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ON THE BOUNDEDNESS OF HARDY'S AVERAGING OPERATORS

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ABSTRACT. In this paper we establish scales of sufficient conditions for the boundedness of Hardy's averaging operators on weighted Lebesgue spaces. The estimations of the operator norms are also obtained. Included in particular are the Erdélyi-Kober operators.

Key words and phrases: Weighted inequalities, Hardy's inequality, Hardy's averaging operator.

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1. INTRODUCTION

In [11], Sinnamon considered the weighted gradient inequality

$$\left\{\int_{\mathbb{R}^n} |g(x)|^q u(x)dx\right\}^{1/q} \le C\left\{\int_{\mathbb{R}^n} |x\cdot\nabla g(x)|^p v(x)dx\right\}^{1/p}, \quad g\in C_0^\infty(\mathbb{R}^n), \tag{1.1}$$

for $1<p<\infty$ and $0<q<\infty$, which is equivalent to

$$\left\{\int_{\mathbb{R}^n} |Qf(x)|^q u(x)dx\right\}^{1/q} \le C\left\{\int_{\mathbb{R}^n} |f(x)|^p v(x)dx\right\}^{1/p}, \tag{1.2}$$

for all $f\in C_0^\infty(\mathbb{R}^n)$. Here $Qf(x)=\int_1^\infty f(xt)dt/t$ is a solution to the equation $x\cdot\nabla(Qf)(x)+f(x)=0$ for $f\in C_0^\infty(\mathbb{R}^n)$. Necessary and sufficient conditions for (1.2) to hold for $0<q\le p<\infty$, $p>1$, were given in [11, Theorem 3.2 & Theorem 3.3]. Moreover, Sinnamon also proved that if $1\le p<q<\infty$, $n>1$, and the weight v is locally integrable on $\mathbb{R}^n$, then (1.2) holds only if $u=0$ almost everywhere. If $p,q>1$, then (1.2) holds for all measurable functions f if and only if

$$\left\{\int_{\mathbb{R}^n} |Pf(x)|^{p^*} v(x)^{1-p^*}dx\right\}^{1/p^*} \le C\left\{\int_{\mathbb{R}^n} |f(x)|^{q^*} u(x)^{1-q^*}dx\right\}^{1/q^*}, \tag{1.3}$$

where $Pf(x)=\int_0^1 t^{n-1}f(xt)dt$ and $1/p+1/p^*=1$, $1/q+1/q^*=1$. The operator P is a special case of Hardy's averaging operator H_k defined as

$$H_kf(x):=\int_0^1 k(t)f(xt)dt, \tag{1.4}$$

where $k:(0,1)\mapsto[0,\infty)$ is a measurable function. Xiao [15] proved that H_k is bounded on $L_p(\mathbb{R}^n)$, $1\le p\le\infty$, if and only if $\int_0^1 k(t)t^{-n/p}dt$ is finite. The operator norm of H_k was also given.

In this paper, we consider the weighted inequality

$$\left\{\int_E (H_kf(x))^q u(x)dx\right\}^{1/q} \le C\left\{\int_E f(x)^p v(x)dx\right\}^{1/p} \tag{1.5}$$

for $0<q\le p<\infty$, $p>1$, and u,v are measurable functions which are positive and finite almost everywhere on E. Here E is a spherical cone in $\mathbb{R}^n$ and $f\in L_{p,v}^+$, which are defined below. In the case $k(t)=t^{n-1}$ and $E=\mathbb{R}^n$, $H_kf(x)$ can be reduced to $Pf(x)$ and necessary and sufficient conditions on u,v for (1.5) to hold for all $f\in L_{p,v}^+$ can be obtained by the results given in [11]. On the other hand, the one-dimensional case of (1.5) was studied by many authors. See [2], [3], [6], [7], [8], [9], [10], [12], and [13] for $k(t)=1$. In the case $k(t)=(1-t)^{\alpha-1}$, see [1] for $0<\alpha<1/p$ and $1<p<q=p/(1-\alpha p)$, and [5] for $\alpha\ge1$ and $1<p\le q<\infty$. Actually in [5], inequality (1.5) was investigated for more general k that is nonincreasing and satisfies $k(ab)\le D(k(a)+k(b))$ for $0<a,b<1$. For some more generalizations of these results see [4], [14], and the references given there.

The purpose of this paper is applying the methods given in [11, Theorem 3.2 & Theorem 3.3] to obtain scales of sufficient conditions on u,v so that (1.5) holds for all $f\in L_{p,v}^+$ with a finite constant C independent of f. The estimation of C is also given. As an application, the operator

$$J_{\sigma\eta}^\alpha f(x)=\int_0^1(1-t^\sigma)^{\alpha-1}t^{\sigma\eta+\sigma-1}f(xt)dt, \quad \sigma>0 \text{ and } \alpha>0,$$

which is called the Erdélyi-Kober operator in the one-dimensional case, is also discussed.

We say that E is a spherical cone in $\mathbb{R}^n$ if each $x \in E$ can be written in the form $x = \xi\sigma$ for some $0 < \xi < \infty$ and some $\sigma \in B$, where B is a given measurable subset of the unit sphere in $\mathbb{R}^n$. We assume that all functions involved in this paper are measurable on their domains. We write $f \in L^+_{p,v}$, $1 < p < \infty$, provided that f is nonnegative on E and $\int_E f(x)^p v(x)dx < \infty$. For $0 < z < \infty$, we define z^* by $1/z + 1/z^* = 1$. We also take $0^0 = \infty^0 = 1$ and $\infty/\infty = 0/0 = 0 \cdot \infty = 0$.

2. MAIN RESULT

Let $0 < q \leq p < \infty$ and $p > 1$. By the results given in [11] we see that (1.5) holds for $k(t) = t^{n-1}$ and $E = \mathbb{R}^n$ if and only if $A < \infty$, where

$$A = \sup_{x\in\mathbb{R}^n} \left(\int_0^1 v(xt)^{1-p^*} t^{n-1} dt \right)^{1/p^*} \left(\int_1^\infty u(xt) t^{n-np-1} dt \right)^{1/p} \tag{2.1}$$

for $p = q$ and

$$\begin{aligned} A &= \left\{ \int_{\mathbb{R}^n} \left(\int_0^1 v(xt)^{1-p^*} t^{n-1} dt \right)^{q(p-1)/(p-q)} \right. \\ &\quad \left. \times \left(\int_1^\infty u(xt) t^{n-nq-1} dt \right)^{q/(p-q)} u(x)dx \right\}^{(p-q)/(pq)} \end{aligned} \tag{2.2}$$

for $p > q$. Now for $1 < s \leq p$, and $\delta \in \mathbb{R}$ we define $A^{pq}_{s\delta}$ as follows:

$$A^{pp}_{s\delta} = \sup_{x\in E} \left(\int_0^1 (t^{s-\delta+n-1} v(xt))^{1-s^*} dt \right)^{(s-1)/p} \left(\int_1^\infty t^{-\delta+n-1} u(xt) dt \right)^{1/p} \tag{2.3}$$

for $p = q$ and

$$\begin{aligned} A^{pq}_{s\delta} &= \left\{ \int_E \left(\int_0^1 (t^{s-\delta+n-1} v(xt))^{1-s^*} dt \right)^{q(s-1)/(p-q)} \right. \\ &\quad \left. \times \left(\int_1^\infty t^{-\delta q/p+n-1} u(xt) dt \right)^{q/(p-q)} u(x)dx \right\}^{(p-q)/(pq)} \end{aligned} \tag{2.4}$$

for $p > q$. If $E = \mathbb{R}^n$ and we choose $s = p$ and $\delta = np$, then $A^{pq}_{s\delta}$ can be reduced to A defined by (2.1) – (2.2). In the one-dimensional case $n = 1$ and $E = (0, \infty)$, $A^{pq}_{s\delta}$ can be reduced to the well-known Muckenhoupt conditions by choosing $s = \delta = p$. See [2], [7], [12], and [13]. The following is our main theorem.

Theorem 2.1. *Let $0 < q \leq p < \infty$ and $p > 1$. Let $k : (0, 1) \mapsto (0, \infty)$. Suppose that there exist $1 < s \leq p$ and $\delta \in \mathbb{R}$ such that $K_{s\delta} < \infty$, where*

$$K_{s\delta} = \begin{cases} \{\int_0^1 k(t)^{p/(p-s)} t^{(s-\delta)/(p-s)} dt\}^{(p-s)/p}, & \text{if } 1 < s < p, \\ \sup_{0<t<1} k(t) t^{1-\delta/p}, & \text{if } s = p. \end{cases} \tag{2.5}$$

If $A^{pq}_{s\delta} < \infty$, then (1.5) holds for all $f \in L^+_{p,v}$ and the best constant C satisfies

$$C \leq \left(\frac{p}{p-q} \right)^{(p-q)/(pq)} s^{1/p} (s^*)^{(s-1)/p} K_{s\delta} A^{pq}_{s\delta}. \tag{2.6}$$

Proof. Let $h^s = f^p$. For $\delta \in \mathbb{R}$, we have

$$H_k f(x) = \int_0^1 k(t) t^{(s-\delta)/p} t^{(\delta-s)/p} h(xt)^{s/p} dt \leq K_{s\delta} \left(\int_0^1 t^{\delta/s-1} h(xt) dt \right)^{s/p}$$

and hence

$$\int_E (H_k f(x))^q u(x)dx \le K_{s\delta}^q \int_E \left(\int_0^1 t^{\delta/s-1} h(xt)dt \right)^{sq/p} u(x)dx$$
$$= K_{s\delta}^q \int_B \int_0^\infty \left(\int_0^1 t^{\delta/s-1} h(\xi\sigma t)dt \right)^{sq/p} u(\xi\sigma)\xi^{n-1} d\xi d\sigma$$
$$= K_{s\delta}^q \int_B \int_0^\infty \left(\int_0^\xi z^{\delta/s-1} h(z\sigma)dz \right)^{sq/p} u(\xi\sigma)\xi^{-\delta q/p+n-1} d\xi d\sigma.$$

Let $\tilde{u}(\xi) = u(\xi\sigma)\xi^{-\delta q/p+n-1}$, $\tilde{v}(z) = z^{s-\delta+n-1}v(z\sigma)$, and define

$$D_{s\delta}(\sigma) = \begin{cases} \sup_{y>0} (\int_y^\infty \tilde{u}(\xi)d\xi)^{1/s} (\int_0^y d\lambda)^{1/s^*}, & \text{if } p = q, \\ \{\int_0^\infty (\int_y^\infty \tilde{u}(\xi)d\xi)^{q/(p-q)} (\int_0^y d\lambda)^{q(s-1)/(p-q)} \tilde{u}(y)dy\}^{(p-q)/(sq)}, & \text{if } p > q, \end{cases}$$

where $d\lambda = \tilde{v}(z)^{1-s^*}dz$. It is well-known that $D_{s\delta}(\sigma) < \infty$ is a necessary and sufficient condition for

$$\left\{ \int_0^\infty \left(\int_0^\xi g(z)dz \right)^{sq/p} \tilde{u}(\xi)d\xi \right\}^{p/(sq)} \le C \left\{ \int_0^\infty g(z)^s \tilde{v}(z)dz \right\}^{1/s} \tag{2.7}$$

to hold for all nonnegative function g and

$$C \le \left(\frac{p}{p-q} \right)^{(p-q)/(sq)} s^{1/s}(s^*)^{1/s^*} D_{s\delta}(\sigma).$$

Here $\{p/(p-q)\}^{(p-q)/(sq)}$ is taken to be 1 when $p = q$. See [12] and [13]. This implies

$$\left\{ \int_E (H_k f(x))^q u(x)dx \right\}^{1/q} \le \left(\frac{p}{p-q} \right)^{(p-q)/(pq)} s^{1/p}(s^*)^{(s-1)/p} K_{s\delta} I^{1/q},$$

where

$$I = \int_B D_{s\delta}(\sigma)^{sq/p} \left(\int_0^\infty h(z\sigma)^s v(z\sigma) z^{n-1}dz \right)^{q/p} d\sigma.$$

If $p = q$, then

$$I \le \left(\sup_{\sigma\in B} D_{s\delta}(\sigma)^s \right) \int_B \int_0^\infty h(z\sigma)^s v(z\sigma)z^{n-1}dzd\sigma = (A_{s\delta}^{pp})^p \int_E f(x)^p v(x)dx.$$

On the other hand, if $0 < q < p < \infty$ and $p > 1$, then by Hölder's inequality we have

$$I \le \left\{ \int_B D_{s\delta}(\sigma)^{sq/(p-q)} d\sigma \right\}^{(p-q)/p} \left\{ \int_B \int_0^\infty h(z\sigma)^s v(z\sigma)z^{n-1}dzd\sigma \right\}^{q/p}$$
$$= (A_{s\delta}^{pq})^q \left\{ \int_E f(x)^p v(x)dx \right\}^{q/p}.$$

This completes the proof. ∎

In the case $k(t) = t^{n-1}$, we choose $s = p$ and $\delta = np$. Then $K_{s\delta} = 1$ and $A_{s\delta}^{pq}$ can be reduced to A defined by (2.1) – (2.2). Therefore we obtain the sufficient part of Sinnamon's results in [11].

As an application, we consider the case $k(t) = (1-t^\sigma)^{\alpha-1}t^{\sigma\eta+\sigma-1}$, where $\sigma > 0$ and $\alpha > 0$. In this case, $H_k f(x)$ can be reduced to

$$J_{\sigma\eta}^\alpha f(x) = \int_0^1 (1-t^\sigma)^{\alpha-1} t^{\sigma\eta+\sigma-1} f(xt)dt.$$

In the one-dimensional case, the operator $J^{\alpha}_{\sigma\eta}$ is called the Erdélyi-Kober operator. If $0 < q \leq p < \infty$, $p > 1$, and $\alpha > 1/p$, then by choosing s and δ so that $1 < s < \min(\alpha p, p)$ and $\delta < (\eta + 1)\sigma p$, we have $K_{s\delta} < \infty$. Therefore $A^{pq}_{s\delta} < \infty$ is a sufficient condition for the boundedness of $J^{\alpha}_{\sigma\eta}$ from $L^{+}_{p,v}$ to $L^{+}_{q,u}$.

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