NECESSARY AND SUFFICIENT CONDITIONS FOR CYCLIC HOMOGENEOUS POLYNOMIAL INEQUALITIES OF DEGREE FOUR IN REAL VARIABLES

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ABSTRACT. In this paper, we give two sets of necessary and sufficient conditions that the inequality \( f_4(x, y, z) \geq 0 \) holds for any real numbers \( x, y, z \), where \( f_4(x, y, z) \) is a cyclic homogeneous polynomial of degree four. In addition, all equality cases of this inequality are analysed. For the particular case in which \( f_4(1, 1, 1) = 0 \), we get the main result in [3]. Several applications are given to show the effectiveness of the proposed methods.

Key words and phrases: Cyclic Homogeneous Inequality, Fourth Degree Polynomial, Three Real Variables, Necessary and Sufficient Conditions.

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1. Introduction

Consider the fourth degree cyclic homogeneous polynomial

\[ f_4(x, y, z) = \sum x^4 + A \sum x^2 y^2 + Bxyz \sum x + C \sum x^3 y + D \sum xy^3, \]

where \( A, B, C, D \) are real constants, and \( \sum \) denotes a cyclic sum over \( x, y \) and \( z \).

The following theorem expresses the necessary and sufficient condition that the inequality \( f_4(x, y, z) \geq 0 \) holds for any real numbers \( x, y, z \) in the particular case when \( f_4(1, 1, 1) = 0 \) (see [3] and [4]):

**Theorem 1.1.** If

\[ 1 + A + B + C + D = 0, \]

then the cyclic inequality \( f_4(x, y, z) \geq 0 \) holds for all real numbers \( x, y, z \) if and only if

\[ 3(1 + A) \geq C^2 + CD + D^2. \]

The corollary below gives only sufficient conditions to have \( f_4(x, y, z) \geq 0 \) for any real numbers \( x, y, z \) (see [3]):

**Corollary 1.2.** If

\[ 1 + A + B + C + D \geq 0 \]

and

\[ 2(1 + A) \geq B + C + D + C^2 + CD + D^2, \]

then the cyclic inequality \( f_4(x, y, z) \geq 0 \) holds for all real numbers \( x, y, z \).

In this paper, we generalize the results in Theorem 1.1 to the case where

\[ 1 + A + B + C + D \geq 0, \]

which is equivalent to the necessary condition \( f_4(1, 1, 1) \geq 0 \).

2. Main Results

We establish two theorems which give necessary and sufficient conditions to have

\[ f_4(x, y, z) \geq 0 \]

for any real numbers \( x, y, z \), where \( f_4(x, y, z) \) is a fourth degree cyclic homogeneous polynomial having the form (1.1).

**Theorem 2.1.** The inequality

\[ f_4(x, y, z) \geq 0 \]

holds for all real numbers \( x, y, z \) if and only if

\[ f_4(t + k, k + 1, kt + 1) \geq 0 \]

for all real \( t \), where \( k \in [0, 1] \) is a root of the polynomial

\[ f(k) = (C - D)k^3 + (2A - B - C + 2D - 4)k^2 - (2A - B + 2C - D - 4)k + C - D. \]

**Remark 2.1.** For \( C = D \), the polynomial \( f(k) \) has the roots 0 and 1, while for \( C \neq D \), \( f(k) \) has three real roots, but only one in \([0, 1]\). To prove this assertion, we see that \( f(0) = -f(1) = C - D \). If \( C > D \), then

\[ f(-\infty) = -\infty, \quad f(0) > 0, \quad f(1) < 0, \quad f(\infty) = \infty, \]

and if \( C < D \), then

\[ f(-\infty) = \infty, \quad f(0) < 0, \quad f(1) > 0, \quad f(\infty) = -\infty. \]
From the proof of Theorem 2.1, we get immediately the equality cases of the inequality \( f_4(x, y, z) \geq 0 \).

**Proposition 2.2.** The inequality \( f_4(x, y, z) \geq 0 \) in Theorem 2.1 becomes an equality if
\[
\frac{x}{t + k} = \frac{y}{k + 1} = \frac{z}{kt + 1}
\]
(or any cyclic permutation), where \( k \in (0, 1] \) is a root of the equation
\[
(C - D)k^3 + (2A - B + C + 2D - 4)k^2 - (2A - B + 2C - D - 4)k + C - D = 0
\]
and \( t \in \mathbb{R} \) is a root of the equation
\[
f_4(t + k, k + 1, kt + 1) = 0.
\]

**Theorem 2.3.** The inequality \( f_4(x, y, z) \geq 0 \) holds for all real numbers \( x, y, z \) if and only if \( g_4(t) \geq 0 \) for all \( t \geq 0 \), where
\[
g_4(t) = 3(2 + A - C - D)t^4 - Ft^3 + 3(4 - B + C + D)t^2 + 1 + A + B + C + D,
\]
\[
F = \sqrt{27(C - D)^2 + E^2}, \quad E = 8 - 4A + 2B - C - D.
\]

**Remark 2.2.** In the special case \( f_4(1, 1, 1) = 0 \), when
\[
1 + A + B + C + D = 0,
\]
from Theorem 2.3 we get Theorem 1.1. The condition \( g_4(t) \geq 0 \) in Theorem 2.3 becomes
\[
(2 + A - C - D)t^4 + (5 + A + 2C + 2D)t^2 \geq \sqrt{(2 - 2A - C - D)^2 + 3(C - D)^2} t^3,
\]
and it holds for all \( t \geq 0 \) if and only if
\[
2 + A - C - D \geq 0,
\]
\[
5 + A + 2C + 2D \geq 0,
\]
\[
2\sqrt{(2 + A - C - D)(5 + A + 2C + 2D)} \geq \sqrt{(2 - 2A - C - D)^2 + 3(C - D)^2}.
\]
The last inequality is equivalent to
\[
3(1 + A) \geq C^2 + D^2 + CD,
\]
which involves
\[
2 + A - C - D \geq 1 - (C + D) + \frac{(C + D)^2}{3} - \frac{CD}{3}
\]
\[
\geq 1 - (C + D) + \frac{(C + D)^2}{3} - \frac{(C + D)^2}{12} = \left(1 - \frac{C + D}{2}\right)^2 \geq 0
\]
and
\[
5 + A + 2C + 2D \geq 4 + 2(C + D) + \frac{(C + D)^2}{3} - \frac{CD}{3}
\]
\[
\geq 4 + 2(C + D) + \frac{(C + D)^2}{3} - \frac{(C + D)^2}{12} = \left(2 + \frac{C + D}{2}\right)^2 \geq 0.
\]
Thus, we obtained the necessary and sufficient condition in Theorem 1.1, namely
\[
3(1 + A) \geq C^2 + CD + D^2.
\]
The following proposition gives the equality cases of the inequality \( f_4(x, y, z) \geq 0 \) for \( F = 0 \).
Theorem 2.3 has the following three possible forms:

\[(x + y + z)^2 [x^2 + y^2 + z^2 + k(xy + yz + zx)] \geq 0, \quad k \in [-1, 2],\]

or

\[x^2 + y^2 + z^2 + k(xy + yz + zx)]^2 \geq 0, \quad k \in (-1, 2),\]

or

\[(x^2 + y^2 + z^2 - xy - yz - zx)[x^2 + y^2 + z^2 + k(xy + yz + zx)] \geq 0, \quad k \in [-1, 2).\]

The following proposition gives the equality cases of the inequality \(f_4(x, y, z) \geq 0\) for \(F > 0\).

Proposition 2.4. For \(F = 0\), assume that the inequality \(f_4(x, y, z) \geq 0\) in Theorem 2.3 becomes an equality for at least a real triple \((x, y, z) \neq (0, 0, 0)\). Then, the inequality \(f_4(x, y, z) \geq 0\) in Theorem 2.3 has the following three possible forms:

\[(x + y + z)^2 [x^2 + y^2 + z^2 + k(xy + yz + zx)] \geq 0, \quad k \in [-1, 2],\]

or

\[x^2 + y^2 + z^2 + k(xy + yz + zx)]^2 \geq 0, \quad k \in (-1, 2),\]

or

\[(x^2 + y^2 + z^2 - xy - yz - zx)[x^2 + y^2 + z^2 + k(xy + yz + zx)] \geq 0, \quad k \in [-1, 2).\]

Remark 2.3. The polynomial

\[f(w) = w^3 - 3w^2 + 3(1 - \alpha^2)w + \frac{2E}{F} \alpha^3 + 3\alpha^2 - 1\]

in Proposition 2.5 has three real roots for any given \(\alpha \geq 0\). This is true if \(f(w_1') \geq 0\) and \(f(w_2') \leq 0\), where \(w_1' = 1 - \alpha\) and \(w_2' = 1 + \alpha\) are the roots of the derivative \(f'(w)\). Indeed, we have

\[f(w_1') = 2 \left(1 + \frac{E}{F}\right) \alpha^3 \geq 0,\]

\[f(w_2') = -2 \left(1 - \frac{E}{F}\right) \alpha^3 \leq 0.\]

Thus, for \(F > 0\), the number of distinct non-zero triples \((x, y, z)\) which satisfy \(f_4(x, y, z) = 0\) is equal to the number of distinct nonnegative roots of the polynomial \(g_4(t)\). Since this number is less than or equal to 2, the equality \(f_4(x, y, z) = 0\) holds for \(x = y = z = 0\) and for at most two distinct triples \((x, y, z)\).

In the special case \(f_4(1, 1, 1) = 0\), when \(1 + A + B + C + D = 0\), from Theorem 2.3 and Remark 2.2 it follows that \(3(1 + A) = C^2 + CD + D^2\) is a necessary condition to have \(f_4(x, y, z) \geq 0\) for all real \(x, y, z\), with equality for at least a real triple \((x, y, z)\) with \(x \neq y\) or \(y \neq z\) or \(z \neq x\). Thus, by Proposition 2.5 we get the following corollary.

Corollary 2.6. Let \(f_4(x, y, z)\) be a fourth degree cyclic homogeneous polynomial such that \(f_4(1, 1, 1) = 0\) and \(f_4(x, y, z) \geq 0\) for all real numbers \(x, y, z\). Let us denote

\[E = 12 - 3(C + D) - 2(C^2 + CD + D^2), \quad F = \sqrt{27(C - D)^2 + E^2},\]

\[\alpha = \sqrt[3]{\frac{3(C + D + 4)^2 + (C - D)^2}{3(C + D - 2)^2 + (C - D)^2}}.\]

For \(F > 0\), the inequality \(f_4(x, y, z) \geq 0\) becomes an equality when \(x = y = z\), and also when \(x, y, z\) satisfy

\[(C - D)(x + y + z)(x - y)(y - z)(z - x) \geq 0\]
and are proportional to the roots $w_1$, $w_2$ and $w_3$ of the polynomial equation

$$w^3 - 3w^2 + 3(1 - \alpha^2)w + \frac{2E}{F} \alpha^3 + 3\alpha^2 - 1 = 0.$$ 

A new special case is the one in which $C = D$, when the homogeneous polynomial $f_4(x, y, z)$ is symmetric. Since $F = |E| = 2|4 - 2A + B - C|$, the polynomial

$$f(w) = w^3 - 3w^2 + 3(1 - \alpha^2)w + \frac{2E}{F} \alpha^3 + 3\alpha^2 - 1$$

in Proposition 2.5 becomes either

$$f(w) = w^3 - 3w^2 + 3(1 - \alpha^2)w + 2\alpha^3 + 3\alpha^2 - 1 = (w - \alpha - 1)^2(w + 2\alpha - 1),$$

or

$$f(w) = w^3 - 3w^2 + 3(1 - \alpha^2)w - 2\alpha^3 + 3\alpha^2 - 1 = (w + \alpha - 1)^2(w - 2\alpha - 1).$$

In both cases, two of the real roots $w_1$, $w_2$ and $w_3$ are equal. Setting $y = z = 1$, the equation $f_4(x, y, z) = 0$ becomes

$$x^4 + 2Cx^3 + (2A + B)x^2 + 2(B + C)x + A + 2C + 2 = 0.$$

So, the following corollary holds.

**Corollary 2.7.** Let

$$f_4(x, y, z) = \sum x^4 + A \sum x^2 y^2 + Bxyz \sum x + C \sum xy(x^2 + y^2)$$

be a fourth degree symmetric homogeneous polynomial such that $4 - 2A + B - C \neq 0$ and $f_4(x, y, z) \geq 0$ for all real numbers $x, y, z$. The inequality $f_4(x, y, z) \geq 0$ becomes an equality when $x/w = y = z$ (or any cyclic permutation), where $w$ is a double real root of the equation

$$w^4 + 2Cw^3 + (2A + B)w^2 + 2(B + C)w + A + 2C + 2 = 0.$$

With regard to the distinct nonnegative roots of the polynomial $g_4(t)$, the following statement holds.

**Proposition 2.8.** Assume that $F > 0$ and $g_4(t) \geq 0$ for all $t \geq 0$. The polynomial $g_4(t)$ in Theorem 2.3 has the following nonnegative real roots:

(i) two pairs of nonnegative roots, namely

$t_1 = t_2 = 0, \ t_3 = t_4 \geq 0$,

if and only if

$$1 + A + B + C + D = 0, \ \ 3(1 + A) = C^2 + CD + D^2;$$

(ii) only one pair of zero roots,

$t_1 = t_2 = 0$,

if and only if

$$1 + A + B + C + D = 0, \ \ 3(1 + A) > C^2 + CD + D^2;$$

(iii) only one pair of positive roots,

$t_1 = t_2 > 0$,

if and only if

$$t = \frac{2\sqrt{2}(2b + \sqrt{b^2 + 12c})}{3\sqrt{b + \sqrt{b^2 + 12c}}},$$

$$a = \frac{2\sqrt{2}(2b + \sqrt{b^2 + 12c})}{3\sqrt{b + \sqrt{b^2 + 12c}}}.$$
where
\[
a = \frac{F}{3(2 + A - C - D)} \geq 0, \quad b = \frac{4 - B + C + D}{2 + A - C - D}, \quad c = \frac{1 + A + B + C + D}{3(2 + A - C - D)} > 0.
\]

**Remark 2.4.** It is much easier to make a thorough study of a cyclic homogeneous polynomial inequality of degree four \( f_4(x, y, z) \geq 0 \) by applying Theorem 2.3 than by applying Theorem 2.1 especially in the case where \( f_4(1, 1, 1) \neq 0 \). For this reason, Theorem 2.1 is more useful for the study of the inequality \( f_4(x, y, z) \geq 0 \) by means of a computer. For example, let us prove by both Theorems 2.1 and 2.3 the well known inequality ([1], [2])

\[
(x^2 + y^2 + z^2)^2 \geq 3(x^3y + y^3z + z^3x), \quad x, y, z \in \mathbb{R}.
\]

We have
\[
f_4(x, y, z) = (x^2 + y^2 + z^2)^2 - 3(x^3y + y^3z + z^3x);
\]
that is,
\[
A = 2, \quad B = 0, \quad C = -3, \quad D = 0.
\]

According to Theorem 2.1, we need to show that \( f_4(t + k, k + 1, kt + 1) \geq 0 \) for all real \( t \), where \( k \approx 0.445042 \) satisfies the equation

\[
k^3 - k^2 - 2k + 1 = 0.
\]

After many calculation, we get
\[
f_4(t + k, k + 1, kt + 1) = (t - 1)^2[(1 - k)(3 - 2k)t^2 + 2(1 - k)(3k - 1)t + 2 - k - 8k^2]
\]
\[
= (1 - k)(3 - 2k)(t - 1)^2 \left( t + \frac{3k - 1}{3 - 2k} \right)^2 \geq 0.
\]

By Proposition 2.2 equality holds for
\[
\frac{x}{t + k} = \frac{y}{k + 1} = \frac{z}{kt + 1}
\]
(or any cyclic permutation), where \( t \in \left\{ 1, \frac{1 - 3k}{3 - 2k} \right\} \); that is, for \( x = y = z \), and also for
\[
\frac{x}{1 - 2k^2} = \frac{y}{(1 + k)(3 - 2k)} = \frac{z}{3 - k - 3k^2}
\]
(or any cyclic permutation).

According to Theorem 2.3, we need to show that \( g_4(t) \geq 0 \) for all \( t \geq 0 \). Indeed, we have
\[
E = 3, \quad F = 6\sqrt{7}, \quad \text{and hence}
\]
\[
g_4(t) = 3t^2(\sqrt{7} t - 1)^2 \geq 0.
\]

Since \( f_4(1, 1, 1) = 0 \), we apply Corollary 2.6 to find the other equality cases. We get \( \alpha = 1/\sqrt{7} \), and the equality conditions
\[
(x + y + z)(x - y)(y - z)(z - x) \leq 0
\]
and
\[
w^3 - 3w^2 + \frac{18}{7}w - \frac{27}{49} = 0,
\]
which lead to the equality case
\[
\frac{x}{\sin^2 \frac{4\pi}{7}} = \frac{y}{\sin^2 \frac{2\pi}{7}} = \frac{z}{\sin^2 \frac{\pi}{7}}
\]
(or any cyclic permutation).
3. Proof of Theorem 2.1

The main idea is to use the linear cyclic substitution

\[ x = a + kb, \quad y = b + kc, \quad z = c + ka, \]

in order to convert the cyclic polynomial \( f_4(x, y, z) \) to a fourth degree symmetric homogeneous polynomial

\[ h_4(a, b, c) = f_4(a + kb, b + kc, c + ka). \]

If this is possible for a real constant \( k \in [0, 1] \), then the inequality \( f_4(x, y, z) \geq 0 \) holds for all real numbers \( x, y, z \) if and only if the inequality \( h_4(a, b, c) \geq 0 \) holds for all real numbers \( a, b, c \). According to Lemma [3.1] below, the inequality \( h_4(a, b, c) \geq 0 \) holds for all real \( a, b, c \) if and only if \( h_4(t, 1, 1) \geq 0 \) for all real \( t \); that is, if and only if

\[ h_4(t, 1, 1) = f_4(t + k, 1 + k, 1 + kt) \]

for all real \( t \). So, we only need to show that the polynomial \( h_4(a, b, c) \) is symmetric if \( k \) is a real root of the polynomial \( f(k) \).

For \( C = D \) and \( k = 0 \), the polynomial \( h_4(a, b, c) \) is clearly symmetric. Consider now that \( C \neq D \). It is easy to show that the expressions \( \sum x^4, \sum x^2y^2, xyz \sum x, \sum x^3y \) and \( \sum xy^3 \) contain respectively the following cyclic expressions \( \sum a^3b \) and \( \sum ab^3 \):

\[
\begin{align*}
\sum x^4 & : 4k \sum a^3b + 4k^3 \sum ab^3, \\
\sum x^2y^2 & : 2k^3 \sum a^3b + 2k \sum ab^3, \\
xyz \sum x & : (k^2 + k) \sum a^3b + (k^3 + k^2) \sum ab^3, \\
\sum x^3y & : (k^4 + 1) \sum a^3b + (3k^2 + k) \sum ab^3, \\
\sum xy^3 & : (k^3 + 3k^2) \sum a^3b + (k^4 + 1) \sum ab^3.
\end{align*}
\]

Therefore, \( h_4(a, b, c) \) contains the expression

\[ E \sum a^3b + F \sum ab^3, \]

where

\[
\begin{align*}
E & = 4k + 2Ak^3 + B(k^2 + k) + C(k^4 + 1) + D(k^3 + 3k^2), \\
F & = 4k^3 + 2Ak + B(k^3 + k^2) + C(3k^2 + k) + D(k^4 + 1).
\end{align*}
\]

Obviously, if \( E = F \), then \( h_4(a, b, c) \) is a symmetric homogeneous polynomial. From

\[
E - F = (C - D)k^4 + (2A - B + D - 4)k^3 - 3(C - D)k^2 - (2A - B + C - 4)k + C - D = (k + 1)f(k),
\]

it follows that \( f(k) = 0 \) involves \( E = F \).

To complete the proof, we still need to show that the equation \( f(k) = 0 \) has at least a root in \([0, 1]\). This is true since \( f(k) \) is a continuous function and \( f(0) = -f(1) = C - D \neq 0 \).

Lemma 3.1. Let \( h_4(a, b, c) \) be a fourth degree symmetric homogeneous polynomial. The inequality

\[ h_4(a, b, c) \geq 0 \]

holds for all real numbers \( a, b, c \) if and only if \( h_4(t, 1, 1) \geq 0 \) for all real \( t \).
Proof. Let \( p = a + b + c, q = ab + bc + ca \) and \( r = abc \). For fixed \( p \) and \( q \), from the known relation
\[
27(a - b)^2(b - c)^2(c - a)^2 = 4(p^2 - 3q)^3 - (2p^3 - 9pq + 27r)^2,
\]
it follows that \( r \) is maximal and minimal when two of \( a, b, c \) are equal. On the other hand, for fixed \( p \) and \( q \), the inequality \( h_4(a, b, c) \geq 0 \) can be written as \( g(r) \geq 0 \), where \( g(r) \) is a linear function. Therefore, \( g(r) \) is minimal when \( r \) is minimal or maximal; that is, when two of \( a, b, c \) are equal. Since the polynomial \( h_4(a, b, c) \) is symmetric, homogeneous and satisfies
\[
h_4(-a, -b, -c) = h_4(a, b, c),
\]
g\( r \) is minimal if and only if \( h_4(t, 1, 1) \geq 0 \) and \( h_4(t, 0, 0) \geq 0 \) for all real \( t \). To complete the proof, it suffices to show that if \( h_4(t, 1, 1) \geq 0 \) for all real \( t \), then \( h_4(t, 0, 0) \geq 0 \) for all real \( t \). Indeed, since \( h_4(a, b, c) \) has the general form
\[
h_4(a, b, c) = A_0 \sum a^4 + A_1 \sum ab(a^2 + b^2) + A_2 \sum a^2b^2 + A_3abc \sum a,
\]
the condition \( h_4(t, 1, 1) \geq 0 \) for all real \( t \) involves \( A_0 \geq 0 \), and hence \( h_4(t, 0, 0) = A_0t^4 \geq 0 \) for all real \( t \). \( \blacksquare \)

4. PROOF OF THEOREM 2.3

Using the substitutions
\[
p = x + y + z, \quad q = xy + yz + zx, \quad r = abc,
\]
we have
\[
xyz \sum x = pr, \quad \sum x^2y^2 = q^2 - 2pq,
\]
\[
\sum x^4 = (\sum x^2)^2 - 2\sum x^2y^2 = (p^2 - 2q)^2 - 2(q^2 - 2pq) = p^4 - 4p^2q + 2q^2 + 4pq,
\]
\[
\sum x^3y + \sum xy^3 = (\sum xy)(\sum x^2) - xyz \sum x = q(p^2 - 2q) - pr,
\]
\[
\sum x^3y - \sum xy^3 = p(x - y)(y - z)(z - x),
\]
\[
27(x - y)^2(y - z)^2(z - x)^2 = 4(p^2 - 3q)^3 - (2p^3 - 9pq + 27r)^2.
\]

Further, we need Lemma 4.1, Lemma 4.2 and Lemma 4.3 below. By Lemma 4.1, the inequality \( f_4(x, y, z) \geq 0 \) holds if and only if
\[
S_4(x, y, z) \geq |(C - D)(x + y + z)(x - y)(y - z)(z - x)|
\]
for all real \( x, y, z \).

Sufficiency. Consider the following two cases: \( p = 0 \) and \( p \neq 0 \).

Case 1: \( p = 0 \). Since \( \sum x^4 = 2q^2, \sum x^2y^2 = q^2 \) and \( \sum x^3y + \sum xy^3 = -2q^2 \), the desired inequality (4.1) becomes
\[
(2 + A - C - D)q^2 \geq 0.
\]
This is true since the hypothesis \( g_4(t) \geq 0 \) for all \( t \geq 0 \) involves \( 2 + A - C - D \geq 0 \).

Case 2: \( p \neq 0 \). Due to homogeneity, we may set \( p = 1 \), which involves \( q \leq 1/3 \). Since
\[
|(x - y)(y - z)(z - x)| = \sqrt{(x - y)^2(y - z)^2(z - x)^2} = \sqrt{\frac{4(1 - 3q)^3 - (2 - 9q + 27r)^2}{27}},
\]
(4.1) becomes
\[
2 - (8 - C - D)q + 2(2 + A - C - D)q^2 + (8 - 4A + 2B - C - D)r \geq \frac{|C - D|}{3\sqrt{3}} \sqrt{4(1 - 3q)^3 - (2 - 9q + 27r)^2}.
\]
As we have shown above, the inequality (4.1) for actually, it suffices to consider that (4.1) holds for all real $x, y, z$, holds for all real $x, y, z$. Lemma 4.1, which is equivalent to $g$, Choosing the triple $t$, where $t$, which is just the hypothesis $g$. Thus, we only need to prove that we get $(4.2)$, where $t$. Substituting $t$, we get $(4.2)$, which implies $q = (1 - t^2)/3$, $t \geq 0$, the inequality turns into $2(2 + A - C - D)t^4 + (16 - 4A + C + D)t^2 - 2 + 2A + C + D + 9Er \geq \geq \sqrt{3}|C - D|\sqrt{4t^6 - (3t^2 - 1 + 27r)^2}$, where $E = 8 - 4A + 2B - C - D$. Applying Lemma 4.2 for (4.2) $\alpha = \sqrt{3}|C - D|$, $\beta = \frac{E}{3}$, $a = 2t^3$, $b = 3t^2 - 1 + 27r$, we get $\sqrt{3}|C - D|\sqrt{4t^6 - (3t^2 - 1 + 27r)^2} \leq \frac{2F t^3}{3} + \frac{E(3t^2 - 1 + 27r)}{3}$. Thus, we only need to prove that $2(2 + A - C - D)t^4 + (16 - 4A + C + D)t^2 - 2 + 2A + C + D + 9Er \geq \geq \frac{2F t^3}{3} + \frac{E(3t^2 - 1 + 27r)}{3}$, which is just the hypothesis $g_4(t) \geq 0$.

Necessity. We need to prove that if (4.1) holds for all real $x, y, z$, then $g_4(t) \geq 0$ for all $t \geq 0$. Actually, it suffices to consider that (4.1) holds for all real $x, y, z$ such that $p = x + y + z = 1$. As we have shown above, the inequality (4.1) for $p = 1$ has the form $2(2 + A - C - D)t^4 + (16 - 4A + C + D)t^2 - 2 + 2A + C + D + 9Er \geq \geq \sqrt{3}|C - D|\sqrt{4t^6 - (3t^2 - 1 + 27r)^2}$, where $E = 8 - 4A + 2B - C - D$. Choosing the triple $(x, y, z)$ as in Lemma 4.3 we get $2(2 + A - C - D)t^4 + (16 - 4A + C + D)t^2 - 2 + 2A + C + D + 9Er \geq \geq \frac{2F t^3}{3} + \frac{E(3t^2 - 1 + 27r)}{3}$, which is equivalent to $g_4(t) \geq 0$.

Lemma 4.1. The inequality $f_4(x, y, z) \geq 0$ holds for all real $x, y, z$ if and only if the inequality $S_4(x, y, z) \geq |(C - D)(x + y + z)(x - y)(y - z)(z - x)|$ holds for all real $x, y, z$, where $S_4(x, y, z) = 2 \sum x^4 + 2A \sum x^2y^2 + 2Bxyz \sum x + (C + D)(\sum x^3y + \sum xy^3)$. Proof. It is easy to show that $2f_4(x, y, z) = S_4(x, y, z) + (C - D)(\sum x^3y - \sum xy^3) = S_4(x, y, z) - (C - D)(x + y + z)(x - y)(y - z)(z - x)$. Sufficiency. According to the hypothesis $S_4(x, y, z) \geq |(C - D)(x + y + z)(x - y)(y - z)(z - x)|$, we have $2f_4(x, y, z) \geq |(C - D)(x + y + z)(x - y)(y - z)(z - x)|$
On the other hand, if \( f \) from the hypothesis \( f(x,y,z) \geq 0 \), we get
\[
S_4(x,y,z) \geq (C - D)(x + y + z)(x - y)(y - z)(z - x).
\]

Since \( f(x,y,z) \geq 0 \) for all real \( x, y, z \), then also \( f(x,z) \geq 0 \) for all real \( x, y, z \). Since
\[
2f_4(x,z) = S_4(x,y,z) - (C - D)(x + y + z)(x - y)(y - z)(z - x),
\]
we get
\[
S_4(x,y,z) \geq (C - D)(x + y + z)(x - y)(y - z)(z - x)
\]
for all real \( x, y, z \). Therefore, we have
\[
S_4(x,y,z) \geq |(C - D)(x + y + z)(x - y)(y - z)(z - x)|.
\]

**Lemma 4.2.** If \( \alpha, \beta, a, b \) are real numbers, \( \alpha \geq 0, a \geq 0 \) and \( a^2 \geq b^2 \), then
\[
\alpha \sqrt{a^2 - b^2} \leq a \sqrt{a^2 + \beta^2} + \beta b,
\]
with equality if and only if
\[
\beta a + b \sqrt{a^2 + \beta^2} = 0.
\]

**Proof.** Since
\[
a \sqrt{a^2 + \beta^2} + \beta b \geq |\beta|a + \beta b \geq |\beta||b| + \beta b \geq 0,
\]
we can write the inequality as
\[
\alpha^2(a^2 - b^2) \leq (a \sqrt{a^2 + \beta^2} + \beta b)^2,
\]
which is equivalent to the obvious inequality
\[
(\beta a + b \sqrt{a^2 + \beta^2})^2 \geq 0.
\]

**Lemma 4.3.** Let \( A, B, C, D, E, F \) be given real constants such that
\[
E = 8 - 4A + 2B - C - D, \quad F = \sqrt{27(C - D)^2 + E^2}.
\]
For any given \( t \geq 0 \), there exists a real triple \( (x, y, z) \) such that
\[
x + y + z = 1, \quad xy + yz + zx = (1 - t^2)/3
\]
and
\[
\sqrt{3}|C - D|\sqrt{4t^6 - (3t^2 - 1 + 27xyz)^2} = \frac{2Ft^3}{3} + \frac{E(3t^2 - 1 + 27xyz)}{3}.
\]

**Proof.** Let \( r = xyz \). From the last relation we get
\[
\left[ \sqrt{3}|C - D|\sqrt{4t^6 - (3t^2 - 1 + 27r)^2} - \frac{E(3t^2 - 1 + 27r)}{3} \right]^2 = \left( \frac{2Ft^3}{3} \right)^2,
\]
\[
\left[ \sqrt{3}|C - D|(3t^2 - 1 + 27r) + \frac{E}{3} \sqrt{4t^6 - (3t^2 - 1 + 27r)^2} \right]^2 = 0,
that is \( f(r) = 0 \), where
\[
f(r) = \sqrt{3}(C - D)(3t^2 - 1 + 27r) + \frac{E}{3}\sqrt{4t^6 - (3t^2 - 1 + 27r)^2}.
\]
We need to prove that for any given \( t \geq 0 \) there exists a real triple \((x, y, z)\) such that \( x + y + z = 1,\ xy + yz + zx = (1 - t^2)/3 \) and \( f(r) = 0 \). According to Theorem 2.3 and its proof in section 4, we have
\[
27(x - y)^2(y - z)^2(z - x)^2 = 4t^6 - (3t^2 - 1 + 27r)^2 \geq 0,
\]
this is true if \( r \in [r_1, r_2] \), where
\[
r_1 = \frac{1}{27}(1 - 3t^2 - 2t^3), \quad r_2 = \frac{1}{27}(1 - 3t^2 + 2t^3).
\]
Therefore, we only need to show that the equation \( f(r) = 0 \) has a root in \([r_1, r_2]\). Indeed, from
\[
f(r_1) = -2\sqrt{3}|C - D|t^3, \quad f(r_2) = 2\sqrt{3}|C - D|t^3, \quad f(r_1)f(r_2) \leq 0,
\]
the desired conclusion follows.

5. PROOF OF PROPOSITION 2.4

We first see that \( F = 0 \) involves
\[
C = D = 4 - 2A + B
\]
and
\[
\frac{1}{3}g_4(t) = (5A - 2B - 6)t^4 + (12 - 4A + B)t^2 + 3 - A + B.
\]
According to Theorem 2.3 and its proof in section 4, we have \( f_4(x, y, z) \geq 0 \) for all real \( x, y, z \), with equality for at least a real triple \((x, y, z) \neq (0, 0, 0)\), only if \( g_4(t) \geq 0 \) for all \( t \geq 0 \) and \( g_4(t) = 0 \) for at least a nonnegative value of \( t \). In our case, we have \( g_4(t) \geq 0 \) for all \( t \geq 0 \) only if \( 5A - 2B - 6 \geq 0 \) and \( 3 - A + B \geq 0 \). We need to consider three cases: \( 5A - 2B - 6 = 0; 5A - 2B - 6 > 0 \) and \( 3 - A + B > 0; 5A - 2B - 6 > 0 \) and \( 3 - A + B = 0 \).

Case 1: \( 5A - 2B - 6 = 0 \). We get
\[
A = 2k + 2, \quad B = 5k + 2, \quad C = D = k + 2, \quad k \in \mathbb{R},
\]
and hence
\[
f_4(x, y, z) = \sum x^4 + 2(5k + 2)xy^2 + (5k + 2)xyz \sum x + (k + 2) \sum xy(x^2 + y^2)
\]
\[
= (x + y + z)^2[x^2 + y^2 + z^2 + k(xy + yz + zx)].
\]
Clearly, the inequality \( f_4(x, y, z) \geq 0 \) holds for all real \( x, y, z \) if and only if \( k \in [-1, 2] \). The same result follows from the condition \( g_4(t) \geq 0 \) for all \( t \geq 0 \), where
\[
g_4(t) = 9(2 - k)t^2 + 9(1 + k).
\]

Case 2: \( 5A - 2B - 6 > 0 \), \( 3 - A + B > 0 \). We have \( g_4(t) \geq 0 \) for all \( t \geq 0 \) and also \( g_4(t) = 0 \) for at least a nonnegative value of \( t \) if and only if \( 12 - 4A + B < 0 \) and
\[
(12 - 4A + B)^2 = 4(5A - 2B - 6)(3 - A + B);
\]
that is,
\[
B = 2(A - 2 \pm \sqrt{A - 2}), \quad A \geq 2.
\]
Putting \( k = \pm \sqrt{A - 2} \), we get
\[
A = k^2 + 2, \quad B = 2k(k + 1), \quad C = D = 2k,
\]
and hence
\[
f_4(x, y, z) = [x^2 + y^2 + z^2 + k(xy + yz + zx)]^2.
\]
From $12 - 4A + B = 2(k + 1)(2 - k) < 0$, we get $k \in (-1, 2)$.

**Case 3:** $5A - 2B - 6 > 0$, $3 - A + B = 0$. We get

$$A = 2 - k, \quad B = -1 - k, \quad C = D = k - 1, \quad k < 2,$$

and hence

$$f_4(x, y, z) = \sum x^4 + (2 - k) \sum x^2 y^2 - (1 + k) x y z \sum x + (k - 1) \sum xy(x^2 + y^2)$$

$$= (x^2 + y^2 + z^2 - xy - yz - zx)[x^2 + y^2 + z^2 + k(xy + yz + zx)].$$

The inequality $f_4(x, y, z) \geq 0$ holds for all real $x, y, z$ if and only if $k \in [-1, 2)$. The same result follows from the condition $g_4(t) \geq 0$ for all $t \geq 0$, where

$$g_4(t) = 9t^2[(2 - k)t^2 + 1 + k].$$

6. **Proof of Proposition 2.3**

By the proof of Theorem 2.3, it follows that the main necessary condition to have $f_4(x, y, z) \geq 0$ for all real $x, y, z$ and $f(x, y, z) = 0$ for at least a real triple $(x, y, z) \neq (0, 0, 0)$ is to have $g_4(t) \geq 0$ for all $t \geq 0$ and $g_4(t) = 0$ for at least a nonnegative value of $t$. Clearly, for $F > 0$, the inequality $g_4(t) \geq 0$ holds for all $t \geq 0$ only if $2 + A - C - D > 0$. We can find all equality cases of the inequality $f_4(x, y, z) \geq 0$ using the above proof of Theorem 2.3. Consider two cases: $x + y + z = 0$ and $x + y + z = 1$.

**Case 1:** $x + y + z = 0$. The inequality (4.1), which is equivalent to $f_4(x, y, z) \geq 0$, becomes

$$(2 + A - C - D)(xy + yz + zx)^2 \geq 0,$$

with equality for $x + y + z = 0$ and $xy + yz + zx = 0$; that is, for $x = y = z = 0$.

**Case 2:** $x + y + z = 1$. According to Lemma 4.1, a first necessary equality condition is

$$(C - D)(x + y + z)(x - y)(y - z)(z - x) \geq 0.$$

In addition, according to Lemma 4.2, it is necessary to have

$$\beta a + b \sqrt{\alpha^2 + \beta^2} = 0,$$

where $\alpha, \beta, a$ and $b$ are given by (4.2). This condition is equivalent to

$$2Et^3 + F(3t^2 - 1 + 27xyz) = 0.$$

Since $x + y + z = 1$ and $xy + yz + zx = (1 - t^2)/3$, where $t$ is any nonnegative root of the polynomial $g_4(t)$, the equality $f_4(x, y, z) = 0$ holds when

$$(C - D)(x + y + z)(x - y)(y - z)(z - x) \geq 0,$$

$$x + y + z = 1, \quad xy + yz + zx = \frac{1 - t^2}{3}, \quad 27xyz = 1 - 3t^2 - \frac{2E}{F}t^3;$$

that is, when $x, y, z$ are proportional to the roots of the equation

$$27w^3 - 27w^2 + 9(1 - t^2)w + \frac{2E}{F}t^3 + 3t^2 - 1 = 0$$

and satisfy $(C - D)(x + y + z)(x - y)(y - z)(z - x) \geq 0$. Substituting $w/3$ for $w$, we get the desired equation

$$w^3 - 3w^2 + 3(1 - t^2)w + \frac{2E}{F}t^3 + 3t^2 - 1 = 0.$$
7. Proof of Proposition [2.8]

Clearly, if \( g_4(t) \geq 0 \) for all \( t \geq 0 \), then \( 2 + A - C - D > 0 \).

(i) If the polynomial \( g_4(t) \) has four nonnegative real numbers \( t_1 \leq t_2 \leq t_3 \leq t_4 \), then the condition \( g_4(t) \geq 0 \) for all \( t \geq 0 \) holds if and only if
\[
0 \leq t_1 = t_2 = a \leq b = t_3 = t_4,
\]
where
\[
g_4(t) = 3(2 + A - C - D)(t - a)^2(t - b)^2.
\]
Since the coefficient of \( t \) is 0 in \( g_4(t) \) and is \( 2ab(a+b) \) in \((t-a)^2(t-b)^2\), it follows that \( a = 0 \) and \( b \geq 0 \). From \( g_4(0) = 0 \), we get \( 1 + A + B + C + D = 0 \), which involves \( 3(1 + A) = C^2 + CD + D^2 \) (see Remark [2.2]).

Reversely, if \( 1 + A + B + C + D = 0 \) and \( 3(1 + A) = C^2 + CD + D^2 \), then
\[
g_4(t) = 3(2 + A - C - D)t^2 \left[ t - \frac{F}{6(2 + A - C - D)} \right]^2,
\]
where \( F \geq 0 \).

(ii) The polynomial \( g_4(t) \) has the double root 0 if and only if \( 1 + A + B + C + D = 0 \), when
\[
g_4(t) = t^2 g(t),
\]
where
\[
g(t) = 3(2 + A - C - D)t^2 - Ft + 3(4 - B + C + D).
\]
Clearly, \( g_4(t) \) has only two nonnegative roots (that are \( t_1 = t_2 = 0 \)) when \( g(t) \) has either negative real roots or complex roots. Since \( F \geq 0 \), \( g(t) \) can not have negative roots, but can have complex roots, when the discriminant of the quadratic polynomial \( g(t) \) is negative; that is,
\[
3(1 + A) > C^2 + CD + D^2.
\]

(iii) Write the inequality \( g_4(t) \geq 0 \) as \( h(t) \geq 0 \), where
\[
h(t) = t^4 - at^3 + bt^2 + c.
\]
In addition, writing \( h(t) \) in the form
\[
h(t) \equiv (t - t_0)^2(t^2 + pt + q), \quad t_0 > 0,
\]
we find
\[
2t_0 - p = a, \quad t_0^2 - 2pt_0 + q = b, \quad pt_0 - 2q = 0, \quad qt_0^2 = c.
\]
From the last three relation, we get
\[
2t_0^2 = b + \sqrt{b^2 + 12c},
\]
\[
6q = \sqrt{b^2 + 12c} - b,
\]
\[
p = \frac{\sqrt{2}(\sqrt{b^2 + 12c} - b)}{3\sqrt{b + \sqrt{b^2 + 12c}}}.
\]
Since \( p > 0 \) and \( q > 0 \), the quadratic polynomial \( t^2 + pt + q \) has no nonnegative real root. Substituting \( t_0, p \) and \( q \) in \( 2t_0 - p = a \), we get
\[
a = 2t_0 - p = 2t_0 - \frac{2q}{t_0} = \frac{2t_0^2 - 2q}{t_0} = \frac{2(2b + \sqrt{b^2 + 12c})}{3t_0} = \frac{2\sqrt{2}(2b + \sqrt{b^2 + 12c})}{3\sqrt{b + \sqrt{b^2 + 12c}}}.
\]
8. APPLICATIONS OF THEOREM 2.3

**Application 1.** If \( x, y, z \) are real numbers, then (5)

\[
(x^2 + y^2 + z^2)^2 + \frac{8}{\sqrt{7}}(x^3y + y^3z + z^3x) \geq 0.
\]

**Proof.** We have

\[
A = 2, \quad B = 0, \quad C = \frac{8}{\sqrt{7}}, \quad D = 0, \quad E = -\frac{8}{\sqrt{7}}, \quad F = 16,
\]

and hence

\[
g_4(t) = 12 \left(1 - \frac{2}{\sqrt{7}}\right) t^4 - 16t^3 + 12 \left(1 + \frac{2}{\sqrt{7}}\right) t^2 + 3 + \frac{8}{\sqrt{7}}
\]

\[
= \frac{2}{\sqrt{7}} \left(t - \frac{3 + \sqrt{7}}{2}\right)^2 \left[6(\sqrt{7} - 2)t^2 + 2(3 - \sqrt{7})t + 1\right].
\]

Since \( g_4(t) \geq 0 \) for all \( t \geq 0 \), the inequality is proved (Theorem 2.3).

To find all equality cases, we apply Proposition 2.5. We see that the polynomial \( g_4(t) \) has only the nonnegative double root \( \alpha = \frac{3 + \sqrt{7}}{2} \). Therefore, equality holds when \( x, y, z \) satisfy

\[
(x + y + z)(x - y)(y - z)(z - x) \geq 0
\]

and are proportional to the roots of the equation

\[
w^3 - 3w^2 - 9 \left(1 + \frac{\sqrt{7}}{2}\right) w + \frac{27}{4} \left(1 + \frac{3}{\sqrt{7}}\right) = 0;
\]

that is, \( x/w_1 = y/w_2 = z/w_3 \) (or any cyclic permutation), where \( w_1 \approx 6.0583, w_2 \approx -3.7007, w_3 \approx 0.6424 \).

**Application 2.** Let \( x, y, z \) be real numbers. If \(-3 \leq k \leq 3\), then (6)

\[
4 \sum x^4 + (9 - k^2)xyz \sum x \geq 2(1 + k) \sum x^3y + 2(1 - k) \sum xy^3.
\]

**Proof.** Applying Theorem 2.3 for

\[
A = 0, \quad B = \frac{9 - k^2}{4}, \quad C = -\frac{1 - k}{2}, \quad D = -\frac{1 + k}{2},
\]

we get \( E = \frac{27 - k^2}{2}, F = \frac{27 + k^2}{2} \) and

\[
4g_4(t) = (t - 1)^2[36t^2 + (9 - k^2)(2t + 1)] \geq 0.
\]

If \(-3 < k < 3\), then the polynomial \( g_4(t) \) has only the nonnegative double root \( t = 1 \). By Proposition 2.5, we get that equality holds when \( x, y, z \) satisfy

\[
k(x + y + z)(x - y)(y - z)(z - x) \leq 0
\]

and are proportional to the roots of the equation

\[
w^3 - 3w^2 + \frac{108}{27 + k^2} = 0.
\]

If \( |k| = 3 \), then the polynomial \( g_4(t) \) has also the double root \( t = 0 \), which leads to the equality case \( x = y = z \).

For instance, if \( k = 1 \), then we get the inequality

\[
x^4 + y^4 + z^4 + 2xyz(x + y + z) \geq x^3y + y^3z + z^3x, \quad x, y, z \in \mathbb{R},
\]
with equality for
\[
\frac{x}{\sin \frac{8\pi}{7}} = \frac{y}{\sin \frac{4\pi}{7}} = \frac{z}{\sin \frac{2\pi}{7}}
\]
(or any cyclic permutation). Also, if \(k = 3\), we get the known inequality (see [1])
\[
x^4 + y^4 + z^4 + xy^3 + yz^3 + zx^3 \geq 2(x^3y + y^3z + z^3x), \quad x, y, z \in \mathbb{R}.
\]
with equality for \(x = y = z\), and also for
\[
x \sin \frac{\pi}{9} = y \sin \frac{7\pi}{9} = z \sin \frac{13\pi}{9}
\]
(or any cyclic permutation).

**Application 3.** Let \(m\) and \(n\) be real numbers. The inequality (17)
\[
\sum x^4 + (m + 3) \sum x^2 y^2 \geq (2 - n) \sum x^3 y + (2 + n) \sum x y^3
\]
holds for all real numbers \(x, y, z\) if and only if \(m \geq 0\) and
\[
|n| \leq \frac{2}{3} \sqrt{(m + 9) \sqrt{m(m + 9)} - m^2}.
\]

**Proof.** We have
\[
A = m + 3, \quad B = 0, \quad C = n - 2, \quad D = -n - 2, \quad E = -4m, \quad F = 2\sqrt{27n^2 + 4m^2},
\]
\[
g_4(t) = 3(m + 9)t^4 - 2\sqrt{27n^2 + 4m^2} t^3 + m.
\]
According to Theorem 2.3, the desired inequality holds if and only if \(g_4(t) \geq 0\) for all \(t \geq 0\). From \(g_4(0) \geq 0\), we get \(m \geq 0\), and by the AM-GM inequality, we have
\[
3(m + 9)t^4 + m \geq 4\sqrt{m(m + 9)^3 t^{12}} = 4\sqrt{(m + 9) \sqrt{m(m + 9)} t^3}.
\]
Therefore, we have \(g_4(t) \geq 0\) for all \(t \geq 0\) if and only if
\[
4\sqrt{(m + 9) \sqrt{m(m + 9)} - 2\sqrt{27n^2 + 4m^2} \geq 0,
\]
which is equivalent to
\[
|n| \leq \frac{2}{3} \sqrt{(m + 9) \sqrt{m(m + 9)} - m^2}.
\]

**Application 4.** If \(x, y, z\) are real numbers, then (18)
\[
(x^2 + y^2 + z^2)^2 + 2(x^3 y + y^3 z + z^3 x) \geq 3(xy^3 + yz^3 + zx^3).
\]

**Proof.** We have
\[
A = 2, \quad B = 0, \quad C = 2, \quad D = -3, \quad E = 1, \quad F = 26,
\]
and hence
\[
g_4(t) = 15t^4 - 26t^3 + 9t^2 + 2 = (t - 1)^2(15t^2 + 4t + 2).
\]
Since \(g_4(t) \geq 0\) for all \(t \geq 0\), the proof is completed (Theorem 2.3).

To analyse the equality cases, we apply Proposition 2.5. Since the polynomial \(g_4(t)\) has the nonnegative double roots 1, we get the equality conditions
\[
(x + y + z)(x - y)(y - z)(z - x) \geq 0
\]
and
\[ w^3 - 3w^2 + \frac{27}{13} = 0, \]
which lead to the equality case \( x/w_1 = y/w_2 = z/w_3 \) (or any cyclic permutation), where \( w_1 \approx -0.7447, w_2 \approx 1.0256, w_3 \approx 2.7191. \)

**Application 5.** If \( x, y, z \) are real numbers, then
\[
10 \sum x^4 + 64 \sum x^2 y^2 \geq 33 \sum xy(x^2 + y^2).
\]

**Proof.** We have
\[
A = \frac{32}{5}, \quad B = 0, \quad C = D = \frac{-33}{10}, \quad E = F = 11,
\]
and hence
\[
5g_4(t) = 225t^4 - 55t^3 - 39t^2 + 4 = (5t + 2)^2(9t^2 - 5t + 1).
\]
Since \( g_4(t) \geq 0 \) for all \( t \geq 0 \), the proof is completed (Theorem 2.3).

Since \( C = D \), according to Corollary 2.7, equality holds when \( x/w = y = z \), where \( w \) is a double real root of the polynomial
\[
h(w) = w^4 + 2Cw^3 + (2A + B)w^2 + 2(B + C)w + A + 2C + 2
= \frac{1}{5}(5w^4 - 33w^3 + 64w^2 - 33w + 9)
= \frac{1}{5}(w - 3)^2(5w^2 - 3w + 1).
\]
Therefore, equality occurs for \( x/3 = y = z \) (or any cyclic permutation).

**REFERENCES**


