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A NOTE ON MIXED NORM SPACES OF ANALYTIC FUNCTIONS

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ABSTRACT. A direct and elementary proof of an estimate of Littlewood is given together with an application concerning the sharpness and strictness of some inclusions in mixed norm spaces of analytic functions.

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1. INTRODUCTION

Let $\mathbb{D}$ be the unit disc in the complex plane and $\mathbb{T}$ its boundary. If $f(z) = f(re^{i\theta})$ is a measurable function in $\mathbb{D}$, then we write as usual

$$M_p(f;r) = \big\|f(r\cdot)\big\|_{L^p(\mathbb{T};dm)}, \quad 0 \le r < 1, \quad 0 < p \le \infty,$$

where dm is the Lebesgue measure on $\mathbb{T}$. The collection of analytic functions $f(z)$, for which $\|f\|_{H^p} = \sup\limits_{0\le r<1} M_p(f;r) < +\infty$, is the usual Hardy space H^p. The quasi-normed space $H(p,q,\alpha)\big(0 < p,q \le \infty, \alpha > 0\big)$ is the set of those functions $f(z)$ analytic in the unit disc $\mathbb{D}$, for which the quasi-norm

$$\|f\|_{p,q,\alpha} = \begin{cases} \left(\displaystyle\int_0^1 (1-r)^{\alpha q-1} M_p^q(f;r)\,dr\right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_{0\le r<1} (1-r)^{\alpha} M_p(f;r), & q = \infty, \end{cases}$$

is finite. If $(1-r)^\alpha M_p(f;r) = o(1)$ as $r \to 1^-$, then we write $f \in H_0(p,\infty,\alpha)$. For $p = q < \infty$ the spaces $H(p,q,\alpha)$ coincide with the well-known weighted Bergman spaces, while $q = \infty$ they are known as growth spaces, and $H_0(p,\infty,\alpha)$ corresponding "little" space.

The mixed norm spaces consisting of harmonic functions will be denoted by $h(p,q,\alpha)$. In [1] among others, some continuous inclusions of Hardy–Littlewood–Flett type in $h(p,q,\alpha)$ are proved in the context of functions n-harmonic in the unit polydisc of $\mathbb{C}^n$.

Theorem 1. *The following inclusions are continuous for any* $\alpha,\beta \in \mathbb{R}$, $0 < p,q \le \infty$*:*

$$\begin{array}{rll}
(i) & h(p,q,\alpha) \subset h(p,q,\beta), & \beta > \alpha, \\
(ii) & h(p,q,\alpha) \subset h(p_0,q,\alpha), & 0 < p_0 < p \le \infty, \\
(iii) & h(p,q,\alpha) \subset h(p,q_0,\alpha), & 0 < q < q_0 \le \infty, \\
(iv) & h(p,q,\alpha) \subset h(p_0,q,\beta), & \beta \ge \alpha + 1/p - 1/p_0,\ 0 < p \le p_0 \le \infty, \\
(v) & h(p,q,\alpha) \subset h(p_0,q_0,\beta), & \beta > \alpha + 1/p,\ 0 < p_0, q_0 \le \infty, \\
(vi) & h(p,q,\alpha) \subset h(p,q_0,\beta), & \beta > \alpha,\ 0 < q_0 \le \infty, \\
(vii) & H^p \subset H\left(p_0, q, \frac{1}{p} - \frac{1}{p_0}\right), & 0 < p < p_0 \le \infty,\ 0 < p \le q \le \infty.
\end{array}$$

Of course, the inclusions (i), (ii) are obvious, while some others are much deeper, for instance, (iii), (iv) and (vii) which were originally proved by Hardy and Littlewood [8, Th.31] and Flett [5, pp.755-756] for functions analytic in the unit disc, see also [4, Th.5.11], [6, Th.3.1], [9].

The purpose of this note is to prove that the inclusions (i)-(vii) for analytic functions are strict and best possible in a certain sense. See Theorem 2 below for the precise formulation.

2. ESTIMATES

Throughout the paper, the capital letters $C(\alpha,\beta,\dots), C_\alpha$ stand for different positive constants depending only on the parameters indicated. For $A, B > 0$ the notation $A \approx B$ denotes the two-sided estimate $C_1 A \le B \le C_2 A$ with some inessential positive constants C_1 and C_2 independent of the variable involved.

The estimates appearing in the next lemma were essentially proved by Littlewood in [10, pp.93-96], see also in [2], [3, p.14]. Such type inequalities are usually proved by means of growth estimates for Taylor coefficients, which were due to Faber and Littlewood [10, pp.93–96], [11, Ch.5,Th.2.31]. Below we give a direct and elementary proof of the estimates avoiding growth estimates for Taylor coefficients.

Lemma 1. *Suppose that* $\alpha,\ \beta \in \mathbb{R}$ *and*

$$J_{\alpha,\beta} = J_{\alpha,\beta}(r) := \int_{-\pi}^{\pi} \left|1 - re^{i\theta}\right|^{-\alpha-1} \left|\log \frac{e}{1 - re^{i\theta}}\right|^{-\beta} d\theta.$$

Then for all $0 \le r < 1$

$$J_{\alpha,\beta} \approx \begin{cases} (1-r)^{-\alpha} \left(\log \dfrac{e}{1-r}\right)^{-\beta}, & \alpha > 0,\ \beta \in \mathbb{R}, \\ 1, & \alpha < 0,\ \beta \in \mathbb{R}, \end{cases} \tag{2.1}$$

$$J_{0,\beta} \approx \begin{cases} \left(\log \dfrac{e}{1-r}\right)^{1-\beta}, & \beta < 1, \\ 1, & \beta > 1, \\ \log\left(e \log \dfrac{e}{1-r}\right), & \beta = 1, \end{cases} \tag{2.2}$$

where the involved constants $C = C(\alpha, \beta) > 0$ *depend only on* α, β.

Proof. It suffices to prove all the estimates only for all r close enough to 1, and moreover for all $z \in \mathbb{D}$ lying in a small neighborhood of 1. For the expression $\left|1 - re^{i\theta}\right| = \sqrt{(1-r)^2 + 4r\sin^2\frac{\theta}{2}}$, we have the simple estimate

$$\frac{1}{\sqrt{2}}\left(1 - r + 2\sqrt{r}\frac{|\theta|}{\pi}\right) \le \left|1 - re^{i\theta}\right| \le 1 - r + |\theta|, \qquad z = re^{i\theta} \in \mathbb{D},$$

in particular,

$$\frac{1}{\pi}(1 - r + |\theta|) \le \left|1 - re^{i\theta}\right| \le 1 - r + |\theta|, \qquad \frac{1}{2} \le r < 1. \tag{2.3}$$

Define the ring sector $E := \left\{z = re^{i\theta} \in \mathbb{D} :\ \frac{9}{10} < r < 1,\ |\theta| < \frac{1}{2}\right\}$, so that $|1 - z| < \dfrac{1}{2}$ $(z \in E)$, and the following inequalities are valid:

$$\begin{aligned} \left|\log \frac{1}{1-z}\right| &\le \log \frac{1}{|1-z|} + \frac{\pi}{2} \le 5 \log \frac{1}{|1-z|}, \qquad z \in E, \\ \left|\log \frac{1}{1-z}\right| &\ge \log \frac{1}{|1-z|} \ge \log \frac{1}{1 - r + |\theta|} \ge \log \frac{5}{3} > \frac{1}{2}, \qquad z \in E. \end{aligned} \tag{2.4}$$

Assuming that $\dfrac{9}{10} < r < 1$ everywhere below and $\alpha > 0$, we begin with the proof of the first estimate in (2.1).

By the estimates (2.3) and (2.4), we obtain

$$\begin{aligned} J_{\alpha,\beta} &= \left(\int_{|\theta|>1/2} + \int_{|\theta|<1/2}\right) \frac{d\theta}{\left|1 - re^{i\theta}\right|^{\alpha+1} \left|\log \frac{e}{1-re^{i\theta}}\right|^{\beta}} \\ &\approx C(\alpha,\beta) + C(\alpha,\beta) \int_0^{1/2} \frac{d\theta}{(1 - r + \theta)^{\alpha+1} \left(\log \frac{1}{1-r+\theta}\right)^{\beta}} \\ &= C(\alpha,\beta) + C(\alpha,\beta) \int_{\log \frac{1}{3/2-r}}^{\log \frac{1}{1-r}} \frac{e^{\alpha t}}{t^{\beta}}\, dt \approx \int_{1}^{\log \frac{1}{1-r}} \frac{e^{\alpha t}}{t^{\beta}}\, dt. \end{aligned} \tag{2.5}$$

Here we have used the inequalities

$$0<\log\frac{5}{3}<\log\frac{1}{3/2-r}<\log 2,\qquad \frac{9}{10}<r<1.$$

Since by l'Hôspital rule,

$$\int_1^x \frac{e^{\alpha t}}{t^\beta}dt \sim \frac{e^x}{\alpha\, x^\beta}\qquad \text{as}\qquad x\to+\infty\quad (\alpha>0),$$

we conclude that

$$J_{\alpha,\beta}\approx \frac{e^{\alpha\log\frac{1}{1-r}}}{\left(\log\frac{1}{1-r}\right)^\beta}=\frac{1}{(1-r)^\alpha\left(\log\frac{1}{1-r}\right)^\beta}$$

for all r sufficiently close to 1. It proves the first inequality in (2.1). The second inequality in (2.1) when $\alpha<0$ follows from (2.5).

We now turn to the proof of (2.2) when $\alpha=0$.

Case $\beta<1$**.** Making use of the estimates (2.3) and (2.4), we deduce that

$$J_{0,\beta}=\int_{|\theta|>1/2}+\int_{|\theta|<1/2}\approx C_\beta+C_\beta\int_0^{1/2}\frac{d\theta}{(1-r+\theta)\left(\log\frac{1}{1-r+\theta}\right)^\beta}$$

$$=C_\beta+C_\beta\left[\left(\log\frac{1}{1-r}\right)^{1-\beta}-\left(\log\frac{1}{3/2-r}\right)^{1-\beta}\right]$$

$$\approx\left(\log\frac{1}{1-r}\right)^{1-\beta}, \tag{2.6}$$

where we have used the inequalities

$$0<\left(\log\frac{5}{3}\right)^{1-\beta}<\left(\log\frac{1}{3/2-r}\right)^{1-\beta}<(\log 2)^{1-\beta}<\left(\frac{1}{2}\log\frac{1}{1-r}\right)^{1-\beta}$$

for all $\frac{9}{10}<r<1$.

Case $\beta=1$**.** In view of (2.3) and (2.4), we obtain for all r close enough to 1

$$J_{0,1}\approx C+C\int_0^{1/2}\frac{d\theta}{(1-r+\theta)\left(\log\frac{1}{1-r+\theta}\right)}$$

$$=C+C\left[\log\left(\log\frac{1}{1-r}\right)-\log\left(\log\frac{1}{3/2-r}\right)\right]$$

$$\approx\log\left(\log\frac{1}{1-r}\right), \tag{2.7}$$

where

$$\log\log\frac{5}{3}<\log\log\frac{1}{3/2-r}<\log\log 2<0,\qquad \frac{9}{10}<r<1.$$

Case $\beta>1$**.** Similarly to (2.5), we have

$$J_{0,\beta}\approx C_\beta+C\int_{\log\frac{1}{3/2-r}}^{\log\frac{1}{1-r}}\frac{1}{t^\beta}\,dt\approx C_\beta+C\int_1^{\log\frac{1}{1-r}}\frac{1}{t^\beta}\,dt\approx 1. \tag{2.8}$$

Combining (2.6)–(2.8), we obtain (2.2). This completes the proof. ∎

3. AN APPLICATION IN MIXED NORM SPACES

Define the following test function

$$F_{b,c}(z) := \big(1-z\big)^{-b}\left(\log\frac{e}{1-z}\right)^{-c}, \qquad z \in \mathbb{D},$$

where $b, c \in \mathbb{R}$. The functions $F_{b,c}$ are very useful as typical functions in many function spaces, see, for example, [2]-[7]. The next lemma gives exact information on $F_{b,c}$ to be in $H(p,q,\alpha)$ or $H_0(p,\infty,\alpha)$.

Lemma 2. *Suppose that* $b,\ c \in \mathbb{R},\ 0 < p \le \infty, 0 < q < \infty, \alpha > 0$. *Then*
(a) $F_{b,c}$ *is in* $H(p,q,\alpha)$ *if and only if* $b < \alpha + \frac{1}{p}, c \in \mathbb{R}$ *or* $b = \alpha + \frac{1}{p}, c > \frac{1}{q}$.
(b) $F_{b,c}$ *is in* $H(p,\infty,\alpha)$ *if and only if* $b < \alpha + \frac{1}{p}, c \in \mathbb{R}$ *or* $b = \alpha + \frac{1}{p}, c \ge 0$.
(c) $F_{b,c}$ *is in* $H_0(p,\infty,\alpha)$ *if and only if* $b < \alpha + \frac{1}{p}, c \in \mathbb{R}$ *or* $b = \alpha + \frac{1}{p}, c > 0$.

Proof. The results follow from corresponding estimates of Lemma 1,

$$M_p(F_{b,c}; r) \approx (1-r)^{-b+1/p}\left(\log\frac{e}{1-r}\right)^{-c}, \qquad 0 \le r < 1,$$

if $1/p < b \le \alpha + 1/p$. ∎

Lemma 2 enables us to prove the sharpness and strictness of the inclusions (i)-(vii) in Theorem 1.

Theorem 2. *Suppose that* $0 < p, q, p_0, q_0 \le \infty,\ \alpha, \beta > 0$ *are arbitrary. Then:*

(i) $H(p,q,\alpha) \subset H(p,q,\beta),\ \beta > \alpha,$ *is strict*.
(ii) $H(p,q,\alpha) \subset H(p_0,q,\alpha),\ p_0 < p,$ *is strict*.
(iii) $H(p,q,\alpha) \subset H(p,q_0,\alpha),\ q < q_0,$ *is strict, and the inclusion* $H(p,q,\alpha) \subset H_0(p,\infty,\alpha)$ *is sharp in the sense that* α *on the right cannot be decreased*.
(iv) $H(p,q,\alpha) \subset H(p_0,q,\beta),\ p \le p_0,$ *holds if and only if* $\beta \ge \alpha + \frac{1}{p} - \frac{1}{p_0}$.
(v) $H(p,q,\alpha) \subset H(\infty,q_0,\beta),\ \beta > \alpha + 1/p,\ q_0 < q,$ *is strict and sharp in the sense that* β *cannot be decreased*.
(vi) $H(p,q,\alpha) \subset H(p,q_0,\beta),\ \beta > \alpha,\ q_0 < q,$ *is strict and sharp in the sense that* β *cannot be decreased*.
(vii) $H^p \subset H(p_0,q,1/p-1/p_0),\ p < p_0, p \le q,$ *is sharp in the sense that it fails for* $p > q$.

Proof. (i) The inclusion (i) is strict because of the function $F_{\alpha+1/p,0}$ for $q < \infty$, and the function $F_{\beta+1/p,0}$ for $q = \infty$.

(ii) The strictness of the inclusion (ii) is proved by the examples $F_{\alpha+1/p,0}$ for $0 < q < \infty$, and $F_{\alpha+1/p_0,0}$ for $q = \infty$.

(iii) The strictness of the inclusion (iii) is proved by the examples $F_{\alpha+1/p,0}$ for $q_0 = \infty$, and $F_{\alpha+1/p,1/q}$ for $q_0 < \infty$. The sharpness of the second inclusion in (iii) means that the inclusion $H(p,q,\alpha) \subset H_0(p,\infty,\alpha-\varepsilon)$ is false for any $0 < p \le \infty, 0 < q < \infty, 0 < \varepsilon < \alpha$. The function $F_{\alpha+1/p-\varepsilon/2,0}(z)$ gives a corresponding example.

(iv) The statement (iv) is proved in [1, p.733].

(v)-(vi) The inclusions (v) and (vi) are strict because of the example $F_{\alpha+1/p,0}$. On the other hand, the inclusions (v) and (vi) are sharp for $q_0 < q$ in the sense that β cannot be decreased. The function $F_{\alpha+1/p,1/q_0}(z)$ gives a suitable example for both inclusions.

(vii) The inclusion (vii) is sharp in the sense that the condition $p \leq q$ is essential, that is for $p > q$ the inclusion (vii) fails. A corresponding example can be provided by the function $F_{1/p,\lambda}(z)$, where $1/p < \lambda < 1/q$. Indeed, $F_{1/p,\lambda}(z) \in H^p$, but $F_{1/p,\lambda}(z)$ is not in $H(p_0, q, 1/p - 1/p_0)$, by Lemma 2. ∎

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