\langle \lambda x - Tx|Tx + \mu x\rangle_A \geq 0$, for all $x \in \mathcal{H}$ then

$$0 \leq ||T||_A - w_A(T) \leq \frac{1}{4} \frac{||\lambda + \mu||^2}{|\lambda - \mu|}$$

**Remark 2.2.** Note that in the literature, the condition $Re\langle \lambda x - Tx|Tx + \mu x\rangle_A \geq 0$, $x \in \mathcal{H}$ means that the operator

$$(T^2 + \mu I)A(\lambda I - T)$$

is accretive.

On squaring (2.2) and replacing $\lambda$ by $\frac{\lambda - \alpha}{2}$, $\alpha$ by $\frac{|\lambda + \alpha|}{2}$, the following corollary follows

**Corollary 2.4.** Let $T \in \mathcal{L}_A(H)$, $\lambda, \mu \in \mathbb{C}$, with $Re\langle \lambda \mu \rangle \leq 0$. If $T$ verifies (2.5), then

$$0 \leq ||T||_A^2 - w_A(T)^2 \leq \frac{||\lambda + \mu||^2}{|\lambda - \mu|^2} ||T||_A^2.$$

and

$$\frac{2\sqrt{-Re\langle \lambda \mu \rangle}}{|\lambda - \mu|} ||T||_A \leq w_A(T)$$

in particular if we choose $\lambda = -\mu > 0$, we get

$$||T||_A = w_A(T).$$

3. **A-NORMAL OPERATORS**

In the following we introduce the notion of $A$-normal operators.

**Definition 3.1.** An operator $T \in \mathcal{L}_A(H)$ is called an $A$-normal operator if $T^2 = TT^4$.

$A$-normal operators may be regarded as a generalization of normal and self-adjoint operators in which $T^2 = T^*$. This last property is realized in particular if $A = I$ or if $T$ and $A$ commute and $A$ has a dense range. The identity operator and the orthogonal projection on $\overline{R(A)}$ are $A$-normal. Moreover, if $T$ is an $A$-normal then $\{TS, T^2 + S, TS = ST, S = S^2\}$ is a set of $A$-normal operators.

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Another characterization is that $T \in \mathcal{L}_A(\mathcal{H})$ is an $A$-normal operator if and only if there are $A$-selfadjoint operators $B, C \in \mathcal{L}_A(\mathcal{H})$ such that $BC = CB$ and $T = B + iC$, $(i^2 = -1)$.

From now on, to simplify notation, we write $P$ instead of $P_{R(A)}$. An important property of $A$-normal operators that will be used frequently in the sequel is the following:

**Theorem 3.1.** A necessary and sufficient condition for an operator $T \in \mathcal{L}_A(\mathcal{H})$ to be $A$-normal is that $R(T^*T) \subset R(A)$ and $\|Tx\|_A = \|T^Tx\|_A$ for every vector $x \in \mathcal{H}$.

Proof. Suppose that $T$ is $A$-normal. It is easily to see that $R(T^*T) = R(T^2T) \subset R(A)$. Moreover, using the fact that $TT^*$ is $A$-selfadjoint, then for $x \in \mathcal{H}$, we obtain,

$$T^2T = TT^* \iff \langle T^2Tx|x \rangle_A = \langle TT^*x|x \rangle_A$$

$$\iff \langle AT^2x|x \rangle = \langle ATT^*x|x \rangle$$

$$\iff \langle TAT^2x|x \rangle = \langle (T^*T)^*Ax|x \rangle$$

$$\iff \langle ATx|Tx \rangle = \langle T^*Ax|T^2x \rangle$$

$$\iff ||Tx||_A = ||T^2x||_A$$

Conversely, if $||Tx||_A = ||T^2x||_A$, then $A(T^2T - TT^*) = 0$, if moreover $R(T^*T) \subset R(A)$, so, it follows $R(T^2T - TT^*) \subset R(A) = N(A)^\perp$ and hence $T^2T - TT^* = 0$, which finishes the proof.

In the next we give some properties on $A$-normal operators.

**Corollary 3.2.** For $T \in \mathcal{L}_A(\mathcal{H})$, the following properties hold

1. If $T$ is $A$-selfadjoint operator then $||T||_A = w_A(T)$.
2. If $T$ is $A$-normal operator then $T^n$ is also for all $n \geq 1$ and $||T||_A = r_A(T)$.
3. Suppose that $N(A)$ is an invariant subspace for $T$ and $\lambda, \mu \in \mathbb{C}$. If $T$ is $A$-normal, then
   (a) $T - \lambda I$ and $T^*$ are $A$-normal.
   (b) $Tx = \lambda x$ yields $T^*x = \overline{\lambda} Px$.
   (c) $M = \{x \in \mathcal{H}/Tx = \lambda x\}$ and $M^{\perp_A}$ are invariant for $T$ and $T^*$.
   (d) $Tx = \lambda x$ and $Ty = \mu y$, $\lambda \neq \mu$ yield $x \perp_A y$ (i.e. $\langle x|y \rangle_A = 0$).

Proof.

1. It is clear that $\sup_{||x||_A = ||y||_A = 1} |\langle Tx|y \rangle_A| \leq ||T||_A$. In the other hand, if we choose $z = \frac{T^*_x}{||T^*_x||_A}$, we obtain

$$||Tx||_A = \langle Tx|z \rangle_A \leq \sup_{||x||_A = ||y||_A = 1} |\langle Tx|y \rangle_A|$$

Moreover, without loss of generality we can suppose $x, y \neq 0$ and that $\langle Tx|y \rangle_A > 0$, then one has

$$\langle T(x + y)|x + y \rangle_A = \langle Tx|x \rangle_A + \langle Tx|y \rangle_A + \langle y|T^2x \rangle_A + \langle Ty|y \rangle_A$$

and

$$\langle T(x - y)|x - y \rangle_A = \langle Tx|x \rangle_A - \langle Tx|y \rangle_A - \langle y|T^2x \rangle_A + \langle Ty|y \rangle_A$$
If $T$ is $A$-selfadjoint then, by parallelogram law

$$
\langle Tx|y\rangle_A = \frac{1}{2} |\langle Tx|y\rangle_A + \langle T^*x|y\rangle_A| = \frac{1}{2} |\langle T(x + y)|x + y\rangle_A - \langle T(x - y)|x - y\rangle_A|
$$

\[\leq \frac{w_A(T)}{2} (||x + y||^2_A + ||x - y||^2_A)\]

\[\leq w_A(T)(||x||^2_A + ||y||^2_A)\]

If we replace $x$ by $\sqrt{\alpha}x$ and $y$ by $\frac{y}{\sqrt{\alpha}}$, where $\alpha = \frac{||x||_A}{||y||_A}$, we get

$$
\langle Tx|y\rangle_A = |\langle Tx|y\rangle_A + \langle T^*x|y\rangle_A| = \frac{w_A(T)}{2} (||x||^2_A + ||y||^2_A)
$$

$$
= w_A(T)||x||_A||y||_A.
$$

which implies, $w_A(T) \leq ||T||_A$ and thus,

$$
||T||_A = \sup \{|\langle Tx|y\rangle_A|; ||x||_A = ||y||_A = 1\} = w_A(T)
$$

(2) Let $n \geq 1$, if $T$ is $A$-normal operator then, $T$ and $T^2$ commute, consequently $T^n$ and $(T^2)^n$ commute. Thus $T^n$ is $A$-normal. Let $x \in \mathcal{H}$, we have

$$
||T^2Tx||^2_A = \langle T^2Tx|T^2Tx\rangle_A = \langle T^2x|T^2x\rangle_A = ||T^2x||^2_A
$$

$$
||Tx||^2_A = \langle Tx|Tx\rangle_A = \langle T^2x|Tx\rangle_A = ||T^2x||_A
$$

Since $T^2T$ is $A$-selfadjoint then by taking the supremum on $||x||_A = 1$ and applying 1. we get

$$
||T||^2_A = \sup_{||x||_A = 1} ||Tx||^2_A = \sup_{||x||_A = 1} \langle T^2Tx\rangle_A
$$

$$
= \sup_{||x||_A = 1} ||T^2Tx||_A
$$

Moreover for all $n \geq 1$ we have

$$
||T^n||^2_A = \langle T^n|T^n\rangle_A = \langle T^2T^{n-1}x|T^{n-1}x\rangle_A \leq ||T^2T^{n-1}x||_A||T^{n-1}x||_A
$$

which implies

$$
||T^n||^2_A \leq ||T^{n+1}||_A||T^{n-1}||_A
$$

Assume that $||T||_A > 0$ then $||T^n||_A > 0$ for all $n \geq 1$ (for $||T||_A = 0$ the desired property is evident) and set $\alpha_n = \frac{||T^{n+1}||_A}{||T^n||_A}$, $n \geq 1$. It is clear that $(\alpha_n)_n$ is an increasing sequence, then it satisfies

$$
\frac{||T^{n+1}||_A}{||T^n||_A} = \alpha_n \geq \alpha_1 = \frac{||T^2||_A}{||T||_A} = \frac{||T||^2_A}{||T||_A} = ||T||_A.
$$

By an induction argument, it follows $||T^n||_A = ||T||^n_A$, for all $n \geq 1$.

Thus $r_A(T) = ||T^n||^\frac{1}{n}_A = ||T||_A$ and the proof is achieved.

(3) (a) Note first that since $N(A)$ is invariant for $T$, then $TP = PT$ and $AP = PA = A$.

Let now $\lambda \in \mathbb{C}$, we have $(T - \lambda I)(T - \lambda I)^* = (T - \lambda I)(T^2 - \lambda T^2 - \lambda T^3 - \lambda T^5 - \lambda T^7 - \lambda T^9 - \lambda T^{11} + \lambda^2 P = T^2T - \lambda T^2 - \lambda T^3 - \lambda T^5 - \lambda T^7 - \lambda T^9 - \lambda T^{11})$. 

\[= (T - \lambda I)(T - \lambda I)^* = T^2T - \lambda T^2 - \lambda T^3 - \lambda T^5 - \lambda T^7 - \lambda T^9 - \lambda T^{11} + \lambda^2 P = (T - \lambda I)^2(T - \lambda I),\]
then \((T - \lambda I)\) is \(A\)-normal.
For all \(x \in \mathcal{H}\), we have also
\[
||((T^2)^*x)||^2_A = \langle (T^2)^*x|(T^2)^*x \rangle_A \\
= \langle PTPx|PTPx \rangle_A \\
= \langle TPx|TPx \rangle_A \\
= ||TPx||^2_A \\
= ||Tx||^2_A = ||T^2x||^2_A
\]

It clear that \(R(T^2(T^2)^*) \subset R(A^2)\), so from Theorem 3.1, it follows that \(T^2\) is \(A\)-normal.

(b) Using (a),
\[
||\sqrt{A}(T^2 - \lambda P)x|| = ||(T^2 - \lambda P)x||_A \\
= ||(T - \lambda I)^2x||_A \\
= ||(T - \lambda I)x||_A = 0
\]
or \(R(T^2 - \lambda P) \subset R(A) = N(A)^1\), then \(T^2x = \lambda P\).

(c) Let \(M = \{x \in \mathcal{H} | Tx = \lambda x\}\). It is clear that \(T(M) \subset M\). Moreover if \(x \in M\) and \(y = T^2x\), then \(Ty = TT^2x = T^2Tx = \lambda T^2x = \lambda y\) yields \(y = Tx \in M\). Hence \(M\) is invariant for both \(T\) and \(T^2\). Using Lemma 2.1 the desired result follows.

(d) Suppose that \(Tx = \lambda x\), \(Ty = \mu y\) with \(0 \neq \lambda \neq \mu\),
\[
\langle x|y \rangle_A = \lambda^{-1}\langle Tx|y \rangle_A = \lambda^{-1}\langle x|T^2y \rangle_A = \lambda^{-1}\mu \langle x|Py \rangle_A = \lambda^{-1}\mu \langle x|y \rangle_A,
\]
then \(\langle x|y \rangle_A = 0\). If \(\lambda = 0\) we permute between \(\lambda\) and \(\mu\) and the proof achieved.

**Question:** If \(T\) is \(A\)-normal, is it true that \(||T||_A = w_A(T)\)?

Note that in the Cauchy-Schwarz inequality i.e.
\[
||\langle u|v \rangle|| \leq ||u|| \times ||v||, \; u, v \in \mathcal{H}
\]
if, we choose \(u = \sqrt{Ax}\) and \(v = \sqrt{Ay}\) we obtain more general formula
\[
||\langle x|y \rangle_A|| \leq ||x||_A \times ||y||_A, \; x, y \in \mathcal{H}
\]
Moreover, for the choices \(Tx\) instead of \(x\) and \(T^2x\) instead of \(y\) with \(x \in \mathcal{H}\), then one gets the following simple inequality for the \(A\)-normal operator \(T\):
\[
||\langle T^2x|x \rangle_A|| \leq ||Tx||^2_A, \; x \in \mathcal{H}
\]

Note that the inequality (3.3) implies in particular that
\[
w_A(T^2) \leq ||T||^2_A.
\]
Note also that the inequality (3.3) becomes an equality if \(T\) is an \(A\)-selfadjoint operator. This property does not remain true for \(A\)-normal operators. Indeed if consider the operators \(\mathcal{H} = \mathbb{C}^2, \; A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathcal{L}(\mathcal{H})^+, \; T = \begin{pmatrix} r & r \\ -r & r \end{pmatrix} \in \mathcal{L}(\mathcal{H})\) for some \(a > 0\) and \(r \neq 0\). It is easy to check that \(T\) admits \(A\)-adjoint operators and by direct computation, we see that \(T\) is an \(A\)-normal operator and that (3.3) is a real inequality.

It is then natural to discuss some estimations of the quantity \(||Tx||^2_A - ||\langle T^2x|x \rangle_A||\) for \(A\)-normal operators and give a measure of the closeness of the two terms involved in (3.3).
Motivated by this problem, we will study in this section some inequalities of $A$-normal operators in semi-Hilbertian spaces by employing some known results for vectors in inner product spaces.

We start with the following result.

**Theorem 3.3.** Let $T \in \mathcal{L}_A(\mathcal{H})$ be an $A$-normal operator; then the inequalities

$$
|\langle Tx|x \rangle_A|^2 \leq \frac{1}{2} (||Tx||_A^2 + ||T^2x||_A) \leq ||Tx||_A^2
$$

(3.4)

hold for all $x \in \mathcal{H}$, $||x||_A = 1$. The constant $\frac{1}{2}$ is the best possible in (3.4).

Proof. The second inequality in (3.4) hold immediately from (3.3). For the first one we use the inequality, which is a consequence of the inequalities (2.3) in [6].

If we choose $e = x$, $||x||_A = 1$, $a = Tx$, and $b = T^2x$, then we obtain

$$
|\langle Tx|x \rangle_A \langle x|T^2x \rangle_A| \leq \frac{1}{2} (||Tx||_A \||T^2x||_A + ||Tx|T^2x \rangle_A)
$$

(3.6)

for all $x \in \mathcal{H}$ and $||x||_A = 1$.

Since $T$ is $A$-normal, then $||Tx||_A = ||T^2x||_A$ and the desired inequality follows from (3.6).

If we suppose now that $T = I$ is the identity operator, then both the two inequalities in (3.4) become equalities, this means that $\frac{1}{2}$ is the best possible constant in (3.4).

The following result is obviously deduced from Theorem 3.3.

**Corollary 3.4.** If $T \in \mathcal{L}_A(\mathcal{H})$ is an $A$-normal operator, then

$$
w_A(T)^2 \leq \frac{1}{2} (||T||_A^2 + w_A(T^2)) \leq ||T||_A^2.
$$

(3.7)

The following result provides an upper bound for the nonnegative quantity

$$
||Tx||_A^2 - |\langle Tx|T^2x \rangle_A|, \ x \in \mathcal{H}
$$

**Theorem 3.5.** Let $T \in \mathcal{L}_A(\mathcal{H})$ be an $A$-normal operator and $\lambda \in \mathbb{C}$, then

$$
0 \leq ||Tx||_A^2 - |\langle Tx|T^2x \rangle_A| \leq \frac{2}{1 + |\lambda|^2} ||Tx - \lambda T^2x||_A^2
$$

(3.8)

for any $x \in \mathcal{H}$.

Proof. For $\lambda = 0$, the inequality in (3.8) is obvious. For $\lambda \neq 0$, we use the Dunkl-Williams inequality [8].

$$
\frac{||a|| \ ||b|| - |\langle a|b \rangle|}{||a|| \ ||b||} \leq \frac{2||a - b||^2}{(||a|| + ||b||)^2}, \ a, b \in \mathcal{H}\{0\}
$$

which shows that

$$
\frac{||a||_A \ ||b||_A - |\langle a|b \rangle|_A}{||a||_A \ ||b||_A} \leq \frac{2||a - b||_A^2}{(||a||_A + ||b||_A)^2}, \ a, b \notin N(A)
$$

(3.9)

Now, taking into account that $T$ is an $A$-normal operator, we choose in (3.9) $a = Tx$ and $b = \lambda T^2x$, $\lambda \neq 0$, $x \notin N(A^\dagger T)$, so from Theorem 3.1 one gets

$$
\frac{||Tx||_A^2 - |\langle Tx|T^2x \rangle_A|}{||Tx||_A^2} \leq \frac{2||Tx - \lambda T^2x||_A^2}{(1 + |\lambda|^2)^2 ||Tx||_A^2}
$$

(3.10)
which immediately implies (3.8).

Since for $A$-normal operators $N(A^{\frac{1}{2}}T) = N(A^{\frac{1}{2}}T^2)$ then, the inequality (3.8) holds also for $x \in N(A^{\frac{1}{2}}T)$ and so the proof is achieved.

**Corollary 3.6.** If $T \in \mathcal{L}_A(\mathcal{H})$ is an A-normal operator, then

$$w_A(T)^2 - w_A(T^2) \leq \frac{1}{2}(||T||^2_A - w_A(T^2)) \leq \frac{1}{1 + ||\lambda||^2}||T - \lambda T^2||^2_A.$$ for all $\lambda \in \mathbb{C}$

The next technic result generalizes Lemma 2.1, [6].

**Lemma 3.1.** Let $a, b \notin N(A)$ and $0 < \varepsilon \leq \frac{1}{2}$, such that

$$0 \leq 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \leq \frac{||a||_A}{||b||_A} \leq 1 - \varepsilon + \sqrt{1 - 2\varepsilon}.$$ Then

(3.10) $$0 \leq ||a||_A \ ||b||_A - Re(a\langle b\rangle)_A \leq \varepsilon ||a - b||^2_A.$$

Using Lemma 3.1, the following similar result may be stated

**Theorem 3.7.** Let $T \in \mathcal{L}_A(\mathcal{H})$ be an A-normal operator, $\lambda \in \mathbb{C}$ and $0 < \varepsilon \leq \frac{1}{2}$ such that

$$0 \leq 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \leq |\lambda| \leq 1 - \varepsilon + \sqrt{1 - 2\varepsilon}.$$ Then

(3.11) $$0 \leq ||Tx||^2_A - |\langle T^2x|_A|| \leq \frac{\varepsilon}{|\lambda|}||Tx - \lambda T^2x||^2_A$$ for any $x \in \mathcal{H}$

Proof. By choosing $a = \lambda T^2x$ and $b = Tx$, $x \notin N(A^{\frac{1}{2}}T^2)$ in Lemma 3.1, we have

$$0 \leq ||\lambda T^2x||_A \ ||Tx||_A - Re(\lambda T^2x|Tx)_A \leq \varepsilon ||\lambda T^2x - Tx||^2_A.$$ or $0 \leq ||Tx||^2_A - |\langle T^2x|_A|| \leq |||Tx||_A \ ||Tx||_A \ ||T^2x||_A$ and $Re(\lambda T^2x|Tx)_A \leq |\lambda| |\langle T^2x|_A||$, $T$ being an A-normal operator, then (3.11) holds for any $x \notin N(A^{\frac{1}{2}}T^2)$.

Since $N(A^{\frac{1}{2}}T^2) = N(A^{\frac{1}{2}}T)$, then for $x \in N(A^{\frac{1}{2}}T)$ it is clear that the inequality (3.11) is checked. Therefore, (3.11) holds for any $x \in \mathcal{H}$.

The following corollary may be stated

**Corollary 3.8.** Let $T \in \mathcal{L}_A(\mathcal{H})$ be an A-normal operator, $\lambda \in \mathbb{C}$ and $0 < \varepsilon \leq \frac{1}{2}$ such that

$$0 \leq 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \leq |\lambda| \leq 1 - \varepsilon + \sqrt{1 - 2\varepsilon}.$$ Then

(3.12) $$0 \leq ||T||^2_A - w_A(T^2) \leq \frac{\varepsilon}{|\lambda|}||T - \lambda T^2||^2_A$$

**Theorem 3.9.** Let $T \in \mathcal{L}_A(\mathcal{H})$ be an A-normal operator and $\lambda \in \mathbb{C} \setminus \{0\}$. Then

(3.13) $$0 \leq ||T||^2_A - w_A(T^2)^2 \leq \frac{1}{|\lambda|^2}||T||^2_A ||T - \lambda T^2||^2_A$$
Proof. We use the following inequality obtained by Dragomir (see [7],(2.10)).

\[ 0 \leq ||a||^2 ||b||^2 - ||a||^2 \leq \frac{1}{|\lambda|^2} ||a||^2 ||a - \lambda b||^2 \]

provided \( a, b \in \mathcal{H} \) and \( \lambda \in \mathbb{C} \setminus \{0\} \).

Immediately on choosing \( a = \sqrt{AT}x \) and \( b = \sqrt{AT^*x} \), one gets,

\[ 0 \leq ||Tx||^2 ||T^*x||_A^2 - ||Tx||^2 ||T^*x - \lambda T^*x||^2_A \leq \frac{1}{|\lambda|^2} ||Tx||^2 ||T^*x - \lambda T^*x||^2_A \]

provided \( x \in \mathcal{H} \) and \( \lambda \in \mathbb{C} \setminus \{0\} \).

Since \( T \) is an \( A \)-normal operator, we obtain

\[ 0 \leq ||Tx||^2_A - ||T^2x||_{A^2}^2 \leq \frac{1}{|\lambda|^2} ||Tx||^2_A ||T^*x - \lambda T^*x||^2._A. \]

Hence the desired result (3.13) is obtained by taking the supremum on \( x \in \mathcal{H} \) with \( ||x||_A = 1 \).

The following Lemma was proved by Mitrinović, Pečarić and Fink in ([10], p544).

**Lemma 3.2.** Let \( a, b \in \mathcal{H} \).

1. If \( p \in (1, 2) \), then

\[ (||a|| + ||b||)^p + ||a|| - ||b|| ||a - b||^p \leq ||a + b||^p + ||a - b||^p \]

2. If \( p \geq 2 \), then

\[ 2(||a||^p + ||b||^p) \leq ||a + b||^p + ||a - b||^p \]

By choosing in Lemma 3.2 \( a = \lambda \sqrt{AT}x \) and \( b = \mu \sqrt{AT^*x} \), for \( \lambda, \mu \in \mathbb{C} \), \( x \in \mathcal{H} \), then taking the supremum over \( x \in \mathcal{H} \), \( ||x||_A = 1 \), we obtain the next result involving the seminorm \( ||\cdot||_A \).

**Theorem 3.10.** Let \( T \in L_A(\mathcal{H}) \) be an \( A \)-normal operator and \( \lambda, \mu \in \mathbb{C} \). Then

1. If \( p \in (1, 2) \), then

\[ (||\lambda|| + ||\mu||^p + ||\lambda|| - ||\mu||^p)||T||^p_A \leq ||\lambda T + \mu T^*||^p_A + ||\lambda T - \mu T^*||^p_A. \]

2. If \( p \geq 2 \), then

\[ 2(||\lambda||^p + ||\mu||^p)||T||^p_A \leq ||\lambda T + \mu T^*||^p_A + ||\lambda T - \mu T^*||^p_A. \]

**Remark 3.1.** In general, for \( T \in L_A(\mathcal{H}), \lambda, \mu \in \mathbb{C} \) and \( p \geq 2 \), we have

\[ w_A \left( \frac{||\lambda||^2T^2T + ||\mu||^2T^2}{2} \right)^{\frac{p}{2}} \leq \frac{1}{4} \left( ||\lambda T + \mu T^*||^p_A + ||\lambda T - \mu T^*||^p_A \right). \]

**References**


