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EXISTENCE RESULTS FOR SECOND ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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ABSTRACT. In this paper we study the existence of solutions for second order impulsive functional differential equations with infinite delay. To obtain our results, we apply fixed point methods.

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1. INTRODUCTION

This paper is concerned with second order impulsive functional differential equations with infinite delay of the form

$$x''(t) = f(t, x_t), \text{ a.e. } t \in J, \ t \neq t_k, k = 1, \dots, m, \tag{1.1}$$

$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, \dots, m, \tag{1.2}$$

$$\Delta x'(t_k) = J_k(x(t_k)), \quad k = 1, \dots, m, \tag{1.3}$$

$$x_0 = \phi \in \mathcal{B}, \quad x'(0) = \eta. \tag{1.4}$$

In problem (1.2)-(1.4), $f : J \times \mathcal{B} \to \mathbb{R}^n$ is an appropriate function, $J = [0, 1]$, $\mathcal{B}$ is an abstract phase space to be specified in the sequel, $\eta \in \mathbb{R}^n$, the impulsive moments $t_1, \dots, t_m$ are such that $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, and, for $k = 1, \dots, m$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$ and the impulse functions $I_k, J_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous. For every $t \in J$, the history function $x_t \in \mathcal{B}$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \leq 0$.

The theory of impulsive differential equations has become an important area of investigation in recent years. Relative to this theory, we only refer the interested reader to [17] and the monographs [1, 15, 19].

In the literature, many existence results for impulsive differential equations are proved under restrictive conditions on the impulse functions. For instance, in [2, 3, 4], in addition to continuity, boundedness condition is required, which is not fulfilled in some important cases such as for linear impulses; in [7, 9, 11], the existence of solutions involves lipschitz condition on the impulses; in [7, 8, 13, 20], sublinear growth is imposed and in [5, 6, 16, 24] other conditions are assumed, just to name a few.

The aim in this paper is to give some existence results of solutions for the initial value problem (1.1)-(1.4) using Schaefer's and Sadovskii's fixed point theorems. In our main results, only the continuity of the impulse functions I_k, J_k, $k = 1, \dots, m$, is required.

2. PRELIMINARIES

In this section, we introduce notations, definitions and results which are used throughout this paper.

Let $L^1(J)$ be the Banach space of measurable functions $x : J \to \mathbb{R}^n$ which are Lebesgue integrable, normed by

$$\|x\|_{L^1} = \int_0^1 |x(t)| dt, \quad x \in L^1(J).$$

By $PC(J)$ we denote the Banach space of functions $x : J \to \mathbb{R}^n$ which are continuous at $t \neq t_k$, left continuous at $t = t_k$, and such that the right limit $x(t_k^+)$ exists, $k = 1, \dots, m$, equipped with the norm

$$\|x\| = \sup\{|x(t)| : t \in J\}, \quad x \in PC(J).$$

Let $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_+ = [0, \infty)$. We assume that the phase space $\mathcal{B}$ is an abstract linear space of functions mapping $\mathbb{R}_-$ into $\mathbb{R}^n$, endowed with a (semi)norm $\| \cdot \|_{\mathcal{B}}$ and satisfying the following fundamental axioms introduced at first by Hale and Kato in [10] (see also [12, 14, 22]):

(A1) There exist functions $K, M : \mathbb{R}_+ \to \mathbb{R}_+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $x : (-\infty, \sigma + a] \to \mathbb{R}^n$, $x_\sigma \in \mathcal{B}$ and x is continuous everywhere on $[\sigma, \sigma + a]$ except for a finite number of points ($\sigma < t_1 < \cdots < t_s < \sigma + a$) at which x has a discontinuity of the first type and is left continuous (x is continuous at

$t \neq t_k$, and left continuous at $t = t_k$, and the right limit $x(t_k^+)$ exists for $k = 1, \ldots, s$), then for every t in $[\sigma, \sigma + a]$ the following conditions hold:
(i) $x_t \in \mathcal{B}$,
(ii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup\{|x(s)| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$.

(A2) For a function x satisfying (A1), the $\mathcal{B}$-valued function $t \mapsto x_t$ is continuous on $[\sigma, \sigma + a] \setminus \{t_1, \ldots, t_s\}$.

(A3) There exists a positive constant H such that $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for all $\phi \in \mathcal{B}$.

(A4) The space $\mathcal{B}$ is complete.

Example 2.1. *Fix a constant α and consider the space $\mathcal{B}_\alpha$ of functions $\phi : \mathbb{R}_- \to \mathbb{R}^n$ such that ϕ is continuous everywhere except for a finite number of points at which it has a discontinuity of the first type and is left continuous, and $\lim_{\theta \to -\infty} e^{\alpha\theta}\phi(\theta)$ exists in $\mathbb{R}^n$. Using the approach of* [12], *with obvious modifications one can show that $\mathcal{B}_\alpha$ with the norm $\|\phi\|_{\mathcal{B}_\alpha} = \sup\{e^{\alpha\theta}|\phi(\theta)| : \theta \leq 0\}$, $\phi \in \mathcal{B}_\alpha$, satisfies the axioms (A1)-(A4) with $H = 1$, $K(t) = \max\{1, e^{-\alpha t}\}$ and $M(t) = e^{-\alpha t}$ for all $t \in \mathbb{R}_+$.*

In the rest of this paper, K and M are the constants defined by $K = \sup\{K(t) : t \in J\}$ and $M = \sup\{M(t) : t \in J\}$.

A function $x \in \mathcal{B} \cap PC(J)$ is said to be a solution of (1.1)-(1.4) if x satisfies the differential equation (1.1) a.e. on $J \setminus \{t_1, \ldots, t_m\}$ and the conditions (1.2)-(1.4).

To establish our main theorems, we need the basic assumptions for the initial value problem (1.1)-(1.4).

(H1) The function $f : J \times \mathcal{B} \to \mathbb{R}^n$ is Carathéodory, that is,
(i) for every $x \in \mathcal{B}$, the function $f(\cdot, x) : J \to \mathbb{R}^n$ is measurable,
(ii) for almost every $t \in J$, the function $f(t, \cdot) : \mathcal{B} \to \mathbb{R}^n$ is continuous.

(H2) There exist a function $q \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,x)| \leq q(t)\psi(\|x\|_{\mathcal{B}}) \text{ for a.e. } t \in J \text{ and all } x \in \mathcal{B}.$$

(H3) For $k = 1, \ldots, m$, the impulse functions $I_k, J_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

Note that the hypotheses (H1) and (H2) are not new, they have been used extensively in the literature on differential equations.

The proofs of our main existence results rely on the following theorems.

Theorem 2.1 (Schaefer Fixed Point Theorem [21, 23])**.** *Let E be a normed space and let $\Gamma : E \to E$ be a completely continuous map, that is, it is a continuous mapping which is compact on each bounded subset of E. If the set $\mathcal{E} = \{x \in E : \lambda x = \Gamma x$ for some $\lambda > 1\}$ is bounded, then Γ has a fixed point.*

Theorem 2.2 (Sadovskii Fixed Point Theorem [18])**.** *Let E be a Banach space and let $\Gamma : E \to E$ be a completely continuous map. If $\Gamma(B) \subset B$ for a nonempty closed, convex and bounded set B of E, then Γ has a fixed point in B.*

3. EXISTENCE RESULTS

In this section we state and prove our existence results for problem (1.1)-(1.4).

Theorem 3.1. *Assume that (H1), (H2) and (H3) hold. Further if*

$$\int_c^\infty \frac{ds}{\psi(s)} = \infty \tag{3.1}$$

where $c = (KH + M)\|\phi\|_{\mathcal{B}} + K|\eta|$, *then the initial value problem (1.1)-(1.4) has a solution on* $(-\infty, 1]$.

Proof. Let $PC_\phi(J) = \{x \in PC(J) : x(0) = \phi(0)\}$. We define the operator $\Gamma : PC_\phi(J) \to PC_\phi(J)$ by

$$(\Gamma x)(t) = \phi(0) + \eta t + \int_0^t\int_0^s f(u, \overline{x}_u)duds + \sum_{0<t_k<t} [I_k(x(t_k)) + (t - t_k)J_k(x(t_k))], \quad x \in PC_\phi(J),\ t \in J, \tag{3.2}$$

where $\overline{x} : (-\infty, 1] \to \mathbb{R}^n$ is such that $\overline{x}_0 = \phi$ and $\overline{x} = x$ on J.

Clearly, Γ is well defined and fixed points of Γ are solutions to the initial value problem (1.1)-(1.4). We will show that Γ satisfies all the conditions of Theorem 2.1. We divide the proof into several steps.

Step 1. Γ is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC_\phi(J)$. Then, for $t \in J$,

$$\begin{aligned} &|(\Gamma x_n)(t) - (\Gamma x)(t)| \\ &\quad\leq \int_0^t\int_0^s |f(u, \overline{x}_{nu}) - f(u, \overline{x}_u)|duds + \sum_{0<t_k<t} |I_k(x_n(t_k)) - I_k(x(t_k))| \\ &\qquad + \sum_{0<t_k<t} (t - t_k)|J_k(x_n(t_k)) - J_k(x(t_k))| \\ &\quad\leq \int_0^1 |f(s, x_{ns}) - f(s, x_s)|ds + \sum_{k=1}^m |I_k(x_n(t_k)) - I_k(x(t_k))| \\ &\qquad + \sum_{k=1}^m (1 - t_k)|J_k(x_n(t_k)) - J_k(x(t_k))|. \end{aligned} \tag{3.3}$$

On one hand, for a.e. $s \in J$, by (H1)(ii) we have $f(s, \overline{x}_{ns}) \to f(s, \overline{x}_s)$ since $\overline{x}_{ns} \to \overline{x}_s$ as $n \to \infty$. On the other hand, let $B = \{u \in PC_\phi(J) : \|u\| \leq r\}$ for some $r > 0$ such that $\|x_n\|, \|x\| \leq r$, for all $n \in \mathbb{N}$. For a.e. $s \in J$, by (A1)(ii) we have

$$\|x_{ns}\|_{\mathcal{B}} \leq K \sup\{|x_n(u)| : u \in [0, s]\} + M\|\overline{x}_{n0}\|_{\mathcal{B}} \leq Kr + M\|\phi\|_{\mathcal{B}}$$

and

$$\|x_s\|_{\mathcal{B}} \leq K \sup\{|x(u)| : u \in [0, s]\} + M\|\overline{x}_0\|_{\mathcal{B}} \leq Kr + M\|\phi\|_{\mathcal{B}}$$

which imply, by (H2),

$$|f(s, \overline{x}_{ns}) - f(s, \overline{x}_s)| \leq q(s)[\psi(\|\overline{x}_{ns}\|_{\mathcal{B}}) + \psi(\|\overline{x}_s\|_{\mathcal{B}})] \leq 2q(s)\psi(Kr + M\|\phi\|_{\mathcal{B}}).$$

Then by the continuity of I_k and J_k, $k = 1, \ldots, m$, and the dominated convergence theorem, from (3.3) we deduce that $\Gamma x_n \to \Gamma x$ as $n \to \infty$; which completes the proof that Γ is continuous on $PC_\phi(J)$.

Now, we will prove that Γ takes bounded sets into relatively compact sets in $PC_\phi(J)$. From the Ascoli-Arzela theorem, it is sufficient to prove that for each bounded subset B of $PC_\phi(J)$, the set ΓB is bounded and is equicontinuous.

Let B be a bounded set in $PC_\phi(J)$. Then there exists a real number $r > 0$ such that $\|x\| \leq r$, for all $x \in B$.

Step 2. The set ΓB is bounded.

Let $x \in B$ and $t \in J$. We will show that there exists a positive constant δ, which does not depend on x and t, such that $|(\Gamma x)(t)| \leq \delta$. After some standard calculations we get

$$\begin{aligned}
|(\Gamma x)(t)| \leq H\|\phi\|_{\mathcal{B}} + |\eta| + &\int_0^1\int_0^1 q(u)\psi(\|\overline{x}_u\|_{\mathcal{B}})duds \\
&+ \sum_{k=1}^{m} \left(|I_k(x(t_k))| + (1-t_k)|J_k(x(t_k))|\right). \\
\leq H\|\phi\|_{\mathcal{B}} &+ |\eta| + \|q\|_{L^1}\psi(Kr + M\|\phi\|_{\mathcal{B}}) \\
&+ \sum_{k=1}^{m} \sup\{|I_k(u)| + (1-t_k)|J_k(u)| : |u| \leq r\} := \delta.
\end{aligned}$$

Step 3. The set ΓB is equicontinuous on J.

Initially, we assume that t and $h \neq 0$ are such that $t, t+h \in J \backslash \{t_1, \ldots, t_m\}$. It is not difficult to get

$$\begin{aligned}
&|(\Gamma x)(t+h) - (\Gamma x)(t)| \\
&\quad \leq h|\eta| + h\|q\|_{L^1}\psi(Kr + M\|\phi\|_{\mathcal{B}}) \\
&\qquad + \sum_{t<t_k<t+h} |I_k(x(t_k)) + (1-t_k)J_k(x(t_k))| + \sum_{0<t_k<t+h} h|J_k(x(t_k))|.
\end{aligned}$$

As $h \to 0$ the right-hand side of the above inequality converges to zero. This proves the equicontinuity on $J \backslash \{t_1, \ldots, t_m\}$.

Now, we assume that $t = t_i$ for some index $i \in \{1, \ldots, m\}$. Let $h \neq 0$ be such that $\{t_k : k \neq i\} \cap [t_i - |h|, t_i + |h|] = \emptyset$. Then we have

$$\begin{aligned}
&|(\Gamma x)(t_i + h) - (\Gamma x)(t_i)| \\
&\qquad \leq h|\eta| + h\|q\|_{L^1}\psi(K\rho + M\|\phi\|_{\mathcal{B}}) + \sum_{0<t_k<t_i+h} h|J_k(x(t_k))|,
\end{aligned}$$

which implies that the left-hand side of the above inequality tends towards zero as $h \to 0$.

Finally, it remains to prove what follows:

Step 4. The set $\mathcal{E} = \{x \in PC_\phi(J) : \lambda x = \Gamma x \text{ for some } \lambda > 1\}$ is bounded.

Let $x \in \mathcal{E}$ and let $\lambda > 1$ be such that $\lambda x = \Gamma x$. Then $x|_{[0,t_1]}$ satisfies, for each $t \in [0, t_1]$,

$$x(t) = \lambda^{-1}\left(\phi(0) + t\eta + \int_0^t\int_0^s f(u, \overline{x}_u)duds\right).$$

It is straightforward to verify that

$$\begin{aligned}
|x(t)| &\leq H\|\phi\|_{\mathcal{B}} + |\eta| + \int_0^t\int_0^s q(u)\psi(\|\overline{x}_u\|_{\mathcal{B}})duds \\
&\leq H\|\phi\|_{\mathcal{B}} + |\eta| + \int_0^t q(s)\psi(K\sup\{|x(u)| : u \in [0,s]\} + M\|\phi\|_{\mathcal{B}})ds.
\end{aligned} \tag{3.4}$$

Introduce the function $v_1(t) = K\sup\{|x(s)| : s \in [0,t]\} + M\|\phi\|_{\mathcal{B}}$, $t \in [0, t_1]$, in (3.4) to obtain

$$|x(t)| \leq H\|\phi\|_{\mathcal{B}} + |\eta| + \int_0^t q(s)\psi(v_1(s))ds.$$

We have also

$$|x(s)| \leq H\|\phi\|_{\mathcal{B}} + |\eta| + \int_0^t q(u)\psi(v_1(u))du, \qquad \forall s \in [0,t]$$

from which we deduce that

$$v_1(t) \leq (KH+M)\|\phi\|_{\mathcal{B}} + K|\eta| + K\int_0^t q(s)\psi(v_1(s))ds.$$

Set

$$w_1(t) = (KH+M)\|\phi\|_{\mathcal{B}} + K|\eta| + K\int_0^t q(s)\psi(v_1(s))ds, \quad \text{for } t \in [0,t_1].$$

Then we have $v_1(t) \leq w_1(t)$ for all $t \in [0,t_1]$. As ψ is nondecreasing, a direct differentiation of w_1 yields

$$\begin{cases} w_1'(t) \leq Kq(t)\psi(w_1(t)), & \text{a.e. } t \in [0,t_1] \\ w_1(0) = (KH+M)\|\phi\|_{\mathcal{B}} + K|\eta| := c. \end{cases}$$

By integration, this gives

$$\int_0^t \frac{w_1'(s)}{\psi(w_1(s))}\,ds \leq K\int_0^t q(s)ds \leq K\|q\|_{L^1}, \quad t \in [0,t_1]. \tag{3.5}$$

By a change of variables, inequality (3.5) implies

$$\int_c^{w_1(t)} \frac{ds}{\psi(s)} \leq K\|q\|_{L^1}, \quad t \in [0,t_1].$$

By (3.1), there is a constant $M_1 > 0$ such that, for all $t \in [0,t_1]$, $w_1(t) \leq M_1$ and therefore $v_1(t) \leq M_1$. We choose M_1 large enough such that $M\|\phi\|_{\mathcal{B}} \leq M_1$ to get

$$\sup\{|x(t)| : t \in [0,t_1]\} \leq \frac{1}{K}(M_1 - M\|\phi\|_{\mathcal{B}}) := \rho_1.$$

Now, consider $x|_{[0,t_2]}$. It satisfies, for each $t \in [0,t_2]$,

$$x(t) = \lambda^{-1}\left(\phi(0) + t\eta + \int_0^t\!\!\int_0^s f(u,\overline{x}_u)duds + I_1(x(t_1)) + (t-t_1)J_1(x(t_1))\right).$$

Therefore,

$$\begin{aligned} |x(t)| &\leq H\|\phi\|_{\mathcal{B}} + |\eta| + \int_0^t\!\!\int_0^s q(u)\psi(\|\overline{x}_u\|_{\mathcal{B}})duds + \overline{IJ}_1 \\ &\leq H\|\phi\|_{\mathcal{B}} + |\eta| + \overline{IJ}_1 + \int_0^t q(s)\psi(K\sup\{|x(r)| : r \in [0,s]\} + M\|\phi\|_{\mathcal{B}})ds \end{aligned} \tag{3.6}$$

where $\overline{IJ}_1 = \sup\{|I_1(u)| + (1-t_1)|J_1(u)| : |u| \leq \rho_1\}$.

Denote $v_2(t) = K\sup\{|x(s)| : s \in [0,t]\} + M\|\phi\|_{\mathcal{B}}$, for $t \in [0,t_2]$. From (3.6) we obtain

$$|x(t)| \leq H\|\phi\|_{\mathcal{B}} + |\eta| + \overline{IJ}_1 + \int_0^t q(s)\psi(v_2(s))ds.$$

We have also

$$|x(s)| \leq H\|\phi\|_{\mathcal{B}} + |\eta| + \overline{IJ}_1 + \int_0^t q(u)\psi(v_2(u))du, \ \forall s \in [0,t]$$

from which we get

$$v_2(t) \leq (KH+M)\|\phi\|_{\mathcal{B}} + K|\eta| + K\overline{IJ}_1 + K\int_0^t q(s)\psi(v_2(s))ds := w_2(t).$$

The function w_2 is such that

$$\begin{cases} w_2'(t) \leq Kq(t)\psi(w_2(t)), & \text{a.e. } t \in [0, t_2] \\ w_2(0) = (KH + M)\|\phi\|_{\mathcal{B}} + K|\eta| + K\overline{IJ}_1. \end{cases}$$

By integration, this yields

$$\int_0^t \frac{w_2'(s)}{\psi(w_2(s))}\, ds \leq K \int_0^t q(s)ds \leq K\|q\|_{L^1}, \quad t \in [0, t_2]$$

which implies

$$\int_{w_2(0)}^{w_2(t)} \frac{ds}{\psi(s)} \leq K\|q\|_{L^1}, \quad t \in [0, t_2].$$

Again, by (3.1), there exists a constant $M_2 > 0$ such that, for all $t \in [0, t_2]$, $w_2(t) \leq M_2$ and then $v_2(t) \leq M_2$. Finally, if we select M_2 such that $M\|\phi\|_{\mathcal{B}} \leq M_2$, we get

$$\sup\{|x(t)| : t \in [0, t_2]\} \leq \frac{1}{K}(M_2 - M\|\phi\|_{\mathcal{B}}) := \rho_2.$$

Continue this process for $x|_{[0,t_3]}, \ldots, x|_J$, we obtain that there exists a constant $\rho > 0$ such that

$$\|x\| \leq \rho.$$

This finish to show that the set $\mathcal{E}$ is bounded in $PC_\phi(J)$.

As a result the conclusion of Theorem 2.1 holds and consequently the initial value problem (1.1)-(1.4) has a solution x on $(-\infty, 1]$. This completes the proof. ∎

With additional restrictive conditions on the impulse functions, condition (3.1) in assumption (H2) in Theorem 3.1 can be weakened to obtain the following result.

Theorem 3.2. *Assume that (H1) and (H2) hold. Additionally we assume that:*

(H3') *The impulse functions $I_k, J_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \ldots, m$, are continuous. Furthermore there exist constants $a_k, b_k, c_k, d_k \in \mathbb{R}_+$, with*

$$\sum_{k=1}^{m}[a_k + (1 - t_k)c_k] < 1,$$

such that $|I_k(x)| \leq a_k|x| + b_k$ and $|J_k(x)| \leq c_k|x| + d_k$, for all $x \in \mathbb{R}^n$.

Then the initial value problem (1.1)-(1.4) has a solution on $(-\infty, 1]$, provided that

$$KC\|q\|_{L^1} < \int_c^\infty \frac{ds}{\psi(s)}$$

where

$$C = \left(1 - \sum_{k=1}^{m}[a_k + (1 - t_k)c_k]\right)^{-1} \tag{3.7}$$

and

$$c = KC\left(H\|\phi\|_{\mathcal{B}} + |\eta| + \sum_{k=1}^{m}[b_k + (1 - t_k)d_k]\right) + M\|\phi\|_{\mathcal{B}}.$$

Proof. Consider the operator $\Gamma : PC_\phi(J) \to PC_\phi(J)$ defined in the proof of Theorem 3.1 by relation (3.2). As the proof that Γ is completely continuous follows the same lines as in the proof that the operator Γ in the proof of Theorem 3.1 possesses the same property, it is omitted. It only remains to show that the set $\mathcal{E} = \{x \in PC_\phi(J) : \lambda x = \Gamma x \text{ for some } \lambda > 1\}$ is bounded.

Let $x \in \mathcal{E}$ and let $\lambda > 1$ be such that $\lambda x = \Gamma x$. Let $t \in J$. After some calculations we get

$$
\begin{aligned}
|x(t)| &= \lambda^{-1}|(\Gamma x)(t)| \\
&\le H\|\phi\|_{\mathcal{B}} + |\eta| + \int_0^t q(s)\psi(\|\overline{x}_s\|_{\mathcal{B}})ds \\
&\quad + \sum_{0<t_k<t}[a_k + (1-t_k)c_k]|x(t_k)| + \sum_{0<t_k<t}[b_k + (1-t_k)d_k] \\
&\le H\|\phi\|_{\mathcal{B}} + |\eta| + \sum_{k=1}^m [b_k + (1-t_k)d_k] \\
&\quad + \int_0^t q(s)\psi(K\sup\{|x(u)| : u \in [0,s]\} + M\|\phi\|_{\mathcal{B}})ds \\
&\quad + \left(\sum_{k=1}^m [a_k + (1-t_k)c_k]\right)\sup\{|x(s)| : s \in [0,t]\}.
\end{aligned} \tag{3.8}
$$

In the right-hand side of (3.8), we introduce the function v defined, for $t \in J$, by $v(t) = K\sup\{|x(s)| : s \in [0,t]\} + M\|\phi\|_{\mathcal{B}}$. We obtain

$$
\begin{aligned}
|x(t)| \le{}& H\|\phi\|_{\mathcal{B}} + |\eta| + \sum_{k=1}^m [b_k + (1-t_k)d_k] \\
&+ \int_0^t q(s)\psi(v(s))ds + \left(\sum_{k=1}^m [a_k + (1-t_k)c_k]\right)\sup\{|x(s)| : s \in [0,t]\}.
\end{aligned}
$$

We have also, for all $s \in [0,t]$,

$$
\begin{aligned}
|x(s)| \le{}& H\|\phi\|_{\mathcal{B}} + |\eta| + \sum_{k=1}^m [b_k + (1-t_k)d_k] \\
&+ \int_0^t q(u)\psi(v(u))du + \left(\sum_{k=1}^m [a_k + (1-t_k)c_k]\right)\sup\{|x(u)| : u \in [0,t]\}
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\sup\{|x(s)| : s \in [0,t]\} \\
&\quad \le C\left(H\|\phi\|_{\mathcal{B}} + |\eta| + \sum_{k=1}^m [b_k + (1-t_k)d_k] + \int_0^t q(s)\psi(v(s))ds\right)
\end{aligned} \tag{3.9}
$$

where C is as in (3.7). From (3.9) we deduce that

$$
v(t) \le KC\left(H\|\phi\|_{\mathcal{B}} + |\eta| + \sum_{0<t_k<t}[b_k + (1-t_k)d_k] + \int_0^t q(s)\psi(v(s))ds\right) + M\|\phi\|_{\mathcal{B}}.
$$

Denote by $w(t)$ the right-hand side of the above inequality. It follows that, $v(t) \le w(t)$ for all $t \in J$ and

$$
\begin{cases}
w'(t) \le KCq(t)\psi(w(t)), & \text{a.e. } t \in J \\
w(0) = KC\left(H\|\phi\|_{\mathcal{B}} + |\eta| + \sum_{k=1}^m [b_k + (1-t_k)d_k]\right) + M\|\phi\|_{\mathcal{B}} := c.
\end{cases}
$$

This yields

$$\int_0^t \frac{w'(s)}{\psi(w(s))}\, ds \leq KC \int_0^t q(s)ds \leq KC\|q\|_{L^1}, \quad t \in J$$

which implies

$$\int_c^{w(t)} \frac{ds}{\psi(s)} \leq KC\|q\|_{L^1}, \quad t \in J.$$

From (3.1) it follows that there exists a constant $\rho > 0$ such that $w(t) \leq \rho$ and then $v(t) \leq \rho$, for all $t \in J$. Finally, if we choose ρ such that $M\|\phi\|_{\mathcal{B}} \leq \rho$, we get

$$\|x\| \leq \frac{1}{K}(\rho - M\|\phi\|_{\mathcal{B}})$$

which finish to show that the set $\mathcal{E}$ is bounded in $PC_\phi(J)$. Now, from Theorem 2.1 we infer the existence of a solution for the initial value problem (1.1)-(1.4). The proof is complete. ∎

If condition (3.1) in Theorem 3.1 is replaced by condition (3.10) below, we obtain a new existence result. This result is not a consequence of Theorem 3.1.

Theorem 3.3. *Under conditions (H1), (H2) and (H3), the initial value problem (1.1)-(1.4) has a solution on* $(-\infty, 1]$*, provided that*

$$K\|q\|_{L^1} \liminf_{r\to+\infty} \frac{\psi(r)}{r} + \sum_{k=1}^{m} \liminf_{r\to+\infty} \frac{\sup\{|I_k(u)| + (1-t_k)|J_k(u)| : |u| \leq r\}}{r} < 1. \tag{3.10}$$

Proof. We claim that there exists $r > 0$ such that $\Gamma(B_r) \subseteq B_r$ where the operator Γ is defined by (3.2) and B_r is the closed ball in $PC_\phi(J)$ with center 0 and radius r. If this property is false, then for each $r > 0$ there exist $x^r \in B_r$ and $t^r \in J$ such that $|(\Gamma x^r)(t^r)| > r$. From this it follows that

$$\begin{aligned}
r &< |(\Gamma x^r)(t^r)| \\
&= \Bigg|\phi(0) + t^r\eta + \int_0^{t^r}\int_0^s f(u, \overline{x^r}_u)duds \\
&\qquad + \sum_{0<t_k<t^r} [I_k(x^r(t_k)) + (t^r - t_k)J_k(x^r(t_k))]\Bigg| \\
&\leq H\|\phi\|_{\mathcal{B}} + |\eta| + \int_0^1\int_0^1 q(u)\psi(\|\overline{x^r}_u\|_{\mathcal{B}})duds \\
&\qquad + \sum_{k=1}^{m} (|I_k(x^r(t_k))| + (1-t_k)|J_k(x^r(t_k))|) \\
&\leq H\|\phi\|_{\mathcal{B}} + |\eta| + \|q\|_{L^1}\psi(Kr + M\|\phi\|_{\mathcal{B}}) \\
&\qquad + \sum_{k=1}^{m} \sup\{|I_k(u)| + (1-t_k)|J_k(u)| : |u| \leq r\}.
\end{aligned}$$

Hence, we obtain

$$1 \le K\|q\|_{L^1} \liminf_{r\to+\infty} \frac{\psi(r)}{r} + \sum_{k=1}^{m} \liminf_{r\to+\infty} \frac{\sup\{|I_k(u)| + (1-t_k)|J_k(u)| : |u| \le r\}}{r},$$

which contradicts (3.10).

Let $r > 0$ be such that $\Gamma : B_r \to B_r$. Arguing as in the proof of Theorem 3.1 we can prove that Γ is completely continuous, and from Theorem 2.2 we conclude that the initial value problem (1.1)-(1.4) has a solution x on $(-\infty, 1]$. The proof is finished. ∎

From the proof of Theorem 3.3, we immediately obtain the following corollaries.

Corollary 3.4. *Assume that (H1) and (H2) hold. In addition, assume that the following condition is satisfied.*

(H3') *The impulse functions $I_k, J_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \dots, m$, are continuous and there exist constants $a_k, b_k, c_k, d_k \in \mathbb{R}_+$ such that $|I_k(x)| \le a_k|x| + b_k$ and $|J_k(x)| \le c_k|x| + d_k$, for all $x \in \mathbb{R}^n$.*

Then the initial value problem (1.1)-(1.4) has a solution on $(-\infty, 1]$, provided that

$$K\|q\|_{L^1} \liminf_{r\to+\infty} \frac{\psi(r)}{r} + \sum_{k=1}^{m} [a_k + (1-t_k)c_k] < 1.$$

Corollary 3.5. *Assume that (H1) and (H2) and the following condition hold.*

(H3") *The impulse functions $I_k, J_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \dots, m$, are continuous and there exist constants $a_k, b_k, c_k, d_k \in \mathbb{R}_+$, $\alpha_k, \beta_k \in [0, 1)$ such that $|I_k(x)| \le a_k|x|^{\alpha_k} + b_k$ and $|J_k(x)| \le c_k|x|^{\beta_k} + d_k$, for all $x \in \mathbb{R}^n$.*

Then the initial value problem (1.1)-(1.4) has a solution on $(-\infty, 1]$, provided that

$$K\|q\|_{L^1} \liminf_{r\to+\infty} \frac{\psi(r)}{r} < 1.$$

REFERENCES

[1] M. BENCHOHRA, J. HENDERSON and S. K. NTOUYAS, *Inpulsive Differential Equations and Inclusions*, Series Contemporary Mathematics and Its Applications, **Vol. 2**, Hindawi Publ. Corp., 2006.

[2] M. BENCHOHRA, J. HENDERSON, S. K. NTOUYAS and A. OUAHAB, Multiple solutions for impulsive semilinear functional and neutral functional differential equations in Hilbert space, *J. Inequal. & Appl.*, (2005), No. 2, pp. 189–205.

[3] M. BENCHOHRA and S. K. NTOUYAS, On second order impulsive functional differential equations in Banach spaces, *J. Appl. Math. & Stoch. Anal.*, **15** (2002), No. 1, pp. 45–52.

[4] M. BENCHOHRA and A. OUAHAB, Multiple solutions for nonresonance impulsive functional differential equations, *Electron. J. Diff. Eqns*, (2003), No. 52, pp. 1–10.

[5] G. CAI, Z. DU and W. GE, On a class of second order impulsive boundary value problem at resonance, *Inter. J. Math. & Math. Sci.*, (2006), Article ID 62512, 11 pages. doi:10.1155/IJMMS/2006/62512

[6] G. CAI and W. GE, Positive solutions for second order impulsive differential equations with dependence on first order derivative, *J. Math. Resear. & Expos.*, **26** (2006), No.4, pp. 725–734.

[7] G. CAI, J. ZHANG and W. GE, The existence of solutions of three-point boundary value problems for second order impulsive differential equations, *Chin. Quart. J. Math.*, **20** (2005), No. 3, pp. 247–257.

[8] B. CHUANZHI and Y. DANDAN, Existence of solutions for second order nonlinear impulsive differential equations with periodic boundary value conditions, *Bound. Value Probl.*, (2007), Article ID 41589, 13 pages. doi:10.1155/2007/41589

[9] M. EDUARDO HERNANDEZ, A second order impulsive Cauchy problems, *Inter. J. Math. & Math. Sci.*, **31** (2002), No. 8, pp. 451–461.

[10] J. K. HALE and J. KATO, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.*, **21** (1978), pp. 11–41.

[11] J. HENDERSON and A. OUAHAB, Local and global existence and uniqueness results for second and higher order impulsive functional differential equations with infinte delay, *Austral. J. Math. Anal. & Appl.*, **4**, 2 (2007), pp. 1–32.

[12] Y. HINO, S. MURAKAMI and T. NAITO, *Functional Differential Equations with Infinite Delay*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, vol. **1473** (1991).

[13] L. JIANLI and S. JIANHUA, Existence of positive solutions for second order impulsive boundary value problems on infinity intervals, *Bound. Value Probl.*, (2006), Article ID 14594, 11 pages. doi:10.1155/BVP/2006/14594

[14] F. KAPPEL and W. SCHAPPACHER, Some considerations to the fundamental theory of infinite delay equations, *J. Differential Equations*, **37** (1980), pp. 141–183.

[15] V. LAKSHMIKANTHAM, D. D. BAINOV and P. S. SIMEONOV, *Theory of Impulsive Differential Equations*, World Scientific Pub. Co., Singapore, 1989.

[16] I. RACHUNKOVA and M. TVRDY, Second order periodic problem with phi-Laplacian and impulses, *Nonlinear Anal., T.M.A.*, **63** (2005), pp. 257–266.

[17] Y. V. ROGOVCHENKO, Impulsive evolution systems: Main results and new trends, *Dynam. Contin., Discrete Impuls. Systems*, **3** (1997), pp. 57–88.

[18] B. N. SADOVSKII, On a fixed point principle, *Funct. Anal. & Appl.*, **1** (1967), No. 2, pp. 74–76.

[19] A. SAMOILENKO and N. PERESYUK, *Differential Equations with Impusive Effectes*, World Scientific Pub. Co., Singapore, 1995.

[20] L. SANHUI and L. JIANLI, Solvability of second order m-point boundary value problems with impulses, *Bound. Value Probl.*, (2007), Article ID 97067, 8 pages. doi:10.1155/2007/97067

[21] H. SCHAEFER, Uber die Methode der a priori-Schranken, *Math. Annal.*, **129** (1955), 415–416 (German).

[22] K. SCHUMACHER, Existence and continuous dependence for functional differential equations with unbounded delay, *Arch. Rational Mech. & Anal.*, **67** (1978), pp. 315–335.

[23] D. R. SMART, *Fixed Point Theorems*, Cambridge Tracts in Mathematics, vol. 66, Cambridge University Press, Cambridge, 1980.

[24] J. TOMECEK, Nonlinear boundary value problem for nonlinear second order differential equations with impulses, *Electron. J. Qual. Theory Differ. Equ.* (2005), No. 10, pp. 1–22.