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NECESSARY AND SUFFICIENT CONDITIONS FOR CYCLIC HOMOGENEOUS POLYNOMIAL INEQUALITIES OF DEGREE FOUR IN REAL VARIABLES

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ABSTRACT. In this paper, we give two sets of necessary and sufficient conditions that the inequality $f_4(x,y,z) \geq 0$ holds for any real numbers x, y, z, where $f_4(x,y,z)$ is a cyclic homogeneous polynomial of degree four. In addition, all equality cases of this inequality are analysed. For the particular case in which $f_4(1,1,1) = 0$, we get the main result in [3]. Several applications are given to show the effectiveness of the proposed methods.

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1. INTRODUCTION

Consider the fourth degree cyclic homogeneous polynomial

$$f_4(x,y,z) = \sum x^4 + A\sum x^2y^2 + Bxyz\sum x + C\sum x^3y + D\sum xy^3, \tag{1.1}$$

where A, B, C, D are real constants, and $\sum$ denotes a cyclic sum over x, y and z.

The following theorem expresses the necessary and sufficient condition that the inequality $f_4(x,y,z) \ge 0$ holds for any real numbers x, y, z in the particular case when $f_4(1,1,1) = 0$ (see [3] and [4]):

Theorem 1.1. *If*

$$1 + A + B + C + D = 0,$$

then the cyclic inequality $f_4(x,y,z) \ge 0$ *holds for all real numbers* x, y, z *if and only if*

$$3(1+A) \ge C^2 + CD + D^2.$$

The corollary below gives only sufficient conditions to have $f_4(x,y,z) \ge 0$ for any real numbers x, y, z (see [3]):

Corollary 1.2. *If*

$$1 + A + B + C + D \ge 0$$

and

$$2(1+A) \ge B + C + D + C^2 + CD + D^2,$$

then the cyclic inequality $f_4(x,y,z) \ge 0$ *holds for all real numbers* x, y, z.

In this paper, we generalize the results in Theorem 1.1 to the case where

$$1 + A + B + C + D \ge 0,$$

which is equivalent to the necessary condition $f_4(1,1,1) \ge 0$.

2. MAIN RESULTS

We establish two theorems which give necessary and sufficient conditions to have

$$f_4(x,y,z) \ge 0$$

for any real numbers x, y, z, where $f_4(x,y,z)$ is a fourth degree cyclic homogeneous polynomial having the form (1.1).

Theorem 2.1. *The inequality*

$$f_4(x,y,z) \ge 0$$

holds for all real numbers x, y, z *if and only if*

$$f_4(t+k, k+1, kt+1) \ge 0$$

for all real t, where $k \in [0,1]$ *is a root of the polynomial*

$$f(k) = (C-D)k^3 + (2A - B - C + 2D - 4)k^2 - (2A - B + 2C - D - 4)k + C - D.$$

Remark 2.1. For $C = D$, the polynomial $f(k)$ has the roots 0 and 1, while for $C \ne D$, $f(k)$ has three real roots, but only one in $[0,1]$. To prove this assertion, we see that $f(0) = -f(1) = C - D$. If $C > D$, then

$$f(-\infty) = -\infty, \quad f(0) > 0, \quad f(1) < 0, \quad f(\infty) = \infty,$$

and if $C < D$, then

$$f(-\infty) = \infty, \quad f(0) < 0, \quad f(1) > 0, \quad f(\infty) = -\infty.$$

From the proof of Theorem 2.1, we get immediately the equality cases of the inequality $f_4(x,y,z) \geq 0$.

Proposition 2.2. *The inequality $f_4(x,y,z) \geq 0$ in Theorem 2.1 becomes an equality if*

$$\frac{x}{t+k} = \frac{y}{k+1} = \frac{z}{kt+1}$$

(or any cyclic permutation), where $k \in (0,1]$ is a root of the equation

$$(C-D)k^3 + (2A-B-C+2D-4)k^2 - (2A-B+2C-D-4)k + C - D = 0$$

and $t \in \mathbb{R}$ is a root of the equation

$$f_4(t+k, k+1, kt+1) = 0.$$

Theorem 2.3. *The inequality*

$$f_4(x,y,z) \geq 0$$

holds for all real numbers x, y, z if and only if $g_4(t) \geq 0$ for all $t \geq 0$, where

$$g_4(t) = 3(2+A-C-D)t^4 - Ft^3 + 3(4-B+C+D)t^2 + 1 + A + B + C + D,$$

$$F = \sqrt{27(C-D)^2 + E^2}, \quad E = 8 - 4A + 2B - C - D.$$

Remark 2.2. In the special case $f_4(1,1,1) = 0$, when

$$1 + A + B + C + D = 0,$$

from Theorem 2.3 we get Theorem 1.1. The condition $g_4(t) \geq 0$ in Theorem 2.3 becomes

$$(2+A-C-D)t^4 + (5+A+2C+2D)t^2 \geq \sqrt{(2-2A-C-D)^2 + 3(C-D)^2}\, t^3,$$

and it holds for all $t \geq 0$ if and only if

$$2 + A - C - D \geq 0,$$

$$5 + A + 2C + 2D) \geq 0,$$

$$2\sqrt{(2+A-C-D)(5+A+2C+2D)} \geq \sqrt{(2-2A-C-D)^2 + 3(C-D)^2}.$$

The last inequality is equivalent to

$$3(1+A) \geq C^2 + D^2 + CD,$$

which involves

$$\begin{aligned} 2 + A - C - D &\geq 1 - (C+D) + \frac{(C+D)^2}{3} - \frac{CD}{3} \\ &\geq 1 - (C+D) + \frac{(C+D)^2}{3} - \frac{(C+D)^2}{12} = \left(1 - \frac{C+D}{2}\right)^2 \geq 0 \end{aligned}$$

and

$$\begin{aligned} 5 + A + 2C + 2D &\geq 4 + 2(C+D) + \frac{(C+D)^2}{3} - \frac{CD}{3} \\ &\geq 4 + 2(C+D) + \frac{(C+D)^2}{3} - \frac{(C+D)^2}{12} = \left(2 + \frac{C+D}{2}\right)^2 \geq 0. \end{aligned}$$

Thus, we obtained the necessary and sufficient condition in Theorem 1.1, namely

$$3(1+A) \geq C^2 + CD + D^2.$$

The following proposition gives the equality cases of the inequality $f_4(x,y,z) \geq 0$ for $F = 0$.

Proposition 2.4. *For $F = 0$, assume that the inequality $f_4(x, y, z) \geq 0$ in Theorem 2.3 becomes an equality for at least a real triple $(x, y, z) \neq (0, 0, 0)$. Then, the inequality $f_4(x, y, z) \geq 0$ in Theorem 2.3 has the following three possible forms:*

$$(x + y + z)^2[x^2 + y^2 + z^2 + k(xy + yz + zx)] \geq 0, \quad k \in [-1, 2],$$

or

$$[x^2 + y^2 + z^2 + k(xy + yz + zx)]^2 \geq 0, \quad k \in (-1, 2),$$

or

$$(x^2 + y^2 + z^2 - xy - yz - zx)[x^2 + y^2 + z^2 + k(xy + yz + zx)] \geq 0, \quad k \in [-1, 2).$$

The following proposition gives the equality cases of the inequality $f_4(x, y, z) \geq 0$ for $F > 0$.

Proposition 2.5. *For $F > 0$, the inequality $f_4(x, y, z) \geq 0$ in Theorem 2.3 becomes an equality if and only if x, y, z satisfy*

$$(C - D)(x + y + z)(x - y)(y - z)(z - x) \geq 0$$

and are proportional to the roots w_1, w_2 and w_3 of the polynomial equation

$$w^3 - 3w^2 + 3(1 - \alpha^2)w + \frac{2E}{F}\alpha^3 + 3\alpha^2 - 1 = 0,$$

where α is any double nonnegative real root of the polynomial $g_4(t)$.

Remark 2.3. The polynomial

$$f(w) = w^3 - 3w^2 + 3(1 - \alpha^2)w + \frac{2E}{F}\alpha^3 + 3\alpha^2 - 1$$

in Proposition 2.5 has three real roots for any given $\alpha \geq 0$. This is true if $f(w_1') \geq 0$ and $f(w_2') \leq 0$, where $w_1' = 1 - \alpha$ and $w_2' = 1 + \alpha$ are the roots of the derivative $f'(w)$. Indeed, we have

$$f(w_1') = 2\left(1 + \frac{E}{F}\right)\alpha^3 \geq 0,$$

$$f(w_2') = -2\left(1 - \frac{E}{F}\right)\alpha^3 \leq 0.$$

Thus, for $F > 0$, the number of distinct non-zero triples (x, y, z) which satisfy $f_4(x, y, z) = 0$ is equal to the number of distinct nonnegative roots of the polynomial $g_4(t)$. Since this number is less than or equal to 2, the equality $f_4(x, y, z) = 0$ holds for $x = y = z = 0$ and for at most two distinct triples (x, y, z).

In the special case $f_4(1, 1, 1) = 0$, when $1 + A + B + C + D = 0$, from Theorem 2.3 and Remark 2.2 it follows that $3(1 + A) = C^2 + CD + D^2$ is a necessary condition to have $f_4(x, y, z) \geq 0$ for all real x, y, z, with equality for at least a real triple (x, y, z) with $x \neq y$ or $y \neq z$ or $z \neq x$. Thus, by Proposition 2.5 we get the following corollary.

Corollary 2.6. *Let $f_4(x, y, z)$ be a fourth degree cyclic homogeneous polynomial such that $f_4(1, 1, 1) = 0$ and $f_4(x, y, z) \geq 0$ for all real numbers x, y, z. Let us denote*

$$E = 12 - 3(C + D) - 2(C^2 + CD + D^2), \quad F = \sqrt{27(C - D)^2 + E^2},$$

$$\alpha = \sqrt{\frac{3(C + D + 4)^2 + (C - D)^2}{3(C + D - 2)^2 + (C - D)^2}}.$$

For $F > 0$, the inequality $f_4(x, y, z) \geq 0$ becomes an equality when $x = y = z$, and also when x, y, z satisfy

$$(C - D)(x + y + z)(x - y)(y - z)(z - x) \geq 0$$

and are proportional to the roots w_1, w_2 and w_3 of the polynomial equation

$$w^3 - 3w^2 + 3(1-\alpha^2)w + \frac{2E}{F}\alpha^3 + 3\alpha^2 - 1 = 0.$$

A new special case is the one in which $C = D$, when the homogeneous polynomial $f_4(x, y, z)$ is symmetric. Since

$$F = |E| = 2|4 - 2A + B - C|,$$

the polynomial

$$f(w) = w^3 - 3w^2 + 3(1-\alpha^2)w + \frac{2E}{F}\alpha^3 + 3\alpha^2 - 1$$

in Proposition 2.5 becomes either

$$f(w) = w^3 - 3w^2 + 3(1-\alpha^2)w + 2\alpha^3 + 3\alpha^2 - 1 = (w - \alpha - 1)^2(w + 2\alpha - 1),$$

or

$$f(w) = w^3 - 3w^2 + 3(1-\alpha^2)w - 2\alpha^3 + 3\alpha^2 - 1 = (w + \alpha - 1)^2(w - 2\alpha - 1).$$

In both cases, two of the real roots w_1, w_2 and w_3 are equal. Setting $y = z = 1$, the equation $f_4(x, y, z) = 0$ becomes

$$x^4 + 2Cx^3 + (2A + B)x^2 + 2(B + C)x + A + 2C + 2 = 0.$$

So, the following corollary holds.

Corollary 2.7. *Let*

$$f_4(x, y, z) = \sum x^4 + A\sum x^2y^2 + Bxyz\sum x + C\sum xy(x^2 + y^2)$$

be a fourth degree symmetric homogeneous polynomial such that $4 - 2A + B - C \neq 0$ and $f_4(x, y, z) \geq 0$ for all real numbers x, y, z. The inequality $f_4(x, y, z) \geq 0$ becomes an equality when $x/w = y = z$ (or any cyclic permutation), where w is a double real root of the equation

$$w^4 + 2Cw^3 + (2A + B)w^2 + 2(B + C)w + A + 2C + 2 = 0.$$

With regard to the distinct nonnegative roots of the polynomial $g_4(t)$, the following statement holds.

Proposition 2.8. *Assume that $F > 0$ and $g_4(t) \geq 0$ for all $t \geq 0$. The polynomial $g_4(t)$ in Theorem 2.3 has the following nonnegative real roots:*

(i) two pairs of nonnegative roots, namely

$$t_1 = t_2 = 0, \quad t_3 = t_4 \geq 0,$$

if and only if

$$1 + A + B + C + D = 0, \quad 3(1 + A) = C^2 + CD + D^2;$$

(ii) only one pair of zero roots,

$$t_1 = t_2 = 0,$$

if and only if

$$1 + A + B + C + D = 0, \quad 3(1 + A) > C^2 + CD + D^2;$$

(iii) only one pair of positive roots,

$$t_1 = t_2 > 0,$$

if and only if

$$a = \frac{2\sqrt{2}(2b + \sqrt{b^2 + 12c})}{3\sqrt{b + \sqrt{b^2 + 12c}}},$$

where

$$a = \frac{F}{3(2+A-C-D)} \geq 0, \quad b = \frac{4-B+C+D}{2+A-C-D}, \quad c = \frac{1+A+B+C+D}{3(2+A-C-D)} > 0.$$

Remark 2.4. It is much easier to make a thorough study of a cyclic homogeneous polynomial inequality of degree four $f_4(x,y,z) \geq 0$ by applying Theorem 2.3 than by applying Theorem 2.1, especially in the case where $f_4(1,1,1) \neq 0$. For this reason, Theorem 2.1 is more useful for the study of the inequality $f_4(x,y,z) \geq 0$ by means of a computer. For example, let us prove by both Theorems 2.1 and 2.3 the well known inequality ([1], [2])

$$(x^2+y^2+z^2)^2 \geq 3(x^3y+y^3z+z^3x), \quad x,y,z \in \mathbb{R}.$$

We have

$$f_4(x,y,z) = (x^2+y^2+z^2)^2 - 3(x^3y+y^3z+z^3x);$$

that is,

$$A = 2, \;\; B = 0, \;\; C = -3, \;\; D = 0.$$

According to Theorem 2.1, we need to show that $f_4(t+k, k+1, kt+1) \geq 0$ for all real t, where $k \approx 0.445042$ satisfies the equation

$$k^3 - k^2 - 2k + 1 = 0.$$

After many calculation, we get

$$\begin{aligned} f_4(t+k,k+1,kt+1) &= (t-1)^2[(1-k)(3-2k)t^2 + 2(1-k)(3k-1)t + 2 - k - 8k^2] \\ &= (1-k)(3-2k)(t-1)^2\left(t + \frac{3k-1}{3-2k}\right)^2 \geq 0. \end{aligned}$$

By Proposition 2.2, equality holds for

$$\frac{x}{t+k} = \frac{y}{k+1} = \frac{z}{kt+1}$$

(or any cyclic permutation), where $t \in \left\{1, \frac{1-3k}{3-2k}\right\}$; that is, for $x = y = z$, and also for

$$\frac{x}{1-2k^2} = \frac{y}{(1+k)(3-2k)} = \frac{z}{3-k-3k^2}$$

(or any cyclic permutation).

According to Theorem 2.3, we need to show that $g_4(t) \geq 0$ for all $t \geq 0$. Indeed, we have $E = 3$, $F = 6\sqrt{7}$, and hence

$$g_4(t) = 3t^2(\sqrt{7}\,t - 1)^2 \geq 0.$$

Since $f_4(1,1,1) = 0$, we apply Corollary 2.6 to find the other equality cases. We get $\alpha = 1/\sqrt{7}$, and the equality conditions

$$(x+y+z)(x-y)(y-z)(z-x) \leq 0$$

and

$$w^3 - 3w^2 + \frac{18}{7}w - \frac{27}{49} = 0,$$

which lead to the equality case

$$\frac{x}{\sin^2 \frac{4\pi}{7}} = \frac{y}{\sin^2 \frac{2\pi}{7}} = \frac{z}{\sin^2 \frac{\pi}{7}}$$

(or any cyclic permutation).

3. PROOF OF THEOREM 2.1

The main idea is to use the linear cyclic substitution

$$x = a + kb, \quad y = b + kc, \quad z = c + ka,$$

in order to convert the cyclic polynomial $f_4(x, y, z)$ to a fourth degree symmetric homogeneous polynomial

$$h_4(a, b, c) = f_4(a + kb, b + kc, c + ka).$$

If this is possible for a real constant $k \in [0, 1]$, then the inequality $f_4(x, y, z) \geq 0$ holds for all real numbers x, y, z if and only if the inequality $h_4(a, b, c) \geq 0$ holds for all real numbers a, b, c. According to Lemma 3.1 below, the inequality $h_4(a, b, c) \geq 0$ holds for all real a, b, c if and only if $h_4(t, 1, 1) \geq 0$ for all real t; that is, if and only if

$$h_4(t, 1, 1) = f_4(t + k, 1 + k, 1 + kt)$$

for all real t. So, we only need to show that the polynomial $h_4(a, b, c)$ is symmetric if k is a real root of the polynomial $f(k)$.

For $C = D$ and $k = 0$, the polynomial $h_4(a, b, c)$ is clearly symmetric. Consider now that $C \neq D$. It is easy to show that the expressions $\sum x^4$, $\sum x^2y^2$, $xyz \sum x$, $\sum x^3y$ and $\sum xy^3$ contain respectively the following cyclic expressions $\sum a^3b$ and $\sum ab^3$:

$$\begin{aligned}
\sum x^4 &: \quad 4k \sum a^3b + 4k^3 \sum ab^3, \\
\sum x^2y^2 &: \quad 2k^3 \sum a^3b + 2k \sum ab^3, \\
xyz \sum x &: \quad (k^2 + k) \sum a^3b + (k^3 + k^2) \sum ab^3, \\
\sum x^3y &: \quad (k^4 + 1) \sum a^3b + (3k^2 + k) \sum ab^3, \\
\sum xy^3 &: \quad (k^3 + 3k^2) \sum a^3b + (k^4 + 1) \sum ab^3.
\end{aligned}$$

Therefore, $h_4(a, b, c)$ contains the expression

$$E \sum a^3b + F \sum ab^3,$$

where

$$E = 4k + 2Ak^3 + B(k^2 + k) + C(k^4 + 1) + D(k^3 + 3k^2),$$
$$F = 4k^3 + 2Ak + B(k^3 + k^2) + C(3k^2 + k) + D(k^4 + 1).$$

Obviously, if $E = F$, then $h_4(a, b, c)$ is a symmetric homogeneous polynomial. From

$$E - F = (C - D)k^4 + (2A - B + D - 4)k^3 - 3(C - D)k^2$$
$$-(2A - B + C - 4)k + C - D = (k + 1)f(k),$$

it follows that $f(k) = 0$ involves $E = F$.

To complete the proof, we still need to show that the equation $f(k) = 0$ has at least a root in $[0, 1]$. This is true since $f(k)$ is a continuous function and $f(0) = -f(1) = C - D \neq 0$.

Lemma 3.1. *Let $h_4(a, b, c)$ be a fourth degree symmetric homogeneous polynomial. The inequality*

$$h_4(a, b, c) \geq 0$$

holds for all real numbers a, b, c if and only if $h_4(t, 1, 1) \geq 0$ for all real t.

Proof. Let $p = a + b + c$, $q = ab + bc + ca$ and $r = abc$. For fixed p and q, from the known relation

$$27(a-b)^2(b-c)^2(c-a)^2 = 4(p^2-3q)^3 - (2p^3 - 9pq + 27r)^2,$$

it follows that r is maximal and minimal when two of a, b, c are equal. On the other hand, for fixed p and q, the inequality $h_4(a, b, c) \geq 0$ can be written as $g(r) \geq 0$, where $g(r)$ is a linear function. Therefore, $g(r)$ is minimal when r is minimal or maximal; that is, when two of a, b, c are equal. Since the polynomial $h_4(a, b, c)$ is symmetric, homogeneous and satisfies $h_4(-a, -b, -c) = h_4(a, b, c)$, $g(r)$ is minimal if and only if $h_4(t, 1, 1) \geq 0$ and $h_4(t, 0, 0) \geq 0$ for all real t. To complete the proof, it suffices to show that if $h_4(t, 1, 1) \geq 0$ for all real t, then $h_4(t, 0, 0) \geq 0$ for all real t. Indeed, since $h_4(a, b, c)$ has the general form

$$h_4(a, b, c) = A_0 \sum a^4 + A_1 \sum ab(a^2 + b^2) + A_2 \sum a^2b^2 + A_3abc \sum a,$$

the condition $h_4(t, 1, 1) \geq 0$ for all real t involves $A_0 \geq 0$, and hence $h_4(t, 0, 0) = A_0t^4 \geq 0$ for all real t. ∎

4. PROOF OF THEOREM 2.3

Using the substitutions

$$p = x + y + z, \quad q = xy + yz + zx, \quad r = abc,$$

we have

$$xyz \sum x = pr, \quad \sum x^2y^2 = q^2 - 2pr,$$

$$\sum x^4 = (\sum x^2)^2 - 2\sum x^2y^2 = (p^2 - 2q)^2 - 2(q^2 - 2pr) = p^4 - 4p^2q + 2q^2 + 4pr,$$

$$\sum x^3y + \sum xy^3 = (\sum xy)(\sum x^2) - xyz \sum x = q(p^2 - 2q) - pr,$$

$$\sum x^3y - \sum xy^3 = p(x-y)(y-z)(z-x),$$

$$27(x-y)^2(y-z)^2(z-x)^2 = 4(p^2-3q)^3 - (2p^3 - 9pq + 27r)^2.$$

Further, we need Lemma 4.1, Lemma 4.2 and Lemma 4.3 below. By Lemma 4.1, the inequality $f_4(x, y, z) \geq 0$ holds if and only if

$$S_4(x, y, z) \geq |(C-D)(x+y+z)(x-y)(y-z)(z-x)| \tag{4.1}$$

for all real x, y, z.

Sufficiency. Consider the following two cases: $p = 0$ and $p \neq 0$.

Case 1: $p = 0$. Since $\sum x^4 = 2q^2$, $\sum x^2y^2 = q^2$ and $\sum x^3y + \sum xy^3 = -2q^2$, the desired inequality (4.1) becomes

$$(2 + A - C - D)q^2 \geq 0.$$

This is true since the hypothesis $g_4(t) \geq 0$ for all $t \geq 0$ involves $2 + A - C - D \geq 0$.

Case 2: $p \neq 0$. Due to homogeneity, we may set $p = 1$, which involves $q \leq 1/3$. Since

$$|(x-y)(y-z)(z-x)| = \sqrt{(x-y)^2(y-z)^2(z-x)^2} = \sqrt{\frac{4(1-3q)^3 - (2 - 9q + 27r)^2}{27}},$$

(4.1) becomes

$$2 - (8 - C - D)q + 2(2 + A - C - D)q^2 + (8 - 4A + 2B - C - D)r \geq$$
$$\geq \frac{|C-D|}{3\sqrt{3}}\sqrt{4(1-3q)^3 - (2 - 9q + 27r)^2}.$$

Substituting $t = \sqrt{1-3q}$, which implies $q = (1-t^2)/3$, $t \geq 0$, the inequality turns into

$$2(2+A-C-D)t^4 + (16-4A+C+D)t^2 - 2 + 2A + C + D + 9Er \geq$$
$$\geq \sqrt{3}|C-D|\sqrt{4t^6 - (3t^2-1+27r)^2},$$

where

$$E = 8 - 4A + 2B - C - D.$$

Applying Lemma 4.2 for

$$\alpha = \sqrt{3}|C-D|, \;\; \beta = \frac{E}{3}, \;\; a = 2t^3, \;\; b = 3t^2 - 1 + 27r, \tag{4.2}$$

we get

$$\sqrt{3}|C-D|\sqrt{4t^6 - (3t^2-1+27r)^2} \leq \frac{2Ft^3}{3} + \frac{E(3t^2-1+27r)}{3}.$$

Thus, we only need to prove that

$$2(2+A-C-D)t^4 + (16-4A+C+D)t^2 - 2 + 2A + C + D + 9Er \geq$$
$$\geq \frac{2Ft^3}{3} + \frac{E(3t^2-1+27r)}{3},$$

which is just the hypothesis $g_4(t) \geq 0$.

Necessity. We need to prove that if (4.1) holds for all real x, y, z, then $g_4(t) \geq 0$ for all $t \geq 0$. Actually, it suffices to consider that (4.1) holds for all real x, y, z such that $p = x + y + z = 1$. As we have shown above, the inequality (4.1) for $p = 1$ has the form

$$2(2+A-C-D)t^4 + (16-4A+C+D)t^2 - 2 + 2A + C + D + 9Er \geq$$
$$\geq \sqrt{3}|C-D|\sqrt{4t^6 - (3t^2-1+27r)^2},$$

where

$$E = 8 - 4A + 2B - C - D.$$

Choosing the triple (x, y, z) as in Lemma 4.3, we get

$$2(2+A-C-D)t^4 + (16-4A+C+D)t^2 - 2 + 2A + C + D + 9Er \geq$$
$$\geq \frac{2Ft^3}{3} + \frac{E(3t^2-1+27r)}{3},$$

which is equivalent to $g_4(t) \geq 0$.

Lemma 4.1. *The inequality $f_4(x, y, z) \geq 0$ holds for all real x, y, z if and only if the inequality*

$$S_4(x, y, z) \geq |(C-D)(x+y+z)(x-y)(y-z)(z-x)|$$

holds for all real x, y, z, where

$$S_4(x, y, z) = 2\sum x^4 + 2A\sum x^2y^2 + 2Bxyz\sum x + (C+D)(\sum x^3y + \sum xy^3).$$

Proof. It is easy to show that

$$2f_4(x, y, z) = S_4(x, y, z) + (C-D)(\sum x^3y - \sum xy^3)$$
$$= S_4(x, y, z) - (C-D)(x+y+z)(x-y)(y-z)(z-x).$$

Sufficiency. According to the hypothesis

$$S_4(x, y, z) \geq |(C-D)(x+y+z)(x-y)(y-z)(z-x)|,$$

we have

$$2f_4(x, y, z) \geq |(C-D)(x+y+z)(x-y)(y-z)(z-x)|$$

$$-(C-D)(x+y+z)(x-y)(y-z)(z-x) \geq 0.$$

Necessity. Since

$$2f_4(x,y,z) = S_4(x,y,z) - (C-D)(x+y+z)(x-y)(y-z)(z-x),$$

from the hypothesis $f_4(x,y,z) \geq 0$, we get

$$S_4(x,y,z) \geq (C-D)(x+y+z)(x-y)(y-z)(z-x).$$

On the other hand, if $f_4(x,y,z) \geq 0$ for all real x,y,z, then also $f_4(x,z,y) \geq 0$ for all real x,y,z. Since

$$2f_4(x,z,y) = S_4(x,y,z) + (C-D)(x+y+z)(x-y)(y-z)(z-x),$$

we get

$$S_4(x,y,z) \geq -(C-D)(x+y+z)(x-y)(y-z)(z-x)$$

for all real x,y,z. Therefore, we have

$$S_4(x,y,z) \geq |(C-D)(x+y+z)(x-y)(y-z)(z-x)|.$$

Lemma 4.2. *If α,β,a,b are real numbers, $\alpha \geq 0$, $a \geq 0$ and $a^2 \geq b^2$, then*

$$\alpha\sqrt{a^2-b^2} \leq a\sqrt{\alpha^2+\beta^2} + \beta b,$$

with equality if and only if

$$\beta a + b\sqrt{\alpha^2+\beta^2} = 0.$$

Proof. Since

$$a\sqrt{\alpha^2+\beta^2} + \beta b \geq |\beta|a + \beta b \geq |\beta||b| + \beta b \geq 0,$$

we can write the inequality as

$$\alpha^2(a^2-b^2) \leq (a\sqrt{\alpha^2+\beta^2} + \beta b)^2,$$

which is equivalent to the obvious inequality

$$(\beta a + b\sqrt{\alpha^2+\beta^2})^2 \geq 0.$$

Lemma 4.3. *Let A,B,C,D,E,F be given real constants such that*

$$E = 8 - 4A + 2B - C - D, \quad F = \sqrt{27(C-D)^2+E^2}.$$

For any given $t \geq 0$, there exists a real triple (x,y,z) such that

$$x+y+z = 1, \quad xy+yz+zx = (1-t^2)/3$$

and

$$\sqrt{3}|C-D|\sqrt{4t^6-(3t^2-1+27xyz)^2} = \frac{2Ft^3}{3} + \frac{E(3t^2-1+27xyz)}{3}.$$

Proof. Let $r = xyz$. From the last relation we get

$$\left[\sqrt{3}|C-D|\sqrt{4t^6-(3t^2-1+27r)^2} - \frac{E(3t^2-1+27r)}{3}\right]^2 = \left(\frac{2Ft^3}{3}\right)^2,$$

$$\left[\sqrt{3}|C-D|(3t^2-1+27r) + \frac{E}{3}\sqrt{4t^6-(3t^2-1+27r)^2}\right]^2 = 0,$$

that is $f(r) = 0$, where

$$f(r) = \sqrt{3}|C - D|(3t^2 - 1 + 27r) + \frac{E}{3}\sqrt{4t^6 - (3t^2 - 1 + 27r)^2}.$$

We need to prove that for any given $t \geq 0$ there exists a real triple (x, y, z) such that $x+y+z = 1$, $xy + yz + zx = (1 - t^2)/3$ and $f(r) = 0$. According to

$$27(x - y)^2(y - z)^2(z - x)^2 = 4t^6 - (3t^2 - 1 + 27r)^2 \geq 0,$$

this is true if $r \in [r_1, r_2]$, where

$$r_1 = \frac{1}{27}(1 - 3t^2 - 2t^3), \quad r_2 = \frac{1}{27}(1 - 3t^2 + 2t^3).$$

Therefore, we only need to show that the equation $f(r) = 0$ has a root in $[r_1, r_2]$. Indeed, from

$$f(r_1) = -2\sqrt{3}|C - D|t^3, \quad f(r_2) = 2\sqrt{3}|C - D|t^3, \quad f(r_1)f(r_2) \leq 0,$$

the desired conclusion follows.

5. Proof of Proposition 2.4

We first see that $F = 0$ involves

$$C = D = 4 - 2A + B$$

and

$$\frac{1}{3}g_4(t) = (5A - 2B - 6)t^4 + (12 - 4A + B)t^2 + 3 - A + B.$$

According to Theorem 2.3 and its proof in section 4, we have $f_4(x, y, z) \geq 0$ for all real x, y, z, with equality for at least a real triple $(x, y, z) \neq (0, 0, 0)$, only if $g_4(t) \geq 0$ for all $t \geq 0$ and $g_4(t) = 0$ for at least a nonnegative value of t. In our case, we have $g_4(t) \geq 0$ for all $t \geq 0$ only if $5A - 2B - 6 \geq 0$ and $3 - A + B \geq 0$. We need to consider three cases: $5A - 2B - 6 = 0$; $5A - 2B - 6 > 0$ and $3 - A + B > 0$; $5A - 2B - 6 > 0$ and $3 - A + B = 0$.

Case 1: $5A - 2B - 6 = 0$. We get

$$A = 2k + 2, \quad B = 5k + 2, \quad C = D = k + 2, \quad k \in \mathbb{R},$$

and hence

$$\begin{aligned} f_4(x, y, z) &= \sum x^4 + 2(k + 1)\sum x^2y^2 + (5k + 2)xyz\sum x + (k + 2)\sum xy(x^2 + y^2) \\ &= (x + y + z)^2[x^2 + y^2 + z^2 + k(xy + yz + zx)]. \end{aligned}$$

Clearly, the inequality $f_4(x, y, z) \geq 0$ holds for all real x, y, z if and only if $k \in [-1, 2]$. The same result follows from the condition $g_4(t) \geq 0$ for all $t \geq 0$, where

$$g_4(t) = 9(2 - k)t^2 + 9(1 + k).$$

Case 2: $5A - 2B - 6 > 0, 3 - A + B > 0$. We have $g_4(t) \geq 0$ for all $t \geq 0$ and also $g_4(t) = 0$ for at least a nonnegative value of t if and only if $12 - 4A + B < 0$ and

$$(12 - 4A + B)^2 = 4(5A - 2B - 6)(3 - A + B);$$

that is,

$$B = 2(A - 2 \pm \sqrt{A - 2}), \quad A \geq 2.$$

Putting $k = \pm\sqrt{A - 2}$, we get

$$A = k^2 + 2, \quad B = 2k(k + 1), \quad C = D = 2k,$$

and hence

$$f_4(x, y, z) = [x^2 + y^2 + z^2 + k(xy + yz + zx)]^2.$$

From $12 - 4A + B = 2(k+1)(2-k) < 0$, we get $k \in (-1, 2)$.
Case 3: $5A - 2B - 6 > 0$, $3 - A + B = 0$. We get

$$A = 2 - k, \quad B = -1 - k, \quad C = D = k - 1, \quad k < 2,$$

and hence

$$\begin{aligned} f_4(x,y,z) &= \sum x^4 + (2-k)\sum x^2y^2 - (1+k)xyz\sum x + (k-1)\sum xy(x^2+y^2) \\ &= (x^2+y^2+z^2-xy-yz-zx)[x^2+y^2+z^2+k(xy+yz+zx)]. \end{aligned}$$

The inequality $f_4(x,y,z) \geq 0$ holds for all real x,y,z if and only if $k \in [-1, 2)$. The same result follows from the condition $g_4(t) \geq 0$ for all $t \geq 0$, where

$$g_4(t) = 9t^2[(2-k)t^2 + 1 + k].$$

6. PROOF OF PROPOSITION 2.5

By the proof of Theorem 2.3 it follows that the main necessary condition to have $f_4(x,y,z) \geq 0$ for all real x,y,z and $f(x,y,z) = 0$ for at least a real triple $(x,y,z) \neq (0,0,0)$ is to have $g_4(t) \geq 0$ for all $t \geq 0$ and $g_4(t) = 0$ for at least a nonnegative value of t. Clearly, for $F > 0$, the inequality $g_4(t) \geq 0$ holds for all $t \geq 0$ only if $2 + A - C - D > 0$. We can find all equality cases of the inequality $f_4(x,y,z) \geq 0$ using the above proof of Theorem 2.3. Consider two cases: $x+y+z = 0$ and $x+y+z = 1$.

Case 1: $x+y+z = 0$. The inequality (4.1), which is equivalent to $f_4(x,y,z) \geq 0$, becomes

$$(2 + A - C - D)(xy+yz+zx)^2 \geq 0,$$

with equality for $x+y+z = 0$ and $xy+yz+zx = 0$; that is, for $x = y = z = 0$.

Case 2: $x+y+z = 1$. According to Lemma 4.1, a first necessary equality condition is

$$(C-D)(x+y+z)(x-y)(y-z)(z-x) \geq 0.$$

In addition, according to Lemma 4.2, it is necessary to have

$$\beta a + b\sqrt{\alpha^2 + \beta^2} = 0,$$

where α, β, a and b are given by (4.2). This condition is equivalent to

$$2Et^3 + F(3t^2 - 1 + 27xyz) = 0.$$

Since $x+y+z = 1$ and $xy+yz+zx = (1-t^2)/3$, where where t is any nonnegative root of the polynomial $g_4(t)$, the equality $f_4(x,y,z) = 0$ holds when

$$(C-D)(x+y+z)(x-y)(y-z)(z-x) \geq 0,$$

$$x+y+z = 1, \quad xy+yz+zx = \frac{1-t^2}{3}, \quad 27xyz = 1 - 3t^2 - \frac{2E}{F}t^3;$$

that is, when x,y,z are proportional to the roots of the equation

$$27w^3 - 27w^2 + 9(1-t^2)w + \frac{2E}{F}t^3 + 3t^2 - 1 = 0$$

and satisfy $(C-D)(x+y+z)(x-y)(y-z)(z-x) \geq 0$. Substituting $w/3$ for w, we get the desired equation

$$w^3 - 3w^2 + 3(1-t^2)w + \frac{2E}{F}t^3 + 3t^2 - 1 = 0.$$

7. PROOF OF PROPOSITION 2.8

Clearly, if $g_4(t) \geq$ for all $t \geq 0$, then $2 + A - C - D > 0$.

(i) If the polynomial $g_4(t)$ has four nonnegative real numbers $t_1 \leq t_2 \leq t_3 \leq t_4$, then the condition $g_4(t) \geq 0$ for all $t \geq 0$ holds if and only if

$$0 \leq t_1 = t_2 = a \leq b = t_3 = t_4,$$

when

$$g_4(t) = 3(2 + A - C - D)(t - a)^2(t - b)^2.$$

Since the coefficient of t is 0 in $g_4(t)$ and is $2ab(a+b)$ in $(t-a)^2(t-b)^2$, it follows that $a = 0$ and $b \geq 0$. From $g_4(0) = 0$, we get $1+A+B+C+D = 0$, which involves $3(1+A) = C^2+CD+D^2$ (see Remark 2.2).

Reversely, if $1 + A + B + C + D = 0$ and $3(1 + A) = C^2 + CD + D^2$, then

$$g_4(t) = 3(2 + A - C - D)t^2 \left[t - \frac{F}{6(2 + A - C - D)}\right]^2,$$

where $F \geq 0$.

(ii) The polynomial $g_4(t)$ has the double root 0 if and only if $1 + A + B + C + D = 0$, when

$$g_4(t) = t^2 g(t),$$

where

$$g(t) = 3(2 + A - C - D)t^2 - Ft + 3(4 - B + C + D).$$

Clearly, $g_4(t)$ has only two nonnegative roots (that are $t_1 = t_2 = 0$) when $g(t)$ has either negative real roots or complex roots. Since $F \geq 0$, $g(t)$ can not have negative roots, but can have complex roots, when the discriminant of the quadratic polynomial $g(t)$ is negative; that is,

$$3(1 + A) > C^2 + CD + D^2.$$

(iii) Write the inequality $g_4(t) \geq 0$ as $h(t) \geq 0$, where

$$h(t) = t^4 - at^3 + bt^2 + c.$$

In addition, writing $h(t)$ in the form

$$h(t) \equiv (t - t_0)^2(t^2 + pt + q), \quad t_0 > 0,$$

we find

$$2t_0 - p = a, \quad t_0^2 - 2pt_0 + q = b, \quad pt_0 - 2q = 0, \quad qt_0^2 = c.$$

From the last three relation, we get

$$2t_0^2 = b + \sqrt{b^2 + 12c},$$

$$6q = \sqrt{b^2 + 12c} - b,$$

$$p = \frac{\sqrt{2}(\sqrt{b^2 + 12c} - b)}{3\sqrt{b + \sqrt{b^2 + 12c}}}.$$

Since $p > 0$ and $q > 0$, the quadratic polynomial $t^2 + pt + q$ has no nonnegative real root. Substituting t_0, p and q in $2t_0 - p = a$, we get

$$a = 2t_0 - p = 2t_0 - \frac{2q}{t_0} = \frac{2t_0^2 - 2q}{t_0}$$
$$= \frac{2(2b + \sqrt{b^2 + 12c})}{3t_0} = \frac{2\sqrt{2}(2b + \sqrt{b^2 + 12c})}{3\sqrt{b + \sqrt{b^2 + 12c}}}.$$

8. APPLICATIONS OF THEOREM 2.3

Application 1. *If x, y, z are real numbers, then* ([5])

$$(x^2+y^2+z^2)^2 + \frac{8}{\sqrt{7}}(x^3y+y^3z+z^3x) \ge 0.$$

Proof. We have

$$A=2,\ \ B=0,\ \ C=8/\sqrt{7},\ \ D=0,\ \ E=-8/\sqrt{7},\ \ F=16,$$

and hence

$$\begin{aligned} g_4(t) &= 12\left(1-\frac{2}{\sqrt{7}}\right)t^4 - 16t^3 + 12\left(1+\frac{2}{\sqrt{7}}\right)t^2 + 3 + \frac{8}{\sqrt{7}} \\ &= \frac{2}{\sqrt{7}}\left(t-\frac{3+\sqrt{7}}{2}\right)^2\left[6(\sqrt{7}-2)t^2+2(3-\sqrt{7})t+1\right]. \end{aligned}$$

Since $g_4(t)\ge 0$ for all $t\ge 0$, the inequality is proved (Theorem 2.3).

To find all equality cases, we apply Proposition 2.5. We see that the polynomial $g_4(t)$ has only the nonnegative double root $\alpha=(3+\sqrt{7})/2$. Therefore, equality holds when x, y, z satisfy

$$(x+y+z)(x-y)(y-z)(z-x)\ge 0$$

and are proportional to the roots of the equation

$$w^3-3w^2-9\left(1+\frac{\sqrt{7}}{2}\right)w+\frac{27}{4}\left(1+\frac{3}{\sqrt{7}}\right)=0;$$

that is, $x/w_1=y/w_2=z/w_3$ (or any cyclic permutation), where $w_1\approx 6.0583$, $w_2\approx -3.7007$, $w_3\approx 0.6424$. ∎

Application 2. *Let x, y, z be real numbers. If $-3\le k\le 3$, then* ([6])

$$4\sum x^4+(9-k^2)xyz\sum x \ge 2(1+k)\sum x^3y + 2(1-k)\sum xy^3.$$

Proof. Applying Theorem 2.3 for

$$A=0,\ \ B=\frac{9-k^2}{4},\ \ C=\frac{-1-k}{2},\ \ D=\frac{-1+k}{2},$$

we get $E=\dfrac{27-k^2}{2}$, $F=\dfrac{27+k^2}{2}$ and

$$4g_4(t)=(t-1)^2[36t^2+(9-k^2)(2t+1)]\ge 0.$$

If $-3<k<3$, then the polynomial $g_4(t)$ has only the nonnegative double root $t=1$. By Proposition 2.5, we get that equality holds when x, y, z satisfy

$$k(x+y+z)(x-y)(y-z)(z-x)\le 0$$

and are proportional to the roots of the equation

$$w^3-3w^2+\frac{108}{27+k^2}=0.$$

If $|k|=3$, then the polynomial $g_4(t)$ has also the double root $t=0$, which leads to the equality case $x=y=z$.

For instant, if $k=1$, then we get the inequality

$$x^4+y^4+z^4+2xyz(x+y+z)\ge x^3y+y^3z+z^3x,\quad x,y,z\in\mathbb{R},$$

with equality for

$$\frac{x}{\sin\dfrac{8\pi}{7}} = \frac{y}{\sin\dfrac{4\pi}{7}} = \frac{z}{\sin\dfrac{2\pi}{7}}$$

(or any cyclic permutation). Also, if $k = 3$, we get the known inequality (see [1])

$$x^4 + y^4 + z^4 + xy^3 + yz^3 + zx^3 \geq 2(x^3y + y^3z + z^3x), \quad x, y, z \in \mathbb{R}.$$

with equality for $x = y = z$, and also for

$$x \sin\frac{\pi}{9} = y \sin\frac{7\pi}{9} = z \sin\frac{13\pi}{9}$$

(or any cyclic permutation). ∎

Application 3. *Let m and n be real numbers. The inequality* ([7])

$$\sum x^4 + (m + 3)\sum x^2y^2 \geq (2 - n)\sum x^3y + (2 + n)\sum xy^3$$

holds for all real numbers x, y, z if and only if $m \geq 0$ and

$$|n| \leq \frac{2}{3}\sqrt{\frac{(m + 9)\sqrt{m(m + 9)} - m^2}{3}}.$$

Proof. We have

$$A = m + 3, \;\; B = 0, \;\; C = n - 2, \;\; D = -n - 2, \;\; E = -4m, \;\; F = 2\sqrt{27n^2 + 4m^2},$$

$$g_4(t) = 3(m + 9)t^4 - 2\sqrt{27n^2 + 4m^2}\, t^3 + m.$$

According to Theorem 2.3, the desired inequality holds if and only if $g_4(t) \geq 0$ for all $t \geq 0$. From $g_4(0) \geq 0$, we get $m \geq 0$, and by the AM-GM inequality, we have

$$3(m + 9)t^4 + m \geq 4\sqrt[4]{m(m + 9)^3t^{12}} = 4\sqrt{(m + 9)\sqrt{m(m + 9)}}\, t^3.$$

Therefore, we have $g_4(t) \geq 0$ for all $t \geq 0$ if and only if

$$4\sqrt{(m + 9)\sqrt{m(m + 9)}} - 2\sqrt{27n^2 + 4m^2} \geq 0,$$

which is equivalent to

$$|n| \leq \frac{2}{3}\sqrt{\frac{(m + 9)\sqrt{m(m + 9)} - m^2}{3}}.$$

∎

Application 4. *If x, y, z are real numbers, then* ([8])

$$(x^2 + y^2 + z^2)^2 + 2(x^3y + y^3z + z^3x) \geq 3(xy^3 + yz^3 + zx^3).$$

Proof. We have

$$A = 2, \;\; B = 0, \;\; C = 2, \;\; D = -3, \;\; E = 1, \;\; F = 26,$$

and hence

$$g_4(t) = 15t^4 - 26t^3 + 9t^2 + 2 = (t - 1)^2(15t^2 + 4t + 2).$$

Since $g_4(t) \geq 0$ for all $t \geq 0$, the proof is completed (Theorem 2.3).

To analyse the equality cases, we apply Proposition 2.5. Since the polynomial $g_4(t)$ has the nonnegative double roots 1, we get the equality conditions

$$(x + y + z)(x - y)(y - z)(z - x) \geq 0$$

and

$$w^3 - 3w^2 + \frac{27}{13} = 0,$$

which lead to the equality case $x/w_1 = y/w_2 = z/w_3$ (or any cyclic permutation), where $w_1 \approx -0.7447$, $w_2 \approx 1.0256$, $w_3 \approx 2.7191$.
∎

Application 5. *If x, y, z are real numbers, then*

$$10\sum x^4 + 64\sum x^2y^2 \geq 33\sum xy(x^2+y^2).$$

Proof. We have

$$A = \frac{32}{5}, \quad B = 0, \quad C = D = \frac{-33}{10}, \quad E = F = 11,$$

and hence

$$5g_4(t) = 225t^4 - 55t^3 - 39t^2 + 4 = (5t+2)^2(9t^2 - 5t + 1).$$

Since $g_4(t) \geq 0$ for all $t \geq 0$, the proof is completed (Theorem 2.3).

Since $C = D$, according to Corollary 2.7, equality holds when

$$\frac{x}{w} = y = z,$$

where w is a double real root of the polynomial

$$\begin{aligned} h(w) &= w^4 + 2Cw^3 + (2A+B)w^2 + 2(B+C)w + A + 2C + 2 \\ &= \frac{1}{5}(5w^4 - 33w^3 + 64w^2 - 33w + 9) \\ &= \frac{1}{5}(w-3)^2(5w^2 - 3w + 1). \end{aligned}$$

Therefore, equality occurs for $x/3 = y = z$ (or any cyclic permutation). ∎

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