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A-NORMAL OPERATORS IN SEMI HILBERTIAN SPACES

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ABSTRACT. In this paper we study some properties and inequalities of A-normal operators in semi-Hilbertian spaces by employing some known results for vectors in inner product spaces. We generalize also most of the inequalities of (α, β)-normal operators discussed in Hilbert spaces [7].

Key words and phrases: A-adjoint, Semi-inner product, Normal operators.

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1. INTRODUCTION

Throughout this paper $\mathcal{H}$ denotes a complex Hilbert space with inner product $\langle .|. \rangle$ and norm $||.||$. $\mathcal{L}(\mathcal{H})$ stands the Banach algebra of all bounded linear operators on $\mathcal{H}$. $I = I_{\mathcal{H}}$ being the identity operator and if $V \subset \mathcal{H}$ is a closed subspace, P_V is the orthogonal projection onto V. For $T \in \mathcal{L}(\mathcal{H})$ its range is denoted by $R(T)$, its null space by $N(T)$, its adjoint by T^* and its spectrum by $\sigma(T)$. The numerical range of T is a subset of the set of complex numbers $\mathbb{C}$ and it is defined by

$$W(T) = \{\langle Tx|x\rangle, \ x \in \mathcal{H}, \ ||x|| = 1\}$$

The spectral radius and the numerical radius and the minimum modulus of T will be denoted respectively by $r(T)$ and $w(T)$ and $\gamma(T)$. They are defined as $r(T) = \sup\{|\lambda|, \ \lambda \in \sigma(T)\}$ and $w(T) = \sup\{|\lambda|, \ \lambda \in W(T)\}$ and $\gamma(T) = \inf\{\|Tx\|, \ x \in N(T)^{\perp}$ and $\|x\| = 1\}$. It is well known that $\gamma(T) > 0$ if and only if $R(T)$ is closed and that $w(T)$ is a norm on the Banach algebra $\mathcal{L}(\mathcal{H})$ (for more detail about the concept of numerical radius, see for example [4],[9]). Moreover for $T \in \mathcal{L}(\mathcal{H})$, we have

$$w(T) \leq ||T|| \leq 2w(T), \tag{1.1}$$

and that for a normal operator T ([3]), one has

$$r(T) = w(T) = ||T|| \tag{1.2}$$

$\mathcal{L}(\mathcal{H})^+$ is the cone of positive operators, i.e.

$$\mathcal{L}(\mathcal{H})^+ = \{A \in \mathcal{L}(\mathcal{H}) : \langle Ax|x\rangle \geq 0, \ \forall \ x \in \mathcal{H} \ \}.$$

Any positive operator $A \in \mathcal{L}(\mathcal{H})^+$ defines a positive semi-definite sesquilinear form

$$\langle \ | \ \rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}, \ \langle x|y\rangle_A = \langle Ax|y\rangle.$$

By $\|.\|_A$ we denote the seminorm induced by $\langle \ | \ \rangle_A$, i.e., $\|x\|_A = \langle x|x\rangle_A^{\frac{1}{2}}$. Note that $\|x\|_A = 0$ if and only if $x \in N(A)$. Then $\|.\|_A$ is a norm on $\mathcal{H}$ if and only if A is an injective operator, and the semi-normed space $(\mathcal{L}(\mathcal{H}), \|.\|_A)$ is complete if and only if $R(A)$ is closed. Moreover $\langle \ | \ \rangle_A$ induces a seminorm on the subspace $\{T \in \mathcal{L}(\mathcal{H}) / \ \exists \ c > 0, \ \|Tx\|_A \leq c\|x\|_A, \ \forall \ x \in \mathcal{H}\}$. For this subspace of operators it holds

$$\|T\|_A = \sup_{x \in \overline{R(A)}, x \neq 0} \frac{\|Tx\|_A}{\|x\|_A} < \infty.$$

Moreover

$$\|T\|_A = \sup\{|\langle Tx|y\rangle_A|; \ x, y \in \mathcal{H} \text{ and } \ \|x\|_A \leq 1, \|y\|_A \leq 1 \ \}.$$

For $x, \ y \in \mathcal{H}$, we say that x and y are A-orthogonal if $\langle x|y\rangle_A = 0$. Note that this definition is a natural extension of the usual notion of orthogonality which represents the I-orthogonality case. For a set $S \subset \mathcal{H}$, its A-orthogonal subspace $S^{\perp_A}$ is given by

$$S^{\perp_A} = \{x \in \mathcal{H}; \ \langle x|y\rangle_A = 0, \ \ \forall \ y \in S\}.$$

Note that $S^{\perp_A} = (AS)^{\perp} = A^{-1}(S^{\perp})$ and since $A(A^{-1}(S) = S \cap R(A)$, then $(S^{\perp_A})^{\perp_A} = \big(S^{\perp} \cap R(A)\big)^{\perp}$. The concept of A-spectral radius, A-numerical radius and A-minimum modulus of an operator are a natural generalization of the spectral radius, the numerical radius and the minimum modulus respectively. In the next, we give the following definition.

Definition 1.1. *Let $T \in \mathcal{L}(\mathcal{H})$. The A-spectral radius, the A-numerical radius and the A-minimum modulus of T are denoted respectively $r_A(T)$, $w_A(T)$ and $\gamma_A(T)$ and they are defined as*

$$r_A(T) = \limsup_{n \to +\infty} ||T^n||_A^{\frac{1}{n}}$$

$$w_A(T) = \sup\{|\langle Tx|x\rangle_A|;\ x \in \mathcal{H},\ ||x||_A = 1\}$$

and

$$\gamma_A(T) = \inf\{\|Tx\|_A;\ x \in N(A^{\frac{1}{2}}T)^{\perp_A} \text{ and } \|x\|_A = 1\}.$$

For any $T,\ S \in \mathcal{L}(\mathcal{H})$, the following properties are immediate:

(1) $w_A(T) \geq 0$ and $w_A(T) = 0$ if and only if $AT = 0$.
(2) $w_A(\lambda T) = |\lambda| w_A(T)$ for any $\lambda \in \mathbb{C}$.
(3) $w_A(T + S) \leq w_A(T) + w_A(S)$.
(4) $\forall\ x \in \mathcal{H},\ |\langle Tx|x\rangle_A| \leq w_A(T)||x||_A^2 \leq \|T\|_A||x||_A^2$ and $\|Tx\|_A \geq \gamma_A(T) d_A(x, N(A^{\frac{1}{2}}T))$ where $d_A(x, V) = \inf\{\|x - y\|_A;\ y \in V\}$ for any $V \subset \mathcal{H}$.

Note that $w_A(.)$ is a seminorm on $\mathcal{L}(\mathcal{H})$ and it is a norm if A is injective. Moreover $w_A(T) \leq ||T||_A$ for any $T \in \mathcal{L}(\mathcal{H})$. The following theorem due to Douglas will be used (see [5] for its proof).

Theorem 1.1. *Let $T, S \in \mathcal{L}(\mathcal{H})$. The following conditions are equivalent.*

(1) $R(S) \subset R(T)$.
(2) *There exists a positive number λ such that $SS^* \leq \lambda TT^*$.*
(3) *There exists $W \in \mathcal{L}(\mathcal{H})$ such that $TW = S$.*

From now on, A denotes a positive operator on $\mathcal{H}$ (i.e. $A \in \mathcal{L}(\mathcal{H})^+$).

Definition 1.2. *Let $T \in \mathcal{L}(\mathcal{H})$, an operator $W \in \mathcal{L}(\mathcal{H})$ is called an A-adjoint of T if*

$$\langle Tu|v\rangle_A = \langle u|Wv\rangle_A \ \text{ for every }\ u, v \in \mathcal{H},$$

or equivalently

$$AW = T^*A;$$

T is called A-selfadjoint if $AT = T^*A$ and it is called A-positive if AT is positive.

By Douglas Theorem, an operator $T \in \mathcal{L}(\mathcal{H})$ admits an A-adjoint if and only if $R(T^*A) \subset R(A)$ and if W is an A-adjoint of T and $AZ = 0$ for some $Z \in \mathcal{L}(\mathcal{H})$ then $W + Z$ is also an A-adjoint of T. Hence neither the existence nor the uniqueness of an A-adjoint operator is guaranteed. In fact an operator $T \in \mathcal{L}(\mathcal{H})$ may admit none, one or many A-adjoints.
From now on, $\mathcal{L}_A(\mathcal{H})$ denotes the set of all $T \in \mathcal{L}(\mathcal{H})$ which admit an A-adjoint, i.e.

$$\mathcal{L}_A(\mathcal{H}) = \{\ T \in \mathcal{L}(\mathcal{H}) :\ R(T^*A) \subset R(A)\ \}.$$

$\mathcal{L}_A(\mathcal{H})$ is a subalgebra of $\mathcal{L}(\mathcal{H})$ which is neither closed nor dense in $\mathcal{L}(\mathcal{H})$.
On the other hand the set of all A-bounded operators in $\mathcal{L}(\mathcal{H})$ (i.e. with respect the seminorm $\|.\|_A$) is

$$\mathcal{L}_{A^{\frac{1}{2}}}(\mathcal{H}) = \{\ T \in \mathcal{L}(\mathcal{H}) :\ T^*R(A^{\frac{1}{2}}) \subset R(A^{\frac{1}{2}})\ \} = \{\ T \in \mathcal{L}(\mathcal{H}) :\ R(A^{\frac{1}{2}}T^*A^{\frac{1}{2}}) \subset R(A)\ \}.$$

Note that $\mathcal{L}_A(\mathcal{H}) \subset \mathcal{L}_{A^{\frac{1}{2}}}(\mathcal{H})$, which shows that if T admits an A-adjoint then it is A- bounded.
Section 2, contains some inequalities giving upper bounds of the difference between the A-norm and A-numerical radius of an A-bounded operator in semi-Hilbertian spaces and under appropriate conditions. In section 3, we introduce the notion of A-normal operators, we prove a characterization involving the A-norm, $||.||_A$, we give some properties on A-normal operators, then we establish new operator norm inequalities. Our inequalities generalize the well known properties for normal operators.

2. INEQUALITIES INVOLVING A-NUMERICAL RADIUS

If $T \in \mathcal{L}(\mathcal{H})$ with $R(T^*A) \subset R(A)$, then T, admits an A-adjoint operator, Moreover there exists a distinguished A-adjoint operator of T, namely, the reduced solution of the equation $AX = T^*A$, i.e. $T^\sharp = A^\dagger T^* A$, where $T^\dagger$ is the Moore-Penrose inverse of T. The A-adjoint operator $T^\sharp$ verifies

$$AT^\sharp = T^*A,\ R(T^\sharp) \subseteq \overline{R(A)} \text{ and } N(T^\sharp) = N(T^*A).$$

In the next we add without proof some important properties of $T^\sharp$ (for more details we refer the reader to [1] and [2]).

Theorem 2.1. *Let $T \in \mathcal{L}_A(\mathcal{H})$. Then*

1. *If $AT = TA$ then $T^\sharp = PT^*$.*
2. *$T^\sharp T$ and $TT^\sharp$ are A-selfadjoint and A-positive.*
3. *$\|T\|_A^2 = \|T^\sharp\|_A^2 = \|T^\sharp T\| = \|TT^\sharp\| = w_A(T^\sharp T) = w_A(TT^\sharp)$.*
4. *$\|S\|_A = \|T^\sharp\|_A$ for every $S \in \mathcal{L}(\mathcal{H})$ which is an A-adjoint of T.*
5. *If $S \in \mathcal{L}_A(\mathcal{H})$ then $ST \in \mathcal{L}_A(\mathcal{H})$, $(ST)^\sharp = T^\sharp S^\sharp$ and $\|TS\|_A = \|ST\|_A$.*
6. *$T^\sharp \in \mathcal{L}_A(\mathcal{H})$, $(T^\sharp)^\sharp = PTP$ and $((T^\sharp)^\sharp)^\sharp = T^\sharp$.*
7. *$\|T^\sharp\| \leq \|S\|$ for every $S \in \mathcal{L}(\mathcal{H})$ which is an A-adjoint of T. Nevertheless, $T^\sharp$ is not in general the unique A-adjoint of T that realizes the minimal norm.*

Lemma 2.1. *Let $T \in \mathcal{L}_A(\mathcal{H})$. If M is an invariant subspace for T and $T^\sharp$, then $M^{\perp_A}$ is also invariant for T and $T^\sharp$.*

Proof. Let $x \in M^{\perp_A}$, and $y \in M$, then $\langle Tx|y\rangle_A = \langle x|T^\sharp y\rangle_A = 0$, since $T^\sharp y \in M$. Thus $Tx \in M^{\perp_A}$, so $T(M^{\perp_A}) \subset M^{\perp_A}$. Similarly, we show that $T^\sharp(M^{\perp_A}) \subset M^{\perp_A}$.

In the following, we establish various inequalities between the operator seminorm $||.||_A$ and the A-numerical radius $w_A(.)$ of operators in semi-Hibertian spaces.

Theorem 2.2. *Let $T \in \mathcal{L}_A(\mathcal{H}), \lambda \in \mathbb{C}$ and $\alpha \geq 0$ are such that $||T - \lambda I||_A \leq \alpha$. Then*

$$(0 \leq)\ \ |\lambda|(||T||_A - w_A(T)) \leq \frac{\alpha^2}{2} \tag{2.1}$$

Moreover, if $|\lambda| > \alpha$ then

$$\sqrt{1 - \frac{\alpha^2}{|\lambda|^2}}||T||_A \leq w_A(T) \leq ||T||_A \tag{2.2}$$

Proof. Since $||T - \lambda I||_A \leq \alpha$ then for $x \in \mathcal{H}$ with $||x||_A = 1$, we have $||Tx - \lambda x||_A \leq \alpha$, or equivalently $||Tx - \lambda x||_A^2 \leq \alpha^2$, which implies that

$$||Tx||_A^2 + |\lambda|^2 \leq 2Re(\overline{\lambda}\langle Tx|x\rangle_A) + \alpha^2 \leq 2|\lambda||\langle Tx|x\rangle_A| + \alpha^2$$

By taking the supremum over $x \in \mathcal{H},\ ||x||_A = 1$, it follows

$$2|\lambda|||T||_A \leq ||T||_A^2 + |\lambda|^2 \leq 2|\lambda|w_A(T) + \alpha^2 \tag{2.3}$$

Hence the desired inequality (2.1) is obtained.
Now if $|\lambda| > \alpha$, on dividing with $|\lambda|^2$ in (2.3) we obtain

$$\frac{||T||_A^2}{|\lambda|^2} + 1 \leq 2\frac{w_A(T)}{|\lambda|} + \frac{\alpha^2}{|\lambda|^2}$$

then by using an elementary inequality, we deduce

$$2\sqrt{1-\frac{\alpha^2}{|\lambda|^2}}\frac{||T||_A}{|\lambda|} \leq \frac{||T||_A^2}{|\lambda|^2} + 1 - \frac{\alpha^2}{|\lambda|^2} \leq 2\frac{w_A(T)}{|\lambda|}$$

from which the inequality (2.2) is easily holds.

Remark 2.1. *Note that for $T \in \mathcal{L}_A(\mathcal{H}), \lambda \in \mathbb{C}$ and $|\lambda| > \alpha \geq 0$ such that $||T - \lambda I||_A \leq \alpha$, (1.1) and (2.2) lead a refinement and improve (1.1) and they provide the following inequalities*

$$w_A(T) \leq ||T||_A \leq \sqrt{\frac{|\lambda|^2}{|\lambda|^2 - \alpha^2}} w_A(T) \leq 2w_A(T), \text{ if } \frac{\alpha}{|\lambda|} \leq \frac{\sqrt{3}}{2}$$

Using the fact that for $x, y, z \in \mathcal{H}$, one has

$$Re\langle y - x | x - z\rangle_A \geq 0 \Leftrightarrow ||x - \frac{y+z}{2}||_A \leq \frac{1}{2}||y - z||_A$$

and by applying Theorem 2.2, (2.1), the following corollary is immediately deduced.

Corollary 2.3. *Let $T \in \mathcal{L}_A(\mathcal{H}), \lambda, \mu \in \mathbb{C},\ \lambda \neq \mu$. If $Re\langle \lambda x - Tx | Tx + \mu x\rangle_A \geq 0$, for all $x \in \mathcal{H}$ then*

$$(0 \leq)\ \ ||T||_A - w_A(T) \leq \frac{1}{4}\frac{|\lambda + \mu|^2}{|\lambda - \mu|} \tag{2.4}$$

Remark 2.2. *Note that in the literature, the condition $Re\langle \lambda x - Tx | Tx + \mu x\rangle_A \geq 0,\ x \in \mathcal{H}$ means that the operator*

$$(T^\sharp + \overline{\mu} I)A(\lambda I - T) \text{ is accretive}. \tag{2.5}$$

On squaring (2.2) and replacing λ by $\frac{\lambda-\alpha}{2}$, α by $\frac{|\lambda+\alpha|}{2}$, the following corollary follows

Corollary 2.4. *Let $T \in \mathcal{L}_A(\mathcal{H}), \lambda, \mu \in \mathbb{C}$, with $Re(\lambda\overline{\mu}) \leq 0$. If T verifies (2.5), then*

$$(0 \leq)\ \ ||T||_A^2 - w_A(T)^2 \leq |\frac{\lambda + \mu}{\lambda - \mu}|^2 ||T||_A^2. \tag{2.6}$$

and

$$\frac{2\sqrt{-Re(\lambda\overline{\mu})}}{|\lambda - \mu|}||T||_A \leq w_A(T)$$

in particular if we choose $\lambda = -\mu > 0$, we get

$$||T||_A = w_A(T). \tag{2.7}$$

3. A-NORMAL OPERATORS

In the following we introduce the notion of A-normal operators.

Definition 3.1. *An operator $T \in \mathcal{L}_A(\mathcal{H})$ is called an A-normal operator if $T^\sharp T = TT^\sharp$.*

A-normal operators may be regarded as a generalization of normal and self-adjoint operators in which $T^\sharp = T^*$. This last property is realized in particular if $A = I$ or if T and A commute and A has a dense range [1].
The identity operator and the orthogonal projection on $\overline{R(A)}$ are A-normal. Moreover, if T is an A-normal then $\{TS,\ T + S /\ TS = ST,\ S = S^\sharp\}$ is a set of A-normal operators.

Another characterization is that $T \in \mathcal{L}_A(\mathcal{H})$ is an A-normal operator if and only if there are A-selfadjoint operators $B, C \in \mathcal{L}_A(\mathcal{H})$ such that $BC = CB$ and $T = B + iC,\ (i^2 = -1)$.

From now on, to simplify notation, we write P instead of $P_{\overline{R(A)}}$. An important property of A-normal operators that will be used frequently in the sequel is the following:

Theorem 3.1. *A necessary and sufficient condition for an operator $T \in \mathcal{L}_A(\mathcal{H})$ to be A-normal is that $R(TT^\sharp) \subset \overline{R(A)}$ and $||Tx||_A = ||T^\sharp x||_A$ for every vector $x \in \mathcal{H}$.*

Proof. Suppose that T is A-normal. It is easily to see that $R(TT^\sharp) = R(T^\sharp T) \subset \overline{R(A)}$. Moreover, using the fact that $TT^\sharp$ is A-selfadjoint, then for $x \in \mathcal{H}$, we obtain,

$$\begin{aligned} T^\sharp T = TT^\sharp &\Rightarrow \langle T^\sharp Tx|x\rangle_A = \langle TT^\sharp x|x\rangle_A \\ &\Leftrightarrow \langle AT^\sharp Tx|x\rangle = \langle ATT^\sharp x|x\rangle \\ &\Leftrightarrow \langle T^*ATx|x\rangle = \langle (TT^\sharp)^*Ax|x\rangle \\ &\Leftrightarrow \langle ATx|Tx\rangle = \langle T^*Ax|T^\sharp x\rangle \\ &\Leftrightarrow ||Tx||_A = ||T^\sharp x||_A \end{aligned}$$

Conversely, if $||Tx||_A = ||T^\sharp x||_A$, then $A(T^\sharp T - TT^\sharp) = 0$, if moreover $R(TT^\sharp) \subset \overline{R(A)}$, so, it follows $R(T^\sharp T - TT^\sharp) \subset \overline{R(A)} = N(A)^\perp$ and hence $T^\sharp T - TT^\sharp = 0$, which finishes the proof.

In the next we give some properties on A-normal operators.

Corollary 3.2. *For $T \in \mathcal{L}_A(\mathcal{H})$, the following properties hold*

(1) *If T is A-selfadjoint operator then $||T||_A = w_A(T)$.*
(2) *If T is A-normal operator then T^n is also for all $n \geq 1$ and $||T||_A = r_A(T)$.*
(3) *Suppose that $N(A)$ is an invariant subspace for T and $\lambda, \mu \in \mathbb{C}$. If T is A-normal, then*
   (a) *$T - \lambda I$ and $T^\sharp$ are A-normal.*
   (b) *$Tx = \lambda x$ yields $T^\sharp x = \overline{\lambda} Px$.*
   (c) *$M = \{x \in \mathcal{H}/Tx = \lambda x\}$ and $M^{\perp_A}$ are invariant for T and $T^\sharp$.*
   (d) *$Tx = \lambda x$ and $Ty = \mu y,\ \lambda \neq \mu$ yield $x \perp_A y\ (i.e.\ \langle x|y\rangle_A = 0)$.*

Proof.

(1) It is clear that $\sup_{||x||_A = ||y||_A = 1} |\langle Tx|y\rangle_A| \leq ||T||_A$. In the other hand, if we choose $z = \frac{Tx}{||Tx||_A}$, we obtain

$$||Tx||_A = \langle Tx|z\rangle_A| \leq \sup_{||x||_A = ||y||_A = 1} |\langle Tx|y\rangle_A|$$

Moreover, without loss of generality we can suppose $x, y \neq 0$ and that $\langle Tx|y\rangle_A > 0$, then one has

$$\langle T(x+y)|x+y\rangle_A = \langle Tx|x\rangle_A + \langle Tx|y\rangle_A + \langle y|T^\sharp x\rangle_A + \langle Ty|y\rangle_A$$

and

$$\langle T(x-y)|x-y\rangle_A = \langle Tx|x\rangle_A - \langle Tx|y\rangle_A - \langle y|T^\sharp x\rangle_A + \langle Ty|y\rangle_A$$

If T is A-selfadjoint then, by parallelogram law

$$\begin{aligned} |\langle Tx|y\rangle_A| = \frac{1}{2}|\langle Tx|y\rangle_A + \langle T^\sharp x|y\rangle_A| &= \frac{1}{2}|\langle T(x+y)|x+y\rangle_A - \langle T(x-y)|x-y\rangle_A| \\ &\leq \frac{w_A(T)}{2}(||x+y||_A^2 + ||x-y||_A^2) \\ &\leq w_A(T)(||x||_A^2 + ||y||_A^2) \end{aligned}$$

If we replace x by $\sqrt{\alpha}x$ and y by $\frac{y}{\sqrt{\alpha}}$, where $\alpha = \frac{||y||_A}{||x||_A}$, we get

$$\begin{aligned} |\langle Tx|y\rangle_A| &= |\langle Tx|y\rangle_A + \langle T^\sharp x|y\rangle_A| \\ &\leq \frac{w_A(T)}{2}(||x||_A^2 + ||y||_A^2) \\ &= w_A(T)||x||_A||y||_A \end{aligned}$$

which implies, $w_A(T) \leq ||T||_A$ and thus,

$$||T||_A = \sup\{|\langle Tx|y\rangle_A|;\ ||x||_A = ||y||_A = 1\} = w_A(T)$$

(2) Let $n \geq 1$, if T is A-normal operator then, T and $T^\sharp$ commute, consequently T^n and $(T^\sharp)^n$ commute. Thus T^n is A-normal.
Let $x \in \mathcal{H}$, we have

$$||T^\sharp Tx||_A^2 = \langle T^\sharp Tx|T^\sharp Tx\rangle_A = \langle T^2x|T^2x\rangle_A = ||T^2x||_A^2$$

$$||Tx||_A^2 = \langle Tx|Tx\rangle_A = \langle T^\sharp Tx|x\rangle_A$$

Since $T^\sharp T$ is A-selfadjoint then by taking the supremum on $||x||_A = 1$ and applying 1. we get

$$\begin{aligned} ||T||_A^2 = \sup_{||x||_A=1} ||Tx||_A^2 &= \sup_{||x||_A=1} \langle T^\sharp Tx|x\rangle_A \\ &= ||T^\sharp T||_A = \sup_{||x||_A=1} ||T^\sharp Tx||_A \\ &= \sup_{||x||_A=1} ||T^2x||_A = ||T^2||_A \end{aligned}$$

Moreover for all $n \geq 1$ we have

$$||T^nx||_A^2 = \langle T^nx|T^nx\rangle_A = \langle T^\sharp T^nx|T^{n-1}x\rangle_A \leq ||T^\sharp T^nx||_A.||T^{n-1}x||_A$$

which implies

$$||T^n||_A^2 \leq ||T^{n+1}||_A.||T^{n-1}||_A$$

Assume that $||T||_A > 0$ then $||T^n||_A > 0$ for all $n \geq 1$ (for $||T||_A = 0$ the desired property is evident) and set $\alpha_n = \frac{||T^{n+1}||_A}{||T^n||_A},\ n \geq 1$. It is clear that $(\alpha_n)_n$ is an increasing sequence, then it satisfies

$$\frac{||T^{n+1}||_A}{||T^n||_A} = \alpha_n \geq \alpha_1 = \frac{||T^2||_A}{||T||_A} = \frac{||T||_A^2}{||T||_A} = ||T||_A.$$

By an induction argument, it follows $||T^n||_A = ||T||_A^n$, for all $n \geq 1$.
Thus $r_A(T) = ||T^n||_A^{\frac{1}{n}} = ||T||_A$ and the proof is achieved .

(3) (a) Note first that since $N(A)$ is invariant for T, then $TP = PT$ and $AP = PA = A$. Let now $\lambda \in \mathbb{C}$, we have $(T - \lambda I)(T - \lambda I)^\sharp = (T - \lambda I)(T^\sharp - \overline{\lambda}P) = TT^\sharp - \lambda T^\sharp - \lambda T^\sharp - T\overline{\lambda}P + |\lambda|^2P = T^\sharp T - \lambda T^\sharp - \overline{\lambda}PT + |\lambda|^2P = (T - \lambda I)^\sharp(T - \lambda I)$,

then $(T-\lambda I)$ is A-normal.
For all $x \in \mathcal{H}$, we have also

$$\begin{aligned} ||(T^\sharp)^\sharp x||_A^2 &= \langle (T^\sharp)^\sharp x|(T^\sharp)^\sharp x\rangle_A \\ &= \langle PTPx|PTPx\rangle_A \\ &= \langle TPx|TPx\rangle_A \\ &= ||TPx||_A^2 \\ &= ||Tx||_A^2 = ||T^\sharp x||_A^2 \end{aligned}$$

It clear that $R(T^\sharp(T^\sharp)^\sharp) \subset \overline{R(A)}$, so from Theorem 3.1, it follows that $T^\sharp$ is A-normal.

(b) Using (a),

$$\begin{aligned} ||\sqrt{A}(T^\sharp - \lambda P)x|| &= ||(T^\sharp - \lambda P)x||_A \\ &= ||(T-\lambda I)^\sharp x||_A \\ &= ||(T-\lambda I)x||_A = 0 \end{aligned}$$

or $R(T^\sharp - \lambda P) \subset \overline{R(A)} = N(A)^\perp$, then $T^\sharp x = \lambda P x$

(c) Let $M = \{x \in \mathcal{H}/Tx = \lambda x\}$. It is clear that $T(M) \subset M$. Moreover if $x \in M$ and $y = T^\sharp x$, then $Ty = TT^\sharp x = T^\sharp Tx = \lambda T^\sharp x = \lambda y$ yields $y = Tx \in M$. Hence M is invariant for both T and $T^\sharp$. Using Lemma 2.1 the desired result follows.

(d) Suppose that $Tx = \lambda x,\ Ty = \mu y$ with $0 \neq \lambda \neq \mu$,
$\langle x|y\rangle_A = \lambda^{-1}\langle Tx|y\rangle_A = \lambda^{-1}\langle x|T^\sharp y\rangle_A = \lambda^{-1}\mu\langle x|Py\rangle_A = \lambda^{-1}\mu\langle x|y\rangle_A$,
then $\langle x|y\rangle_A = 0$. If $\lambda = 0$ we permute between λ and μ and the proof achieved.

Question: If T is A-normal, is it true that $||T||_A = w_A(T)$?

Note that in the Cauchy-Schwarz inequality i.e.

$$|\langle u|v\rangle| \leq ||u||\ ||v||,\ u, v \in \mathcal{H} \tag{3.1}$$

if, we choose $u = \sqrt{A}x$ and $v = \sqrt{A}y$ we obtain more general formula

$$|\langle x|y\rangle_A| \leq ||x||_A\ ||y||_A,\ x, y \in \mathcal{H} \tag{3.2}$$

Moreover, for the choices Tx instead of x and $T^\sharp x$ instead of y with $x \in \mathcal{H}$, then one gets the following simple inequality for the A-normal operator T:

$$|\langle T^2x|x\rangle_A| \leq ||Tx||_A^2,\ x \in \mathcal{H} \tag{3.3}$$

Note that the inequality (3.3) implies in particular that

$$w_A(T^2) \leq ||T||_A^2.$$

Note also that the inequality (3.3) becomes an equality if T is an A-selfadjoint operator. This property does not remain true for A-normal operators. Indeed if consider the operators $\mathcal{H} = \mathbb{C}^2,\ A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathcal{L}(\mathcal{H})^+,\quad T = \begin{pmatrix} r & r \\ -r & r \end{pmatrix} \in \mathcal{L}(\mathcal{H})$ for some $a > 0$ and $r \neq 0$. It is easy to check that T admits A-adjoint operators and by direct computation, we see that T is an A-normal operator and that (3.3) is a real inequality.

It is then naturel to discuss some estimations of the quantity $||Tx||_A^2 - |\langle T^2x|x\rangle_A|$ for A-normal operators and give a measure of the closeness of the two terms involved in (3.3).

Motivated by this problem, we will study in this section some inequalities of A-normal operators in semi-Hilbertian spaces by employing some known results for vectors in inner product spaces.
We start with the following result.

Theorem 3.3. *Let $T \in \mathcal{L}_A(\mathcal{H})$ be an A-normal operator, then the inequalities*

$$|\langle Tx|x\rangle_A|^2 \leq \frac{1}{2}(||Tx||_A^2 + |\langle T^2x|x\rangle_A|) \leq ||Tx||_A^2 \tag{3.4}$$

hold for all $x \in \mathcal{H},\ ||x||_A = 1$. The constant $\frac{1}{2}$ is the best possible in (3.4).

Proof. The second inequality in (3.4) hold immediately from (3.3). For the first one we use the inequality, which is a consequence of the inequalities (2.3) in [6].

$$|\langle a|e\rangle_A\ \langle e|b\rangle_A| \leq \frac{1}{2}(||a||_A\ ||b||_A + |\langle a|b\rangle_A|) \tag{3.5}$$

provided a, b, e are vectors in $\mathcal{H}$ and $||e||_A = 1$.
If we choose $e = x,\ ||x||_A = 1,\ a = Tx,$ and $b = T^\sharp x$, then we obtain

$$|\langle Tx|x\rangle_A\ \langle x|T^\sharp x\rangle_A| \leq \frac{1}{2}(||Tx||_A\ ||T^\sharp x||_A + |\langle Tx|T^\sharp x\rangle_A|) \tag{3.6}$$

for all $x \in \mathcal{H}$ and $||x||_A = 1$.
Since T is A-normal, then $||Tx||_A = ||T^\sharp x||_A$ and the desired inequality follows from (3.6). If we suppose now, that $T = I$ is the identity operator, then both the two inequalities in (3.4) become equalities, this means that $\frac{1}{2}$ is the best possible constant in (3.4).

The following result is obviously deduced from Theorem 3.3.

Corollary 3.4. *If $T \in \mathcal{L}_A(\mathcal{H})$ is an A-normal operator, then*

$$w_A(T)^2 \leq \frac{1}{2}(||T||_A^2 + w_A(T^2)) \leq ||T||_A^2. \tag{3.7}$$

The following result provides an upper bound for the nonnegative quantity

$$||Tx||_A^2 - |\langle T^2x|x\rangle_A|,\ x \in \mathcal{H}$$

Theorem 3.5. *Let $T \in \mathcal{L}_A(\mathcal{H})$ be an A-normal operator and $\lambda \in \mathbb{C}$, then*

$$0 \leq ||Tx||_A^2 - |\langle T^2x|x\rangle_A| \leq \frac{2}{1+|\lambda|^2}||Tx - \lambda T^\sharp x||_A^2 \tag{3.8}$$

for any $x \in \mathcal{H}$.

Proof. For $\lambda = 0$, the inequality in (3.8) is obvious. For $\lambda \neq 0$, we use the Dunkl-Williams inequality [8],

$$\frac{||a||\ ||b|| - |\langle a|b\rangle|}{||a||\ ||b||} \leq \frac{2||a-b||^2}{(||a|| + ||b||)^2},\ a, b \in \mathcal{H}\backslash\{0\}$$

which shows that

$$\frac{||a||_A\ ||b||_A - |\langle a|b\rangle_A|}{||a||_A\ ||b||_A} \leq \frac{2||a-b||_A^2}{(||a||_A + ||b||_A)^2},\ a, b \notin N(A) \tag{3.9}$$

Now, taking into account that T is an A-normal operator, we choose in (3.9) $a = Tx$ and $b = \lambda T^\sharp x,\ \lambda \neq 0,\ x \notin N(A^{\frac{1}{2}}T)$, so from Theorem 3.1, one gets

$$\frac{||Tx||_A^2 - |\langle Tx|T^\sharp x\rangle_A|}{||Tx||_A^2} \leq \frac{2||Tx - \lambda T^\sharp x||_A^2}{(1+|\lambda|^2)^2||Tx||_A^2}$$

which immediately implies (3.8).
Since for A-normal operators $N(A^{\frac{1}{2}}T) = N(A^{\frac{1}{2}}T^{\sharp})$ then, the inequality (3.8) holds also for $x \in N(A^{\frac{1}{2}}T)$ and so the proof is achieved.

Corollary 3.6. *If $T \in \mathcal{L}_A(\mathcal{H})$ is an A-normal operator, then*

$$w_A(T)^2 - w_A(T^2) \leq \frac{1}{2}(||T||_A^2 - w_A(T^2)) \leq \frac{1}{1+|\lambda|^2}||T - \lambda T^{\sharp}||_A^2.$$

for all $\lambda \in \mathbb{C}$

The next technic result generalizes Lemma 2.1, [6].

Lemma 3.1. *Let $a, b \notin N(A)$ and $0 < \varepsilon \leq \frac{1}{2}$, such that*

$$0 \leq 1 - \varepsilon - \sqrt{1-2\varepsilon} \leq \frac{||a||_A}{||b||_A} \leq 1 - \varepsilon + \sqrt{1-2\varepsilon}.$$

Then

$$0 \leq ||a||_A\, ||b||_A - Re\langle a|b\rangle_A \leq \varepsilon ||a-b||_A^2. \tag{3.10}$$

Using Lemma 3.1, the following similar result may be stated

Theorem 3.7. *Let $T \in \mathcal{L}_A(\mathcal{H})$ be an A-normal operator, $\lambda \in \mathbb{C}$ and $0 < \varepsilon \leq \frac{1}{2}$ such that*

$$0 \leq 1 - \varepsilon - \sqrt{1-2\varepsilon} \leq |\lambda| \leq 1 - \varepsilon + \sqrt{1-2\varepsilon}.$$

Then

$$0 \leq ||Tx||_A^2 - |\langle T^2x|x\rangle_A| \leq \frac{\varepsilon}{|\lambda|}||Tx - \lambda T^{\sharp}x||_A^2 \tag{3.11}$$

for any $x \in \mathcal{H}$

Proof. By choosing $a = \lambda T^{\sharp}x$ and $b = Tx,\ x \notin N(A^{\frac{1}{2}}T)$ in Lemma 3.1, we have

$$0 \leq ||\lambda T^{\sharp}x||_A\, ||Tx||_A - Re\langle \lambda T^{\sharp}x|Tx\rangle_A \leq \varepsilon ||\lambda T^{\sharp}x - Tx||_A^2.$$

or $0 \leq ||Tx||_A^2 - |\langle T^2x|x\rangle_A|$, $||Tx||_A = ||T^{\sharp}x||$ and $Re\langle \lambda T^{\sharp}x|Tx\rangle_A \leq |\lambda||\langle T^2x|x\rangle_A|$, T being an A-normal operator, then (3.11) holds for any $x \notin N(A^{\frac{1}{2}}T)$.
Since $N(A^{\frac{1}{2}}T^{\sharp}) = N(A^{\frac{1}{2}}T)$, then for $x \in N(A^{\frac{1}{2}}T)$ it is clear that the inequality (3.11) is checked. Therefore, (3.11) holds for any $x \in \mathcal{H}$.

The following corollary may be stated

Corollary 3.8. *Let $T \in \mathcal{L}_A(\mathcal{H})$ be an A-normal operator, $\lambda \in \mathbb{C}$ and $0 < \varepsilon \leq \frac{1}{2}$ such that*

$$0 \leq 1 - \varepsilon - \sqrt{1-2\varepsilon} \leq |\lambda| \leq 1 - \varepsilon + \sqrt{1-2\varepsilon}.$$

Then

$$0 \leq ||T||_A^2 - w_A(T^2) \leq \frac{\varepsilon}{|\lambda|}||T - \lambda T^{\sharp}||_A^2 \tag{3.12}$$

Theorem 3.9. *Let $T \in \mathcal{L}_A(\mathcal{H})$ be an A-normal operator and $\lambda \in \mathbb{C} \setminus \{0\}$. Then*

$$0 \leq ||T||_A^4 - w_A(T^2)^2 \leq \frac{1}{|\lambda|^2}||T||_A^2||T - \lambda T^{\sharp}||_A^2 \tag{3.13}$$

Proof. We use the following inequality obtained by Dragomir (see [7],(2.10)).

$$0 \leq ||a||^2||b||^2 - |\langle a|b\rangle|^2 \leq \frac{1}{|\lambda|^2}||a||^2||a-\lambda b||^2 \tag{3.14}$$

provided $a, b \in \mathcal{H}$ and $\lambda \in \mathbb{C} \setminus \{0\}$.
Immediately on choosing $a = \sqrt{A}Tx$ and $b = \sqrt{A}T^\sharp x$, one gets,

$$0 \leq ||Tx||_A^2||T^\sharp x||_A^2 - |\langle Tx|T^\sharp x\rangle_A|^2 \leq \frac{1}{|\lambda|^2}||Tx||_A^2||Tx-\lambda T^\sharp x||_A^2$$

provided $x \in \mathcal{H}$ and $\lambda \in \mathbb{C} \setminus \{0\}$.
Since T is an A-normal operator, we obtain

$$0 \leq ||Tx||_A^4 - |\langle T^2x|x\rangle_A|^2 \leq \frac{1}{|\lambda|^2}||Tx||_A^2||Tx-\lambda T^\sharp x||_A^2.$$

Hence the desired result (3.13) is obtained by taking the supremum on $x \in \mathcal{H}$ with $||x||_A = 1$.
The following Lemma was proved by Mitrinović, Pečarić and Fink in ([10], p544).

Lemma 3.2. *Let* $a, b \in \mathcal{H}$*,*

(1) *If* $p \in (1,2)$*, then*

$$(||a|| + ||b||)^p + |\,||a|| - ||b||\,|^p \leq ||a+b||^p + ||a-b||^p \tag{3.15}$$

(2) *If* $p \geq 2$*, then*

$$2(||a||^p + ||b||^p) \leq ||a+b||^p + ||a-b||^p \tag{3.16}$$

By choosing in Lemma 3.2 $a = \lambda\sqrt{A}Tx$ and $b = \mu\sqrt{A}T^\sharp x$, for $\lambda, \mu \in \mathbb{C},\ x \in \mathcal{H}$, then taking the supremum over $x \in \mathcal{H},\ ||x||_A = 1$, we obtain the next result involving the seminorm $||.||_A$.

Theorem 3.10. *Let* $T \in \mathcal{L}_A(\mathcal{H})$ *be an* A*-normal operator and* $\lambda, \mu \in \mathbb{C}$*. Then*

(1) *If* $p \in (1,2)$*, then*

$$[(|\lambda| + |\mu|)^p + |\,|\lambda| - |\mu|\,|^p]||T||_A^p \leq ||\lambda T + \mu T^\sharp||_A^p + ||\lambda T - \mu T^\sharp||_A^p. \tag{3.17}$$

(2) *If* $p \geq 2$*, then*

$$2(|\lambda|^p + |\mu|^p)||T||_A^p \leq ||\lambda T + \mu T^\sharp||_A^p + ||\lambda T - \mu T^\sharp||_A^p. \tag{3.18}$$

Remark 3.1. *In general, for* $T \in \mathcal{L}_A(\mathcal{H})$*,*$\lambda, \mu \in \mathbb{C}$ *and* $p \geq 2$*, we have*

$$w_A\Big(\frac{|\lambda|^2T^\sharp T + |\mu|^2TT^\sharp}{2}\Big)^{\frac{p}{2}} \leq \frac{1}{4}\Big(||\lambda T + \mu T^\sharp||_A^p + ||\lambda T - \mu T^\sharp||_A^p\Big). \tag{3.19}$$

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