The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 9, Issue 1, Article 21, pp. 1-12, 2012

ON GENERALIZED TRIANGLE INEQUALITY IN p-FREÉCHET SPACES, $0 < p < 1$

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Received 9 January, 2009; accepted 4 November, 2009; published 29 June, 2012.

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ABSTRACT. In this paper generalized triangle inequality and its reverse in a p-Fréchet space where, $0 < p < 1$ are obtained.

Key words and phrases: p-Fréchet space, p-norm, triangle inequality.

2000 *Mathematics Subject Classification.* 26D15.

ISSN (electronic): 1449-5910

1. INTRODUCTION

A norm inequality for n vectors in a normed linear space obtained by Pêcarić and Rajić in [11] is given by

$$\max_{k\in\{1,2,\ldots,n\}}\left\{\frac{1}{\|x_k\|}\left[\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}|\|x_j\|-\|x_k\||\right]\right\}$$
$$\leq\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\leq\min_{k\in\{1,2,\ldots,n\}}\left\{\frac{1}{\|x_k\|}\left[\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}|\|x_j\|-\|x_k\||\right]\right\} \tag{1.1}$$

provided x_j are nonzero vectors in a normed linear space $(X,\|.\|)$ over the filed $\mathbb{K}$ ($\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$) and $j\in\{1,2,...,n\}$. In order to provide generalization to the above inequality S.S Dragomir [4] gave the following inequality for n vectors

$$\max_{k\in\{1,2,\ldots,n\}}\left\{|\alpha_k|\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}|\alpha_j-\alpha_k|\,\|x_j\|\right\}$$
$$\leq\left\|\sum_{j=1}^{n}\alpha_jx_j\right\|\leq\min_{k\in\{1,2,\ldots,n\}}\left\{|\alpha_k|\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}|\alpha_j-\alpha_k|\,\|x_j\|\right\} \tag{1.2}$$

where x_j are vectors in a normed linear space $(X,\|.\|)$ over the filed $\mathbb{K}$ ($\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$) and $\alpha_j\in\mathbb{K}$, $j\in\{1,2,...,n\}$. Readers can easily verify that the choice of $\alpha_k=\frac{1}{\|x_k\|}$, $\|x_k\|\neq0$, $k\in\{1,2,...,n\}$, in (1.2) gives the Pêcarić and Rajić inequality given above by (1.1). Pêcarić and Rajić inequality also gives the following refinement and reverse established by M. Kato et al. in [8]

$$\min_{k\in\{1,2,\ldots,n\}}\{\|x_k\|\}\left[n-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right]$$
$$\leq\sum_{j=1}^{n}\|x_j\|-\left\|\sum_{j=1}^{n}x_j\right\|\leq\max_{k\in\{1,2,\ldots,n\}}\{\|x_k\|\}\left[n-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right] \tag{1.3}$$

The other choice of the $\alpha_k=\|x_k\|$, $k\in\{1,2,...,n\}$, in (1.2) gives the following result

$$\max_{k\in\{1,2,\ldots,n\}}\left\{\|x_k\|\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}|\|x_j\|-\|x_k\||\,\|x_j\|\right\}$$
$$\leq\left\|\sum_{j=1}^{n}\|x_j\|\,x_j\right\|\leq\min_{k\in\{1,2,\ldots,n\}}\left\{\|x_k\|\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}|\|x_j\|-\|x_k\||\,\|x_j\|\right\} \tag{1.4}$$

which in turn implies another refinement and reverse of the generalized triangle inequality given below

$$(0 \leq) \frac{\sum_{j=1}^{n} \|x_j\|^2 - \left\| \sum_{j=1}^{n} x_j \|x_j\| \right\|}{\max_{k \in \{1,2,\dots,n\}} \{\|x_k\|\}}$$

$$\leq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\| \leq \frac{\sum_{j=1}^{n} \|x_j\|^2 - \left\| \sum_{j=1}^{n} x_j \|x_j\| \right\|}{\min_{k \in \{1,2,\dots,n\}} \{\|x_k\|\}} \tag{1.5}$$

Motivated by the above results (1.1)-(1.5), the main purpose of the present paper is to establish all these results in a p-Fréchet space, where $0 < p < 1$.

2. PRELIMINARIES

It is well known that an F-space $(X, +, ., \|.\|)$ is a linear space (over the field $\mathbb{K} = \mathbb{R}$ or $K = \mathbb{C}$) such that $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$, $\|x\| = 0$ if and only if $x = 0$, $\|\lambda x\| \leq |\lambda| \|x\|$, for all scalars λ with $|\lambda| \leq 1, x \in X$, and with respect to the metric $D(x, y) = \|x - y\|$, X is a complete metric space (see e.g. [3, p. 52] or [7]). Obviously D is invariant under translations. In addition, if there exists $0 < p < 1$ with $\|\lambda x\| = |\lambda|^p \|x\|$, for all $\lambda \in \mathbb{K}$ and $x \in X$, then $\|.\|$ will be called a p-norm and X will be called p-Fréchet space. (This is only a slight abuse of terminology. Note that in e.g. [2] these spaces are called p-Banach spaces). In this case, it is immediate that $D(\lambda x, \lambda y) = |\lambda|^p D(x, y)$, for all $x, y \in X$, $\lambda \in \mathbb{K}$. It is known that F-spaces are not necessarily locally convex spaces. Three classical examples of p-Fréchet spaces, non-locally convex, are the Hardy space H_p with $0 < p < 1$ that consists in the class of all analytic functions $f : \mathbb{D} \longrightarrow \mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C}; |z| \leq 1\}$ with the property

$$\|f\| = \frac{1}{2\pi} \sup\{ \int_0^{2\pi} |f(re^{it})|^p dt, r \in [0, 1)\} < +\infty$$

the sequences space l^p

$$l^p = \{x = (x_n)_n; \|x\| = \sum_{n=1}^{\infty} |x_n| < \infty\}$$

for $0 < p < 1$, and the $L^p[0, 1]$, $0 < p < 1$, given by

$$L^p[0, 1] = \{f : [0, 1] \longrightarrow \mathbb{R}; \|f\| = \int_0^1 |f(t)|^p dt < \infty\}$$

More generally, we may consider $L^p(\Omega, \Sigma, \mu)$, $0 < p < 1$, based on a general measure space (Ω, Σ, μ), with the p-norm given by $\|f\| = \int_\Omega |f|^p d\mu$. Some important characteristics of the F-spaces are given by the following remark.

Remark 2.1. Three fundamental results in Functional Analysis hold for F-spaces too : the Principle of Uniform Boundedness (see e.g. [3, p. 52]), the Open Mapping Theorem and the Closed Graph Theorem (see e.g. [7, p. 9-10]). But on the other hand, the Hahn-Banach Theorem fails in non-locally convex F-spaces. More exactly, if in an F-space the Hahn-Banach theorem holds, then that space is necessarily locally convex space (see e.g. [7, Chapter 4]).

3. MAIN RESULTS

Everywhere in this section, $(X, +, ., \|.\|)$ is a p-Fréchet space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $0 < p < 1$ unless otherwise specified. We use the same technique as in [4] to establish our results. Following theorem gives another form of (1.2) in a p-Fréchet space X.

Theorem 3.1. *If $x_j \in X$, $\alpha_j \in \mathbb{K}$, $j \in \{1, 2, ..., n\}$ and $0 < p < 1$, then*

$$\max_{k\in\{1,2,...,n\}}\left\{|\alpha_k|^p\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{j=1}^{n}|\alpha_j-\alpha_k|^p\|x_j\|\right\}$$
$$\leq \left\|\sum_{j=1}^{n}\alpha_j x_j\right\| \leq \min_{k\in\{1,2,...,n\}}\left\{|\alpha_k|^p\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}|\alpha_j-\alpha_k|^p\|x_j\|\right\}$$

Proof. For any $k \in \{1, 2, ..., n\}$ we observe that

$$\sum_{j=1}^{n}\alpha_j x_j = \alpha_k\left(\sum_{j=1}^{n}x_j\right) + \sum_{j=1}^{n}(\alpha_j-\alpha_k)x_j$$

Taking the p-norm on both sides, we have

$$\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| = \left\|\alpha_k\left(\sum_{j=1}^{n}x_j\right) + \sum_{j=1}^{n}(\alpha_j-\alpha_k)x_j\right\|$$

By using the triangle inequality and properties of p-norm, we get

$$\begin{aligned}\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| &\leq \left\|\alpha_k\left(\sum_{j=1}^{n}x_j\right)\right\| + \left\|\sum_{j=1}^{n}(\alpha_j-\alpha_k)x_j\right\| \\ &\leq \left\|\alpha_k\left(\sum_{j=1}^{n}x_j\right)\right\| + \sum_{j=1}^{n}\|(\alpha_j-\alpha_k)x_j\| \\ &= |\alpha_k|^p\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}|\alpha_j-\alpha_k|^p\|x_j\| \\ &\leq \max_{j\in\{1,2,..,n\}}\left\{|\alpha_k|^p\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}|\alpha_j-\alpha_k|^p\|x_j\|\right\} \qquad (3.1)\end{aligned}$$

Also we observe that

$$\sum_{j=1}^{n}\alpha_j x_j = \alpha_k\left(\sum_{j=1}^{n}x_j\right) - \sum_{j=1}^{n}(\alpha_k-\alpha_j)x_j$$

Taking p-norm on both sides

$$\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| = \left\|\alpha_k\left(\sum_{j=1}^{n}x_j\right) - \sum_{j=1}^{n}(\alpha_k-\alpha_j)x_j\right\|$$

By utilizing continuity and properties of p-norm, we obtain

$$\begin{aligned}\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| &\geq \left\|\alpha_k\left(\sum_{j=1}^{n}x_j\right)\right\| - \left\|\sum_{j=1}^{n}(\alpha_j-\alpha_k)\,x_j\right\| \\ &\geq \left\|\alpha_k\left(\sum_{j=1}^{n}x_j\right)\right\| - \sum_{j=1}^{n}\|(\alpha_j-\alpha_k)\,x_j\| \\ &= |\alpha_k|^p\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{j=1}^{n}|\alpha_j-\alpha_k|^p\,\|x_j\| \\ &\geq \min_{j\in\{1,2,..,n\}}\left\{|\alpha_k|^p\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}|\alpha_j-\alpha_k|^p\,\|x_j\|\right\} \qquad (3.2)\end{aligned}$$

(3.1) and (3.2) together complete the proof of the Theorem. ∎

Now we give a Lemma which will be helpful in the sequel.

Lemma 3.2. *Let* $0 < p \leq 1$ *and* $\boldsymbol{a} = (a_1, a_2, ..., a_n)$, $\boldsymbol{b} = (b_1, b_2, ..., b_n) \in \mathbb{R}^n$, $a_j \geq 0$, $b_j \geq 0$. *Then*

$$\sum_{j=1}^{n}(a_j+b_j)^p \leq \sum_{j=1}^{n}a_j^p + \sum_{j=1}^{n}b_j^p$$

Proof. Here we can prove that the inequality holds for only one j. That is

$$(a_j+b_j)^p \leq a_j^p + b_j^p \qquad (3.3)$$

whenever $0 < p \leq 1$ and $a_j \geq 0$, $b_j \geq 0$, and then get the result by taking the finite sum over all $j = 1, 2, ..., n$. When $p = i$, then the result is obviously true. So assume that $0 < p < 1$. Consider the function

$$f(t) = 1 + t^p - (1+t)^p,\, t \geq 0$$

Then

$$f^{'}(t) = pt^{p-1} - p(1+t)^{p-1},\, t \geq 0$$

Since $p - 1 < 0$, hence $f^{'}(t) \geq 0$, $t \geq 0$. Thus

$$f(t) \geq 0,\, t \geq 0$$

That is

$$(1+t)^{p-1} \leq t^{p-1},\, t \geq 0$$

If $b_j = 0$ then $(a_j+b_j)^p \leq a_j^p + b_j^p$ is true with equality sign, so let $b_j > 0$ and take $t = \frac{a_j}{b_j}$ in (3.3), we have

$$\left(1+\frac{a_j}{b_j}\right)^{p-1} \leq \left(\frac{a_j}{b_j}\right)^{p-1},b_j > 0$$

so that

$$(a_j+b_j)^p \leq a_j^p + b_j^p$$

Summing over $j = 1, 2, ..., n$, we obtain

$$\sum_{j=1}^{n}(a_j+b_j)^p \leq \sum_{j=1}^{n}a_j^p + \sum_{j=1}^{n}b_j^p$$

∎

The following corollary gives another form of generalized triangle inequality (1.1) and its reverse, developed by Pêcarić and Rajić, in a p-Fréchet space X.

Corollary 3.3. *If* $x_k \in X$, $\|x_k\| \neq 0$, $k \in \{1, 2, ..., n\}$ *and* $0 < p < 1$, *then*

$$\max_{k\in\{1,2,\ldots,n\}}\left\{\frac{1}{\|x_k\|^p}\left[\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}\left|\|x_j\|-\|x_k\|\right|^p\|x_j\|^{1-p}\right]\right\}$$
$$\leq\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\leq\min_{k\in\{1,2,\ldots,n\}}\left\{\frac{1}{\|x_k\|^p}\left[\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}\left|\|x_j\|-\|x_k\|\right|^p\|x_j\|^{1-p}\right]\right\}$$

Proof. If we replace α_k by $\frac{1}{\|x_k\|}$, $\|x_k\| \neq 0$, $k \in \{1, 2, ..., n\}$ in the first inequality of Theorem 3.1 we get

$$\begin{aligned}\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| &\geq \left|\frac{1}{\|x_k\|}\right|^p\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}\left|\frac{1}{\|x_j\|}-\frac{1}{\|x_k\|}\right|^p\|x_j\|\\ &=\frac{1}{\|x_k\|^p}\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}\frac{1}{\|x_k\|^p}\left|\|x_j\|-\|x_k\|\right|^p\|x_j\|^{1-p}\\ &\geq\max_{k\in\{1,2,\ldots,n\}}\left\{\frac{1}{\|x_k\|^p}\left[\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}\left|\|x_j\|-\|x_k\|\right|^p\|x_j\|^{1-p}\right]\right\}\end{aligned}\tag{3.4}$$

Similarly if we replace α_k by $\frac{1}{\|x_k\|}$, $\|x_k\| \neq 0$, $k \in \{1, 2, ..., n\}$ in the second inequality of Theorem 3.1 we get

$$\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\leq\min_{k\in\{1,2,\ldots,n\}}\left\{\frac{1}{\|x_k\|^p}\left[\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}\left|\|x_j\|-\|x_k\|\right|^p\|x_j\|^{1-p}\right]\right\}\tag{3.5}$$

Combining (3.4) and (3.5) complete the proof of the corollary. ∎

Now we give a slight different but almost the similar refinement and reverse of generalized triangle inequality and its reverse obtained by M. Kato et al in a p-Fréchet space X.

Corollary 3.4. *If* $x_k \in X$, $\|x_k\| \neq 0$, $k \in \{1, 2, ..., n\}$ *and* $0 < p < 1$, *then*

$$\min_{k\in\{1,2,\ldots,n\}}\{\|x_k\|^p\}\left[-\sum_{j=1}^{n}\|x_j\|^{1-p}-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right]$$
$$\leq\sum_{j=1}^{n}\|x_j\|-\left\|\sum_{j=1}^{n}x_j\right\|\leq\max_{k\in\{1,2,\ldots,n\}}\{\|x_k\|^p\}\left[\sum_{j=1}^{n}\|x_j\|^{1-p}-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right]+2\sum_{j=1}^{n}\|x_j\|$$

Proof. From the second inequality of Corollary 3.3 we have

$$\begin{aligned}\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| &\le \frac{1}{\|x_k\|^p}\left[\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}|\|x_j\|-\|x_k\||^p\|x_j\|^{1-p}\right]\\ &\le \frac{1}{\|x_k\|^p}\left[\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}(\|x_j\|^p+\|x_k\|^p)\|x_j\|^{1-p}\right] \text{ by Lemma 3.2}\\ &= \frac{1}{\|x_k\|^p}\left[\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}\left(\|x_j\|+\|x_k\|^p\|x_j\|^{1-p}\right)\right]\\ &= \frac{1}{\|x_k\|^p}\left[\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}\|x_j\|+\|x_k\|^p\sum_{j=1}^{n}\|x_j\|^{1-p}\right]\\ &= \frac{1}{\|x_k\|^p}\left\|\sum_{j=1}^{n}x_j\right\|+\frac{1}{\|x_k\|^p}\sum_{j=1}^{n}\|x_j\|+\sum_{j=1}^{n}\|x_j\|^{1-p}\end{aligned}$$

and hence

$$\sum_{j=1}^{n}\|x_j\|-\left\|\sum_{j=1}^{n}x_j\right\|\le \max_{k\in\{1,2,\dots,n\}}\{\|x_k\|^p\}\left[\sum_{j=1}^{n}\|x_j\|^{1-p}-\left\|\sum_{j-1}^{n}\frac{x_j}{\|x_j\|}\right\|\right]+2\sum_{j=1}^{n}\|x_j\| \tag{3.6}$$

From the first part of Corollary 3.3 we get

$$\begin{aligned}\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| &\ge \frac{1}{\|x_k\|^p}\left[\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}|\|x_j\|-\|x_k\||^p\|x_j\|^{1-p}\right]\\ &\ge \frac{1}{\|x_k\|^p}\left[\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}(\|x_j\|^p+\|x_k\|^p)\|x_j\|^{1-p}\right] \text{ by Lemma 3.2}\\ &= \frac{1}{\|x_k\|^p}\left[\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}\left(\|x_j\|+\|x_k\|^p\|x_j\|^{1-p}\right)\right]\\ &= \frac{1}{\|x_k\|^p}\left[\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}\|x_j\|-\|x_k\|^p\sum_{j=1}^{n}\|x_j\|^{1-p}\right]\\ &= \frac{1}{\|x_k\|^p}\left\|\sum_{j=1}^{n}x_j\right\|-\frac{1}{\|x_k\|^p}\sum_{j=1}^{n}\|x_j\|-\sum_{j=1}^{n}\|x_j\|^{1-p}\end{aligned}$$

and therefore we get

$$\sum_{j=1}^{n}\|x_j\|-\left\|\sum_{j=1}^{n}x_j\right\|\ge \min_{k\in\{1,2,\dots,n\}}\{\|x_k\|^p\}\left[-\sum_{j=1}^{n}\|x_j\|^{1-p}-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right] \tag{3.7}$$

If we combine (3.6) and (3.7) then proof of the corollary is complete. ∎

Another refinement and reverse of generalized triangle inequality in a p-Fréchet space X is given in the following corollary

Corollary 3.5. *If* $x_k \in X$, $\|x_k\| \neq 0$, $k \in \{1, 2, ..., n\}$ *and* $0 < p < 1$, *then*

$$\frac{-\sum\limits_{j=1}^{n}\|x_j\|^{p+1} - \left\|\sum\limits_{j=1}^{n}\|x_j\|\,x_j\right\|}{\max\limits_{k\in\{1,2,\ldots,n\}}\{\|x_k\|^p\}}$$

$$\leq \sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}\|x_j\|\right\| \leq \frac{\sum\limits_{j=1}^{n}\|x_j\|^{p+1} - \left\|\sum\limits_{j=1}^{n}\|x_j\|\,x_j\right\|}{\min\limits_{k\in\{1,2,\ldots,n\}}\{\|x_k\|^p\}} + 2\sum_{j=1}^{n}\|x_j\|$$

Proof. If we replace α_k by $\|x_k\|$, $\|x_k\| \neq 0$, $k \in \{1, 2, ..., n\}$ in the first inequality of Theorem 3.1 we get

$$\begin{aligned}\left\|\sum_{j=1}^{n}\|x_j\|\,x_j\right\| &\geq \|x_k\|^p\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{j=1}^{n}\left|\|x_j\| - \|x_k\|\right|^p\|x_j\| \\ &\geq \|x_k\|^p\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{j=1}^{n}\left(\|x_j\|^p + \|x_k\|^p\right)\|x_j\| \text{ by Lemma 3.2} \\ &= \|x_k\|^p\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{j=1}^{n}\left(\|x_j\|^{p+1} + \|x_k\|^p\|x_j\|\right) \\ &= \|x_k\|^p\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{j=1}^{n}\|x_j\|^{p+1} - \|x_k\|^p\sum_{j=1}^{n}\|x_j\|\end{aligned}$$

From the above inequality we obtain

$$\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\| \geq \frac{-\sum\limits_{j=1}^{n}\|x_j\|^{p+1} - \left\|\sum\limits_{j=1}^{n}\|x_j\|\,x_j\right\|}{\|x_k\|^p} \tag{3.8}$$

Now by taking the second inequality of Theorem 3.1 and replacing α_k by $\|x_k\|$, $\|x_k\| \neq 0$, $k \in \{1, 2, ..., n\}$, we get

$$\begin{aligned}\left\|\sum_{j=1}^{n}\|x_j\|\,x_j\right\| &\leq \|x_k\|^p\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}\left|\|x_j\| - \|x_k\|\right|^p\|x_j\| \\ &\leq \|x_k\|^p\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}\left(\|x_j\|^p + \|x_k\|^p\right)\|x_j\| \text{ by Lemma 3.2} \\ &= \|x_k\|^p\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}\left(\|x_j\|^{p+1} + \|x_k\|^p\|x_j\|\right) \\ &= \|x_k\|^p\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}\|x_j\|^{p+1} + \|x_k\|^p\sum_{j=1}^{n}\|x_j\|\end{aligned}$$

Therefore we have

$$\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\| \leq \frac{\sum_{j=1}^{n}\|x_j\|^{p+1} - \left\|\sum_{j=1}^{n}\|x_j\|\,x_j\right\|}{\|x_k\|^p} + 2\sum_{j=1}^{n}\|x_j\| \tag{3.9}$$

If we combine (3.8) and (3.9) then proof of the corollary is complete. ∎

Now if in the following corollary:

Corollary 3.6. [6, Corollary 4.4, p.30] *Let* $\lambda_1, \lambda_2, ..., \lambda_n$ *be positive numbers and* $x_1, x_2, ..., x_n$ *be nonzero elements in a normed space* X. *Then*

$$\left\|\sum_{j=1}^{n}x_j\right\|^p \leq \left[\left\|\sum_{j=1}^{n}x_j\right\| + \left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\min_{1\leq j\leq n}\|x_j\|\right]^p \leq \left(\sum_{j=1}^{n}\lambda_j\right)^{p-1}\sum_{j=1}^{n}\frac{\|x_j\|^p}{\lambda_j^{p-1}} \tag{3.10}$$

if we take $\lambda_1 = \lambda_2 = ... = \lambda_n = 1$ the we have the following result:

$$\left\|\sum_{j=1}^{n}x_j\right\|^s \leq n^{s-1}\left(\sum_{j=1}^{n}\|x_j\|^s\right), n \geq 2, s \geq 1 \tag{3.11}$$

for all nonzero elements $x_1, x_2, ..., x_n \in X$, where X is a normed space. By using the above result now we give the more general form of the generalized triangle inequality

Theorem 3.7. *If* $x_k \in X$, $\|x_k\| \neq 0$, $k \in \{1, 2, ..., n\}$, $n \geq 2$, $s \geq 1$ *and* $0 < p < 1$, *then*

$$\begin{aligned}&\min_{k\in\{1,2,...,n\}}\{\|x_k\|^{ps}\}\left[-\sum_{j=1}^{n}\|x_j\|^{s(1-p)} - n^{1-s}\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|^s\right] + \left[(2n)^{1-s} - 1\right]\left\|\sum_{j=1}^{n}x_j\right\|^s \\ &\leq \sum_{j=1}^{n}\|x_j\|^s - \left\|\sum_{j=1}^{n}x_j\right\|^s \leq \max_{k\in\{1,2,...,n\}}\{\|x_k\|^{ps}\}\left[\sum_{j=1}^{n}\|x_j\|^{s(1-p)} - (2n)^{1-s}\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|^s\right] + 2\sum_{j=1}^{n}\|x_j\|^s\end{aligned} \tag{3.12}$$

Proof. Since

$$\sum_{j=1}^{n}\frac{x_j}{\|x_j\|} = \frac{1}{\|x_k\|}\sum_{j=1}^{n}x_j + \sum_{j=1}^{n}\left(\frac{1}{\|x_j\|} - \frac{1}{\|x_k\|}\right)x_j$$

Therefore, by taking p-norm and using the properties of p-norm, we have

$$\begin{aligned}
\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|^s &= \left\|\frac{1}{\|x_k\|}\sum_{j=1}^{n}x_j + \sum_{j=1}^{n}\left(\frac{1}{\|x_j\|}-\frac{1}{\|x_k\|}\right)x_j\right\|^s \\
&\le 2^{s-1}\left[\left\|\frac{1}{\|x_k\|}\sum_{j=1}^{n}x_j\right\|^s + \left\|\sum_{j=1}^{n}\left(\frac{1}{\|x_j\|}-\frac{1}{\|x_k\|}\right)x_j\right\|^s\right] \text{ By (3.11)} \\
&\le 2^{s-1}\left[\frac{1}{\|x_k\|^{ps}}\left\|\sum_{j=1}^{n}x_j\right\|^s + n^{s-1}\sum_{j=1}^{n}\left|\frac{1}{\|x_j\|}-\frac{1}{\|x_k\|}\right|^{ps}\|x_j\|^s\right] \text{ By (3.11)} \\
&= 2^{s-1}\left[\frac{1}{\|x_k\|^{ps}}\left\|\sum_{j=1}^{n}x_j\right\|^s + \frac{n^{s-1}}{\|x_k\|^{ps}}\sum_{j=1}^{n}\left|\|x_j\|-\|x_k\|\right|^{ps}\|x_j\|^{s(1-p)}\right] \\
&\le 2^{s-1}\left[\frac{1}{\|x_k\|^{ps}}\left\|\sum_{j=1}^{n}x_j\right\|^s + \frac{n^{s-1}}{\|x_k\|^{ps}}\sum_{j=1}^{n}\left(\|x_j\|^{ps}+\|x_k\|^{ps}\right)\|x_j\|^{s(1-p)}\right] \text{ by Lemma 3.2} \\
&= 2^{s-1}\left[\frac{1}{\|x_k\|^{ps}}\left\|\sum_{j=1}^{n}x_j\right\|^s + \frac{n^{s-1}}{\|x_k\|^{ps}}\sum_{j=1}^{n}\|x_j\|^s + n^{s-1}\sum_{j=1}^{n}\|x_j\|^{s(1-p)}\right] \\
&= \frac{2^{s-1}}{\|x_k\|^{ps}}\left\|\sum_{j=1}^{n}x_j\right\|^s + \frac{(2n)^{s-1}}{\|x_k\|^{ps}}\sum_{j=1}^{n}\|x_j\|^s + (2n)^{s-1}\sum_{j=1}^{n}\|x_j\|^{s(1-p)} \\
&\le \frac{(2n)^{s-1}}{\|x_k\|^{ps}}\left\|\sum_{j=1}^{n}x_j\right\|^s + \frac{(2n)^{s-1}}{\|x_k\|^{ps}}\sum_{j=1}^{n}\|x_j\|^s + (2n)^{s-1}\sum_{j=1}^{n}\|x_j\|^{s(1-p)}
\end{aligned}$$

Therefore we have

$$(2n)^{1-s}\|x_k\|^{ps}\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|^s \le \left\|\sum_{j=1}^{n}x_j\right\|^s + \sum_{j=1}^{n}\|x_j\|^s + \|x_k\|^{ps}\sum_{j=1}^{n}\|x_j\|^{s(1-p)}$$

And from the above inequality we get

$$\sum_{j=1}^{n}\|x_j\|^s - \left\|\sum_{j=1}^{n}x_j\right\|^s \le \|x_k\|^{ps}\left[\sum_{j=1}^{n}\|x_j\|^{s(1-p)} - (2n)^{1-s}\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|^s\right] + 2\sum_{j=1}^{n}\|x_j\|^s$$

Which implies the second inequality in (3.12)

Also we observe that

$$\sum_{j=1}^{n}\frac{x_j}{\|x_j\|} = \frac{1}{\|x_k\|}\sum_{j=1}^{n}x_j - \sum_{j=1}^{n}\left(\frac{1}{\|x_k\|}-\frac{1}{\|x_j\|}\right)x_j$$

Taking p-norm and by using the continuity and properties of p-norm we have

$$\begin{aligned}
\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|^s &= \left\|\frac{1}{\|x_k\|}\sum_{j=1}^{n}x_j - \sum_{j=1}^{n}\left(\frac{1}{\|x_k\|}-\frac{1}{\|x_j\|}\right)x_j\right\|^s \\
&\geq 2^{1-s}\left\|\frac{1}{\|x_k\|}\sum_{j=1}^{n}x_j\right\|^s - \left\|\sum_{j=1}^{n}\left(\frac{1}{\|x_k\|}-\frac{1}{\|x_j\|}\right)x_j\right\|^s \text{ By (3.11)} \\
&\geq 2^{1-s}\left\|\frac{1}{\|x_k\|}\sum_{j=1}^{n}x_j\right\|^s - n^{s-1}\sum_{j=1}^{n}\left\|\left(\frac{1}{\|x_k\|}-\frac{1}{\|x_j\|}\right)x_j\right\|^s \text{ By (3.11)} \\
&\geq \frac{2^{1-s}}{\|x_k\|^{ps}}\left\|\sum_{j=1}^{n}x_j\right\|^s - n^{s-1}\sum_{j=1}^{n}\left|\frac{1}{\|x_k\|}-\frac{1}{\|x_j\|}\right|^{ps}\|x_j\|^s \\
&= \frac{2^{1-s}}{\|x_k\|^{ps}}\left\|\sum_{j=1}^{n}x_j\right\|^s - \frac{n^{s-1}}{\|x_k\|^{ps}}\sum_{j=1}^{n}\left|\|x_k\|-\|x_j\|\right|^{ps}\|x_j\|^{s(1-p)} \\
&\geq \frac{2^{1-s}}{\|x_k\|^{ps}}\left\|\sum_{j=1}^{n}x_j\right\|^s - \frac{n^{s-1}}{\|x_k\|^{ps}}\sum_{j=1}^{n}\left(\|x_k\|^{ps}+\|x_j\|^{ps}\right)\|x_j\|^{s(1-p)} \text{ by Lemma 3.2} \\
&= \frac{2^{1-s}}{\|x_k\|^{ps}}\left\|\sum_{j=1}^{n}x_j\right\|^s - n^{s-1}\sum_{j=1}^{n}\|x_j\|^{s(1-p)} - \frac{n^{s-1}}{\|x_k\|^{ps}}\sum_{j=1}^{n}\|x_j\|^s
\end{aligned}$$

From this inequality we obtain

$$\begin{aligned}
&\sum_{j=1}^{n}\|x_j\|^s - \left\|\sum_{j=1}^{n}x_j\right\|^s \\
&\geq \|x_k\|^{ps}\left[-\sum_{j=1}^{n}\|x_j\|^{s(1-p)} - n^{1-s}\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|^s\right] + \left[(2n)^{1-s}-1\right]\left\|\sum_{j=1}^{n}x_j\right\|^s
\end{aligned}$$

Which implies the first inequality in(3.12) This completes the proof of the theorem as well. ∎

REFERENCES

[1] T. AOKI, Locally bounded linear topological spaces, *Proc. Imp. Acad. Tokyo.*, **18** (1942), pp. 588–594.

[2] A. BAYOUMI, Foundations of Complex Analysis in Non-Locally Convex Spaces, *North-Holland Mathematics Studies*, vol. 193, Elsevier (Amsterdam, 2003).

[3] N. DANFORD AND J.T SCHWARTZ, *Linear Operators, Part I*, Interscience, New York, 1964.

[4] S. S. DRAGOMIR, Some inequalities in normed algebra, *J. Inequal.Pure & Appl. Math.*, **9** (2008), Issue 1, Article 5, pp. 1-10.

[5] H. HUDZIK AND T.R. LANDES, Characteristic of convexity of Köthe function spaces, *Math. Ann.*, **294** (1992), pp. 117-124.

[6] C. Y. HSU, S. Y. SHAW, H. J. WONG, Refinements of generalized triangle inequalities, *J. Math. Anal. Appl.*, **344** (2008), pp. 17-31.

[7] N.J. KALTON, N.T. PECK AND J.W. ROBERTS, Refinements of generalized triangle inequalities, *An F-Space Sampler, London Math. Soc.*, Lecture Notes **89**, Cambridge University Press, Cambridge, 1984.

[8] K.-I. MITANI, K.-S. SAITO, M.I. KATO AND T. TAMURA, On sharp triangle inequalities in Banach spaces, *J. Math. Anal. Appl.*, **336** (2007), pp. 1178-1186.

[9] L. MALIGRANDA, Simple norm inequalities, *Amer. Math. Monthly*, **113** (2006), pp. 256-260.

[10] L. MALIGRANDA, Some remarks on the triangle inequality for norms, *Banach J. Math. Anal.*, **2** (2008), no. 2, pp. 31-41.

[11] J. PEČARIĆ AND R. RAJIĆ , The Dunkl-Williams inequality with n elements in normed linear spaces, *Math. Ineq. & Appl.*, **10** (2) (2007), pp. 461-470.

[12] S. ROLEWICZ, On a certain class of linear metric spaces, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astrono. Phys.*, **5** (1957), pp. 471-473.

[13] C. WU AND Y. LI, On the triangle inequality in Quasi Banach Spaces, *J. Inequal. Pure & Appl. Math.*, Volume **9** (2008), Issue 2, Article 41, pp. 4.