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STABILITY PROBLEMS FOR GENERALIZED ADDITIVE MAPPINGS AND EULER-LAGRANGE TYPE MAPPINGS

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ABSTRACT. We introduce a generalized additivity of a mapping between Banach spaces and establish the Ulam type stability problem for a generalized additive mapping. The obtained results are somewhat different from the Ulam type stability result of Euler-Lagrange type mappings obtained by H. -M. Kim, K. -W. Jun and J. M. Rassias.

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1. INTRODUCTION

We are interested in the following stability problem: Given an approximately additive mapping, can one find the strictly additive mapping near it? This problem is proposed by S. M. Ulam in 1940 and may be famous as the Ulam stability problem (cf. [6]). Next year, D. H. Hyers [2] gave an affirmative answer to this problem for additive mappings between Banach spaces. T. Aoki [1] and Th. M. Rassias [4] obtained generalized results of Hyers' theorem which allow the Cauchy difference to be unbounded.

H. -M. Kim, K. -W. Jun and J. M. Rassias [3] established an Ulam type stability problem for Euler-Lagrange type mappings of a real normed space into a real Banach space. However such mappings can be changed into the Jensen type mappings under a suitable transformation. Then we introduce a generalized additive mapping which generalizes Jensen type mappings and establish the Ulam type stability problem for such mappings. The obtained Ulam type stability results for Euler-Lagrange type mappings are somewhat different from their results.

2. A ULAM TYPE STABILITY RESULT FOR COMMUTATIVE DIAGRAM

Let X be a set with a binary operation $\circ$, Y a complete metric space with a continuous binary operation $\diamond$ and f a mapping of X into Y. In [5], S.-E. Takahasi, T. Miura and H. Takagi obtained an Ulam type stability result for the following commutative diagram applying Banach's fixed point theorem:

$$\begin{array}{ccc} X \times X & \xrightarrow{\ \circ\ } & X \\ {\scriptstyle f\times f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y \times Y & \xrightarrow[\ \diamond\ ]{} & Y. \end{array}$$

We will describe this stability result more precisely. If σ and τ are self-maps of X and Y, respectively, and if ε is a nonnegative function on $X \times X$, we can define the following two quantities:

$$N_{\sigma,\varepsilon} = \inf\{K \geq 0 : \varepsilon(\sigma x, \sigma x') \leq K\varepsilon(x, x') \ (x, x' \in X)\},$$
$$L_\tau = \inf\{K \geq 0 : d(\tau y, \tau y') \leq K d(y, y') \ (y, y' \in Y)\}.$$

Using these quantities, they showed the following stability results in [5].

Theorem A. *[cf.* [5, Corollary 3.2]*] Suppose that the square operator : $x \to x \circ x$ is an automorphism of X with inverse σ and that the square operator $\tau : y \to y \diamond y$ is an endomorphism of Y with $L_\tau < \infty$. Let ε be a nonnegative function on $X \times X$ such that $N_{\sigma,\varepsilon} < \infty$ and $L_\tau N_{\sigma,\varepsilon} < 1$. If a mapping $f : X \to Y$ satisfies*

$$d(f(x \circ x'), f(x) \diamond f(x')) \leq \varepsilon(x, x') \qquad (\forall x, x' \in X), \tag{2.1}$$

then there exists a mapping $f_\infty : X \to Y$ such that

$$f_\infty(x \circ x') = f_\infty(x) \diamond f_\infty(x') \qquad (\forall x, x' \in X), \tag{2.2}$$

$$d(f(x), f_\infty(x)) \leq \frac{N_{\sigma,\varepsilon}}{1 - L_\tau N_{\sigma,\varepsilon}} \varepsilon(x, x) \qquad (\forall x \in X). \tag{2.3}$$

If $g : X \to Y$ satisfies (2.2) *and*

$$\sup_{\varepsilon(x,x)\neq 0} \frac{d(f(x), g(x))}{\varepsilon(x, x)} < \infty, \tag{2.4}$$

then $g = f_\infty$.

Theorem B. *[cf.* [5, Corollary 3.5]*] Suppose that the square operator $\sigma : x \to x \circ x$ is an endomorphism of X and that the square operator $y \to y \diamond y$ is an automorphism of Y with inverse τ satisfying $L_\tau < \infty$. Let ε be a nonnegative function on $X \times X$ such that $N_{\sigma,\varepsilon} < \infty$ and $L_\tau N_{\sigma,\varepsilon} < 1$. If a mapping $f : X \to Y$ satisfies* (2.1)*, then there exists a mapping $f_\infty : X \to Y$ which satisfies* (2.2) *and*

$$d(f(x), f_\infty(x)) \leq \frac{L_\tau}{1 - L_\tau N_{\sigma,\varepsilon}} \varepsilon(x, x) \qquad (\forall x \in X). \tag{2.5}$$

If $g : X \to Y$ satisfies (2.2) *and* (2.4)*, then $g = f_\infty$.*

Remark 2.1. We would like to state that the definition of $\alpha_{\sigma,\varepsilon}$ given in [5, p. 425] is not described correctly. This should be described as follows:

$$\alpha_{\sigma,\varepsilon} = \inf\{K \geq 0 : \varepsilon(\sigma x, \sigma x') \leq K \varepsilon(x, x') \ (x, x' \in X)\}.$$

Remark 2.2. The uniqueness of f_∞ in Theorems A and B follows easily from the proof of [5, Theorem 3.1].

3. A STABILITY OF GENERALIZED ADDITIVE MAPPINGS

Denote by $\mathbb{K}$ either the field $\mathbb{R}$ of all real numbers or the complex field $\mathbb{C}$ of all complex numbers and fix $a, b, c, d \in \mathbb{K}$. Let X and Y be a normed space over $\mathbb{K}$ and a Banach space over $\mathbb{K}$, respectively and fix $x_0 \in X$ and $y_0 \in Y$. A mapping $f : X \to Y$ is called $(a, b, c, d; x_0, y_0)$-additive if

$$f(ax + bx' + x_0) = cf(x) + df(x') + y_0 \qquad (x, x' \in X).$$

In case of $x_0 = y_0 = 0$, f is simply called (a, b, c, d)-additive.

We consider the Ulam type stability problem for such generalized additive mappings of X into Y.

Theorem 3.1. *Let ε be a nonnegative function on $X \times X$ and suppose that*

(i) $a + b \neq 0$.

(ii) $\exists K \geq 0 : K|c+d| < 1$ and $\varepsilon(x, x') \leq K\varepsilon((a+b)x + x_0, (a+b)x' + x_0)$ $(\forall x, x' \in X)$.

If a mapping $f : X \to Y$ satisfies

$$\|f(ax + bx' + x_0) - cf(x) - df(x') - y_0\| \leq \varepsilon(x, x') \qquad (\forall x, x' \in X), \tag{3.1}$$

then there exists a unique $(a, b, c, d; x_0, y_0)$-additive mapping $f_\infty : X \to Y$ such that

$$\|f(x) - f_\infty(x)\| \leq \frac{K}{1 - K|c + d|} \varepsilon(x, x) \qquad (\forall x \in X). \tag{3.2}$$

Proof. Define binary operations $\circ$ on X and $\diamond$ on Y by

$$x \circ x' = ax + bx' + x_0 \quad (x, x' \in X) \qquad \text{and} \qquad y \diamond y' = cy + dy' + y_0 \quad (y, y' \in Y).$$

Then we can easily see that these operations are continuous and the corresponding square operators

$$x \to x \circ x = (a + b)x + x_0 \quad (x \in X) \qquad \text{and} \qquad y \to y \diamond y = (c + d)y + y_0 \quad (y \in Y)$$

are endomorphic. Also from (i), the square operator $x \to x \circ x$ has the inverse σ which is given by $\sigma x = (a + b)^{-1}(x - x_0)$ $(x \in X)$. Then we have from (ii) that

$$\varepsilon(\sigma x, \sigma x') \leq K\varepsilon(x, x') \qquad (\forall x, x' \in X)$$

and hence $N_{\sigma,\varepsilon} \leq K$. Also denote by τ the corresponding square operator to $\diamond$. Since $L_\tau = |c+d|$, it follows from (ii) that $L_\tau N_{\sigma,\varepsilon} < 1$ and

$$\frac{N_{\sigma,\varepsilon}}{1 - L_\tau N_{\sigma,\varepsilon}} \leq \frac{K}{1 - K|c+d|}.$$

Then the desired result follows immediately from Theorem A. ∎

Theorem 3.2. *Let ε be a nonnegative function on $X \times X$ and suppose that*

(i) $c + d \neq 0$.
(ii) $\exists K \geq 0 : K < |c+d|$ and $\varepsilon((a+b)x + x_0, (a+b)x' + x_0) \leq K\varepsilon(x,x')$ $(\forall x, x' \in X)$.

If a mapping $f : X \to Y$ satisfies (3.1)*, then there exists a unique $(a,b,c,d;x_0,y_0)$-additive mapping $f_\infty : X \to Y$ such that*

$$\|f(x) - f_\infty(x)\| \leq \frac{1}{|c+d| - K}\,\varepsilon(x,x) \qquad (\forall x \in X). \tag{3.3}$$

Proof. Let $\circ$ and $\diamond$ be as in the proof of Theorem 3.1 and denote by σ the corresponding square operator to $\circ$. Then we have from (ii) that

$$\varepsilon(\sigma x, \sigma x') \leq K\varepsilon(x,x') \qquad (\forall x, x' \in X)$$

and hence $N_{\sigma,\varepsilon} \leq K$. Also from (i), the square operator $y \to y \diamond y$ has the inverse τ which is given by $\tau y = (c+d)^{-1}(y - y_0)$ $(y \in Y)$. Since $L_\tau = |c+d|^{-1}$, it follows from (ii) that $L_\tau N_{\sigma,\varepsilon} < 1$ and

$$\frac{L_\tau}{1 - L_\tau N_{\sigma,\varepsilon}} \leq \frac{|c+d|^{-1}}{1 - K|c+d|^{-1}} = \frac{1}{|c+d| - K}.$$

Then the desired result follows immediately from Theorem B. ∎

4. A STABILITY OF EULER-LAGRANGE TYPE MAPPINGS

We consider the following two Euler-Lagrange type mappings f and g of a normed space X into a Banach space Y satisfying

$$f(ax + bx') + f(ax - bx') + 2af(-x) = 0 \qquad (\forall x, x' \in X) \tag{4.1}$$

and

$$g(ax + bx') - g(ax - bx') + 2bg(-x') = 0 \qquad (\forall x, x' \in X), \tag{4.2}$$

respectively. Here a, b are nonzero fixed numbers in $\mathbb{K}$.

The following result is a consequence of Theorem 3.1.

Corollary 4.1 (cf. [3, Theorem 2.4])**.** *Let ε be a nonnegative function on $X \times X$ and suppose*

$$\exists K \geq 0 : K < |a| \text{ and } \varepsilon(-ax, -ax') \leq K\varepsilon(x,x') \qquad (\forall x \in X). \tag{4.3}$$

If a mapping $f : X \to Y$ satisfies

$$\|f(ax + bx') + f(ax - bx') + 2af(-x)\| \leq \varepsilon(x,x') \qquad (\forall x, x' \in X), \tag{4.4}$$

then there exists a unique mapping $f_\infty : X \to Y$ satisfying (4.1) *and*

$$\|f(x) - f_\infty(x)\| \leq \frac{K}{2(|a| - K)}\,\varepsilon\left(\frac{x}{a}, 0\right) \qquad (\forall x \in X). \tag{4.5}$$

Proof. Put $u = ax + bx', v = ax - bx'$ for each $x, x' \in X$. Under these transformations, (4.4) changes into the following estimate

$$\left\| f\left(\frac{u+v}{-2a}\right) - \frac{f(u)+f(v)}{-2a} \right\| \le \varepsilon_1(u,v) \qquad (\forall u, v \in X), \tag{4.6}$$

where

$$\varepsilon_1(u,v) = \frac{1}{2|a|}\varepsilon\left(\frac{u+v}{2a}, \frac{u-v}{2b}\right) \qquad (\forall u, v \in X).$$

Moreover put $\lambda = \frac{1}{-2a}$. Then by (4.3), $K|2\lambda| < 1$ and

$$\begin{aligned}\varepsilon_1(x,x') &= \frac{1}{2|a|}\varepsilon\left(\frac{x+x'}{2a}, \frac{x-x'}{2b}\right)\\ &\le \frac{K}{2|a|}\varepsilon\left(\frac{x+x'}{-2a^2}, \frac{x-x'}{-2ab}\right)\\ &= K\varepsilon_1\left(\frac{x}{-a}, \frac{x'}{-a}\right)\\ &= K\varepsilon_1(2\lambda x, 2\lambda x')\end{aligned}$$

holds for all $x, x' \in X$. Therefore by Theorem 3.1, there exists a unique $(\lambda, \lambda, \lambda, \lambda)$-additive mapping $f_\infty : X \to Y$ such that

$$\|f(x) - f_\infty(x)\| \le \frac{K}{1 - 2|\lambda|K}\,\varepsilon_1(x,x) \qquad (\forall x \in X). \tag{4.7}$$

However we can easily see the following assertions:

(i) f_∞ is $(\lambda, \lambda, \lambda, \lambda)$-additive if and only if f_∞ satisfies (4.1).
(ii) (4.7) is equivalent to (4.5).

This completes the proof. ∎

The following result is a consequence of Theorem 3.2.

Corollary 4.2 (cf. [3, Theorem 2.4])**.** *Let ε be a nonnegative function on $X \times X$ and suppose*

$$\exists K \ge 0 : K < \frac{1}{|a|} \text{ and } \varepsilon(x,x') \le K\varepsilon(-ax, -ax') \qquad (\forall x, x' \in X). \tag{4.8}$$

If a mapping $f : X \to Y$ satisfies (4.4)*, then there exists a unique mapping $f_\infty : X \to Y$ satisfying* (4.1) *and*

$$\|f(x) - f_\infty(x)\| \le \frac{1}{2(1 - K|a|)}\,\varepsilon\left(\frac{x}{a}, 0\right) \qquad (\forall x \in X). \tag{4.9}$$

Proof. Put $u = ax + bx', v = ax - bx'$ for each $x, x' \in X$. Under these transformations, (4.4) changes into the following estimate

$$\left\| f\left(\frac{u+v}{-2a}\right) - \frac{f(u)+f(v)}{-2a} \right\| \le \varepsilon_1(u,v) \qquad (\forall u, v \in X), \tag{4.10}$$

where

$$\varepsilon_1(u,v) = \frac{1}{2|a|}\varepsilon\left(\frac{u+v}{2a}, \frac{u-v}{2b}\right) \qquad (\forall u, v \in X).$$

Moreover put $\lambda = \frac{1}{-2a}$. Then by (4.8), $K < |2\lambda|$ and

$$\begin{aligned}\varepsilon_1(2\lambda x, 2\lambda x') &= \frac{1}{2|a|}\varepsilon\left(\frac{2\lambda(x+x')}{2a}, \frac{2\lambda(x-x')}{2b}\right)\\ &= \frac{1}{2|a|}\varepsilon\left(\frac{x+x'}{-2a^2}, \frac{x-x'}{-2ab}\right)\\ &\leq \frac{K}{2|a|}\varepsilon\left(\frac{x+x'}{2a}, \frac{x-x'}{2b}\right)\\ &= K\varepsilon_1(x,x')\end{aligned}$$

holds for all $x, x' \in X$. Therefore by Theorem 3.2, there exists a unique $(\lambda,\lambda,\lambda,\lambda)$-additive mapping $f_\infty : X \to Y$ such that

$$\|f(x) - f_\infty(x)\| \leq \frac{1}{2|\lambda| - K}\varepsilon_1(x,x) \qquad (\forall x \in X). \tag{4.11}$$

However we can easily see the following assertions:

(i) f_∞ is $(\lambda,\lambda,\lambda,\lambda)$-additive if and only if f_∞ satisfies (4.1).
(ii) (4.11) is equivalent to (4.9).

This completes the proof. ∎

Remark 4.1. There are two transformations: $u = ax + bx', v = -x'$ and $u = ax - bx', v = -x'$ except the transformation treated in the proofs of the above corollaries. However, if we apply these transformations, then the corresponding results are complicated and hence we don't feel beauty.

Remark 4.2. The following assertions follow immediately from Corollaries 4.1 and 4.2.

(i) (cf. [3, Corollary 2.5]). Let $\delta, p \geq 0$ and $q > 0$ with $p + q \neq 1$ and suppose $a, b \neq 0$. If a mapping $f : X \to Y$ satisfies

$$\|f(ax + bx') + f(ax - bx') + 2af(-x)\| \leq \delta\|x\|^p\|x'\|^q \qquad (\forall x, x' \in X),$$

then $f(ax + bx') + f(ax - bx') + 2af(-x) = 0$ holds for all $x, x' \in X$.

In fact, put $\varepsilon(x, x') = \delta\|x\|^p\|x'\|^q$ for each $x, x' \in X$. Since $q > 0$, it follows that $\varepsilon(x, 0) = 0$ for all $x \in X$. Note that (4.3) and (4.8) are equivalent to $|a|^{p+q} < |a|$ and $|a| < |a|^{p+q}$, respectively. Since $a \neq 0$ and $p + q \neq 1$, it follows that either (4.3) or (4.8) holds. Then our assertion follows immediately from Corollaries 4.1 and 4.2.

(ii) (cf. [3, Corollary 2.8]). Let $\delta \geq 0$ and suppose $a, b \neq 0$ and $|a| \neq 1$. If a mapping $f : X \to Y$ satisfies

$$\|f(ax + bx') + f(ax - bx') + 2af(-x)\| \leq \delta \qquad (\forall x, x' \in X),$$

then there exists a unique function $f_\infty : X \to Y$ such that

$$f_\infty(ax + bx') + f_\infty(ax - bx') + 2af_\infty(-x) = 0 \qquad (\forall x, x' \in X),$$

$$\|f(x) - f_\infty(x)\| \leq \frac{\delta}{2||a| - 1|} \qquad (\forall x \in X).$$

In fact, put $\varepsilon(x, x') = \delta$ for each $x, x' \in X$. Note that (4.3) and (4.8) are equivalent to $|a| > 1$ and $|a| < 1$, respectively. Since $|a| \neq 1$, it follows that either (4.3) or (4.8) holds. Then our assertion follows immediately from Corollaries 4.1 and 4.2.

The following result is a consequence of Theorem 3.1.

Corollary 4.3 (cf. [3, Theorem 2.6]). *Let ε be a nonnegative function on $X \times X$ and suppose that*

(i) $\pm b \neq -\frac{1}{2}$.
(ii) $\exists K \geq 0 : K|1 \pm 2b| < 1$ *and*

$$\varepsilon(x, x') \leq K\varepsilon((1 \pm 2b)x, (1 \pm 2b)x') \qquad (\forall x, x' \in X).$$

If a mapping $f : X \to Y$ satisfies

(4.12) $$\|f(ax + bx') - f(ax - bx') + 2bf(-x')\| \leq \varepsilon(x, x') \qquad (\forall x, x' \in X),$$

then there exists a unique mapping $f_\infty : X \to Y$ satisfying (4.2) *and*

(4.13) $$\|f(x) - f_\infty(x)\| \leq \frac{K}{1 - K|1 \pm 2b|}\, \varepsilon\left(\frac{1 \pm b}{a}x, -x\right) \qquad (\forall x \in X)$$

(*double-sign corresponds*).

Proof. The case of $+$. Put $u = ax + bx', v = -x'$ for each $x, x' \in X$. Under these transformations, (4.12) is changed into the following estimate

(4.14) $$\|f(u + 2bv) - f(u) - 2bf(v)\| \leq \varepsilon_2(u, v) \qquad (\forall u, v \in X),$$

where

$$\varepsilon_2(u, v) = \varepsilon\left(\frac{u + bv}{a}, -v\right) \qquad (\forall u, v \in X).$$

Moreover put $\lambda_1 = 1, \lambda_2 = 2b, \lambda_3 = 1$ and $\lambda_4 = 2b$. By (i), we have $\lambda_1 + \lambda_2 \neq 0$. Also by (ii) $K|\lambda_3 + \lambda_4| < 1$ and

$$\begin{aligned}\varepsilon_2(x, x') &= \varepsilon\left(\frac{x + bx'}{a}, -x'\right)\\ &\leq K\varepsilon\left(\frac{(1 + 2b)(x + bx')}{a}, -(1 + 2b)x'\right)\\ &= K\varepsilon_2((1 + 2b)x, (1 + 2b)x')\\ &= K\varepsilon_2((\lambda_1 + \lambda_2)x, (\lambda_1 + \lambda_2)x')\end{aligned}$$

holds for all $x, x' \in X$. Therefore by Theorem 3.1, there exists a unique $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-additive mapping $f_\infty : X \to Y$ such that

(4.15) $$\|f(x) - f_\infty(x)\| \leq \frac{K}{1 - K|\lambda_3 + \lambda_4|}\, \varepsilon_2(x, x) \qquad (\forall x \in X).$$

However we can easily see the following assertions:

(i) f_∞ is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-additive if and only if f_∞ satisfies (4.2).
(ii) (4.15) is equivalent to (4.13).

This completes the proof for the case of $+$.

The case of $-$. By using the transformation : $u = ax - bx', v = -x'$, we obtain the desired result in the same way. ∎

The following result is a consequence of Theorem 3.2.

Corollary 4.4 (cf. [3, Theorem 2.6])**.** *Let ε be a nonnegative function on $X \times X$ and suppose that*

(i) $\pm b \neq -\frac{1}{2}$.
(ii) $\exists K \geq 0 : K < |1 \pm 2b|$ *and*

$$\varepsilon((1 \pm 2b)x, (1 \pm 2b)x') \leq K\varepsilon(x, x') \qquad (\forall x, x' \in X).$$

If a mapping $f : X \to Y$ *satisfies* (4.12)*, then there exists a unique mapping* $f_\infty : X \to Y$ *satisfying* (4.2) *and*

$$\|f(x) - f_\infty(x)\| \le \frac{1}{|1 \pm 2b| - K}\,\varepsilon\left(\frac{1 \pm b}{a}x, -x\right) \qquad (\forall x \in X) \tag{4.16}$$

(*double-sign corresponds*).

Proof. The case of $+$. Put $u = ax + bx', v = -x'$ for each $x, x' \in X$. Under these transformations, (4.12) is changed into (4.14). Moreover put $\lambda_1 = 1, \lambda_2 = 2b, \lambda_3 = 1$ and $\lambda_4 = 2b$. By (i) we have $\lambda_3 + \lambda_4 \neq 0$. Also by (ii), $K < |\lambda_3 + \lambda_4|$ and

$$\begin{aligned}\varepsilon_2((\lambda_1 + \lambda_2)x, (\lambda_1 + \lambda_2)x') &= \varepsilon_2((1 + 2b)x, (1 + 2b)x') \\ &= \varepsilon\left(\frac{(1 + 2b)x + b(1 + 2b)x'}{a}, -(1 + 2b)x'\right) \\ &\le K\varepsilon\left(\frac{x + bx'}{a}, -x'\right) \\ &= K\varepsilon_2(x, x')\end{aligned}$$

for all $x, x' \in X$. Therefore by Theorem 3.2, there exists a unique $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-additive mapping $f_\infty : X \to Y$ such that

$$\|f(x) - f_\infty(x)\| \le \frac{1}{|\lambda_3 + \lambda_4| - K}\,\varepsilon_2(x, x) \qquad (\forall x \in X). \tag{4.17}$$

However we can easily see the following assertions:

(i) f_∞ is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-additive if and only if f_∞ satisfies (4.2).
(ii) (4.17) is equivalent to (4.16).

This completes the proof for the case of $+$.

The case of $-$. By using the transformation : $u = ax - bx', v = -x'$, we obtain the desired result in the same way. ∎

Remark 4.3. Let $p, q, \delta \ge 0, p + q \neq 1$, $\pm b \neq -\frac{1}{2}$ and $a, b \neq 0$ and suppose that a mapping $f : X \to Y$ satisfies

$$\|f(ax + bx') - f(ax - bx') + 2bf(-x')\| \le \delta\|x\|^p\|x'\|^q \qquad (\forall x, x' \in X).$$

Then the following assertions follow immediately from Corollaries 4.3 and 4.4 (cf. [3, Corollaries 2.7 and 2.8]).

(i) If $1 \pm b \neq 0$, then there exists a unique mapping $f_\infty : X \to Y$ satisfying (4.2) and

$$\|f(x) - f_\infty(x)\| \le \frac{\delta|1 \pm b|^p}{|a|^p\,\left||1 \pm 2b|^{p+q} - |1 \pm 2b|\right|}\|x\|^{p+q} \qquad (\forall x \in X)$$

(double-sign corresponds).

(ii) If $p = q = 0$, then there exists a unique mapping $f_\infty : X \to Y$ satisfying (4.2) and

$$\|f(x) - f_\infty(x)\| \le \frac{\delta}{|1 - |1 \pm 2b||} \qquad (\forall x \in X)$$

(double-sign corresponds).

(iii) If either $1 + b = 0$ and $p > 0$ or $1 - b = 0$ and $p > 0$, then f satisfies (4.2).

In fact, put $\varepsilon(x, x') = \delta\|x\|^p\|x'\|^q$ for each $x, x' \in X$. Note that (ii) in Corollary 4.3 and (ii) in Corollary 4.4 are equivalent to $|1 \pm 2b| < |1 \pm 2b|^{p+q}$ and $|1 \pm 2b| > |1 \pm 2b|^{p+q}$, respectively (double-sign corresponds). Since $\pm b \neq -\frac{1}{2}, 0$ and $p + q \neq 1$, it follows that either (ii) in Corollary 4.3 or (ii) in Corollary 4.4 holds (double-sign corresponds). Then the assertion (i) follows immediately from Corollaries 4.3 and 4.4. If $\varepsilon(x, x') = \delta$ for each $x, x' \in X$, then the assertion (ii) holds in the same way. Also if $1 \pm b = 0$ and $p > 0$, then $|1 \pm b|^p = 0$ (double-sign corresponds) and hence we can easily see that the assertion (iii) holds by the same consideration in the above (i).

REFERENCES

[1] T. AOKI, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, **2** (1950), pp. 64–66.

[2] D. H. HYERS, On the stability of the linear functional equations, *Proc. Nat. Acad. Sci. U. S. A.*, **27** (1941), pp. 222–224.

[3] H.-M. KIM, K.-W. JUN and J.M. RASSIAS, Extended stability problem for alternative Cauchy-Jensen mappings, *J. Inequal. Pure Appl. Math.*, **8**-4 (2007), Art.120, 17 pp.

[4] TH. RASSIAS, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), pp. 297-300.

[5] S.-E. TAKAHASI, T. MIURA and H. TAKAGI, On a Hyers-Ulam-Aoki-Rassias type stability and a fixed point theorem, *J. Nonlinear Convex Anal.*, **11**-2 (2010), pp. 423-439.

[6] S. M. ULAM, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York, 1960.