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ON GENERALIZATION OF HARDY-TYPE INEQUALITIES

K. RAUF, S. PONNUSAMY, AND J. O. OMOLEHIN

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILORIN, ILORIN, NIGERIA.
krauf@unilorin.edu.ng

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY MADRAS, CHENNAI- 600 036, INDIA.
samy@iitm.ac.in

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILORIN, ILORIN, NIGERIA.
omolehin_joseph@yahoo.com

ABSTRACT. This paper is devoted to some new generalization of Hardy-type integral inequalities and the reversed forms. The study is to determine conditions on which the generalized inequalities hold using some known hypothesis. Improvement of some inequalities are also presented.

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1. INTRODUCTION

The following famous classical inequality was proved in 1920 by G. H. Hardy (see, [14]): If $1 < p < \infty$, $A_n = \sum_{k=1}^{n} a_k$ and $a_n = \{a_k\}$ is a sequence of non-negative real numbers, then

$$\sum_{n=1}^{\infty} \left|\frac{1}{n}A_n\right|^p \leq C_p \sum_{n=1}^{\infty} |a_n|^p \tag{1.1}$$

and in 1925 he proved the continuous counterpart:

Theorem 1.1. *Let $f(x)$ be a non-negative p-integrable function defined on $(0,\infty)$, and $p > 1$. Then, f is integrable over the interval $(0,x)$ for each x and the following inequality:*

$$\int_0^{\infty} \left[\frac{1}{x}\Big(\int_0^x f(y)dy\Big)\right]^p dx \leq \Big(\frac{p}{p-1}\Big)^p \int_0^{\infty} f(x)^p dx \tag{1.2}$$

holds, where $\left(\frac{p}{p-1}\right)^p$ is the best possible constant (see [15]).

This inequality was developed in his attempt to provide an elementary proof to the following famous Albert Hilbert double series theorem [17, 43]:

Theorem 1.2. *If $\sum_{m=1}^{\infty} a_m^2 < \infty$ and $\sum_{n=1}^{\infty} b_n^2 < \infty$, where $a_m \geq 0$ and $b_n \geq 0$, then the double series: $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n}$ converges. In particular,*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} \leq \pi\Big(\sum_{m=1}^{\infty} a_m^2\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{\infty} b_n^2\Big)^{\frac{1}{2}} \tag{1.3}$$

.

In his attempt to simplify this theorem, he needed an estimate for arithmetic means of the form:

$$\sum_{n=1}^{\infty} \left|\frac{1}{n}A_n\right|^2 \leq C_2 \sum_{n=1}^{\infty} |a_n|^2$$

with both a_n and A_n are as defined above. This lead him to inequality (1.2).

In 1928, Hardy[16] obtained a generalized form of (1.2), namely that if $p \geq 1$ and $k \neq 1$, then

$$\int_0^{\infty} x^{-k}\left(\int_0^x f(t)dt\right)^p dx \leq \Big(\frac{p}{k-1}\Big)^p \int_0^{\infty} x^{p-k} f(x)^p dx (p \geq 1, k > 1) \tag{1.4}$$

and also the dual form of this inequality

$$\int_0^{\infty} x^{-k}\left(\int_x^{\infty} f(t)dt\right)^p dx \leq \Big(\frac{p}{1-k}\Big)^p \int_0^{\infty} x^{p-k} f(x)^p dx (p \geq 1, k < 1). \tag{1.5}$$

The constant $(\frac{p}{|k-1|})^p$ is the best possible in both cases, see [16] (see also [17], Chapter 9, Theorem 330, p. 245).

Furthermore, Hardy[16] pointed out (see, [17], Chapter 9, Theorem 347, p. 256) that if k and f satisfy the conditions of the above results, then (1.4) and (1.5) hold in the reversed direction with $0 < p \leq 1$.

Thereafter, inequality (1.2) was extended and generalized in many direction, for example, if

$$T : L^p(\mathbb{R}) \to L^p(\mathbb{R})$$

where T is an integral operator of the form:

$$(Tf)(x) = \int_{-\infty}^{x} K(x,y)f(y)dy$$

or

$$(T^*f)(x) = \int_x^\infty K(y,x)f(y)dy$$

then, Hardy's inequality is expressible in the operator form as

$$\int_0^\infty (Tf)(x)^p dx \le A(K,p) \int_0^\infty f(x)^p dx \tag{1.6}$$

where $A(K,p)$ is a constant independent of f, $p > 1$ and $K(x,y) = \frac{1}{x}$ if $y \le x$ and 0 otherwise.

Hardy's inequality has many applications in analysis (see [6]) most especially in the study of Fourier series ([33]), theory of ordinary differential equations ([5]) and in providing bounds to integral operators ([12]). Due to its usefulness, this inequality has been extensively studied and generalized in various directions by a number of researchers. Some of those who have worked on this inequality are: [9, 11, 19, 20, 28, 35, 7].

However, in the early seventies, a new dimension was introduced into inequality (1.2) and emphasis was later shifted to finding the necessary and sufficient conditions on the non-negative weight functions ω and ν such that, the norm inequality:

$$\|\omega Tf\|_p^p \le A(K,p)\|f\nu\|_p^p \tag{1.7}$$

is valid, where $p > 1$, f is a non-negative function defined on $[0,\infty]$, $A(K,p)$ is a constant depending on K and p but independent of the function f and $K(x,y) = \frac{1}{x}$ if $y \le x$ and 0 otherwise. We observed that when $\omega(x) = x^{-1}$ and $K(x,y) = \nu(x) = 1$, $X = \mathbb{R}$ then, inequality (1.7) is equivalent to (1.2).

In particular, Tomaselli [41] and Talenti [40] investigated independently the necessary and sufficient conditions on the non-negative weight functions ω and ν which ensure that the inequality:

$$\left(\int_0^\infty \left(\omega(x)\int_0^x f(t)dt\right)^p dx\right)^{\frac{1}{p}} \le C\left(\int_0^\infty \left(f(x)\nu(x)\right)^p dx\right)^{\frac{1}{p}} \tag{1.8}$$

holds, where f and $C = A(K,p)$ is as defined in (1.7). It can be readily observed that (1.8) reduced to (1.2) with $\omega(x) = x^{-1}$ and $\nu(x) = 1$.

Muchenhoupt [26] studied inequality (1.6) and gave conditions on the non-negative weight functions ω and ν such that (1.7) is valid. He raised the question that given the weight function ω, under what condition will there exist a weight function ν, such that

$$\int_X (Tf)(x)^p \omega d\mu \le \int_X f(x)^p \nu d\mu \tag{1.9}$$

holds for all $f \ge 0$.

In their attempt to simplify this problem, Kerman and Sawyer[22] provided a partial solution to this question and two new open problems were posed. That is, the characterization of weights ω for which there exist $\nu < \infty$ μ-almost everywhere such that (1.9) holds, where T is a sublinear operator and secondly, for $1 < p, q < \infty$, those weight functions ω and ν are to be characterized when T maps $L^p(\nu)$ to $L^q(\omega)$ such that:

$$\left(\int_X (Tf)(x)^q \omega d\mu\right)^{\frac{1}{q}} \le \left(\int_X f(x)^p \nu d\mu\right)^{\frac{1}{p}} \tag{1.10}$$

holds for all $f \ge 0$, $\nu < \infty$, and for every μ-almost everywhere on X. Problem 1 has been treated partially when T is the Hardy-Littlewood maximal function; see [42] while Problem 2 has been treated partially in [36] for the case of fractional integrals.

Rauf and Imoru[31] provided partial solution to the open problems when T is a sublinear operator while Rauf and Omolehin[32] provided partial solution to the same problems in the case in which T is a non-linear integral operator.

Bradley[8] studied Hardy's inequality with mixed norms and showed that the generalized Hardy's inequality:

$$\left(\int_0^\infty \left(\omega(x)\int_0^x f(t)dt\right)^q dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty \left(f(x)\nu(x)\right)^p dx\right)^{\frac{1}{p}}$$

holds for non-negative weight function f defined on $[0,\infty]$ if and only if

$$\sup_{r>0}\left(\int_0^\infty \omega(x)^q dx\right)^{\frac{1}{q}}\left(\int_0^r \nu(x)^{-p'} dx\right)^{\frac{1}{p'}} \equiv K < \infty,$$

where $1 \leq p \leq q \leq \infty, \omega(x)$ and $\nu(x)$ are non-negative weight functions, p and p' are conjugate exponents, and K is a positive constant independent of f.

Beesack and Heining[6] considered the weighted case for negative powers and Heining [18] extended the result to the case where $p, q < 0$ and $0 < p, q < 1$. They investigated the reverse Hardy inequality:

$$\left(\int_0^\infty \left(f(x)\nu(x)\right)^p dx\right)^{\frac{1}{p}} \leq C\left(\int_0^\infty \left(u(x)\int_0^x f(t)d(t)\right)^q dx\right)^{\frac{1}{q}} \tag{1.11}$$

The dual version of (1.11) with necessary and sufficient conditions for the validity of the inequality were also considered.

In 1983, Andersen and Heinig [3] gave conditions on the non-negative weight functions $\omega(x)$ and $\nu(x)$ which ensure that the inequality of the form:

$$\left(\int_{-\infty}^\infty \left((Tf)(x)\omega(x)\right)^q dx\right)^{\frac{1}{q}} \leq C\left(\int_{-\infty}^\infty \left(f(x)\nu(x)\right)^p dx\right)^{\frac{1}{p}} \tag{1.12}$$

holds, where T is an integral operator, f a non-negative function, p and q are as defined above. Inequality (1.12) extended some of the earlier as well as recent extensions on classical Hardy's inequality (see, [10]). If $K(x, y) \equiv 1$ and $p = q$, the inequality (1.12) yields (1.8) from which (1.2) can be obtained.

Opic and Kufner [29] generalized this result to N-dimensional Hardy's inequality:

$$\left(\int_\Omega |f(x)|^q \omega(x)d(x)\right)^{\frac{1}{q}} \leq C\left(\sum_{i=1}^N \int_\Omega \left|\frac{\delta f(x)}{\delta x_i}\right|^p \nu_i(x)dx\right)^{\frac{1}{p}} \tag{1.13}$$

holds, where Ω is a domain in the N-dimensional Euclidean Space $\mathbb{R}^N$, p, q are positive real numbers and $\omega, \nu_1, \nu_2, \ldots, \nu_N$ are weight functions that is measurable and positive almost everywhere in Ω.

Adams [2] investigated special case of (1.13) by considering $\omega(x) = \nu_i(x) \equiv 1$, $i = 1, 2, \ldots, N$ and have

$$\left(\int_\Omega |f(x)|^q dx\right)^{\frac{1}{q}} \leq C\left(\int_\Omega |\nabla f(x)|^p dx\right)^{\frac{1}{p}} \tag{1.14}$$

holds, for continuous function $f(x)$ defined on $(0,\infty)$, $X = (X_1, X_2, \ldots, X_N)$, $\nabla f(x) = \left(\frac{\delta f(x)}{\delta x_1}, \frac{\delta f(x)}{\delta x_2}, \ldots, \frac{\delta f(x)}{\delta x_N}\right)$, $1 < p < N$, $1 < q < \frac{N_p}{N-p}$ and $|\nabla f(x)|^p = \sum_{i=1}^N \left|\frac{\delta f(x)}{\delta x_i}\right|^p$ and Ω is a bounded domain with Lipschitzian boundary $\delta\Omega$ where the admissible values of the parameter q may change. This is called Sobolev inequality.

Another special case of (1.14) was considered in literature with $p = q = 2$ as:

$$\int_\Omega |f(x)|^2 dx \le C^2 \int_\Omega |\nabla f(x)|^2 dx$$

holds. This inequality is called Friedrichs inequality.

Also, for all functions $f(x)$ whose mean value over Ω is zero:

$$\int_\Omega f(x)dx = 0$$

is called the Poincare inequality. See, [29, 30, 34].

Finally, by replacing f with $f^{\frac{1}{p}}$ in inequality (1.2) and letting $p \to \infty$, we have the limiting inequality

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right) dx \le e \int_0^\infty f(x)dx$$

This is called the Knopp's-Poly$\acute{a}$ inequality. For further development, remarks, extensions, generalizations and applications of inequalities (1.2), (1.3), (1.4), (1.5), (1.9), (1.10) and (1.11), see for instance, [1, 4, 24, 25, 27, 13] and the references cited therein.

This work is, therefore, devoted to Hardy-type inequalities and to some modifications and consequences. The aim is to determine conditions on the data of our problem. These were done by introducing n-terms of functions for all $n \in \mathbb{N}$ on a multiple Hardy integral operator and by making one of the weight functions a power function.

Throughout this paper, $p > 1$ except otherwise stated, We shall use f to be integrable or $f \in L$ or $\int f(x)dx$ exists whenever f is measurable and $\int |f(x)|\, dx < \infty$. Hence, if f is an integrable function, then $f \in L$ or $\int f(x)\, dx$ exists whenever f is measurable and $\int |f(x)|\, dx < \infty$.

2. MAIN RESULTS

In this section, we let (X, ζ, μ) be a σ-finite measure space, $K(x, y)$ be a nonnegative and measurable on $X \times X$ and T a positive linear operator defined for nonnegative functions on the measure space. It is on record that [39] and [38] dealt with some weighted inequalities for a multiple Hardy operator T_n of the form:

$$T_n = \int_a^x \int_a^{x_1} \cdots \int_a^{x_n} f(t)\, dt\, dx_n \cdots dx_1$$

They derived sufficient conditions for the validity of the corresponding multiple Hardy inequality.

Theorem 2.1. *Let $p_1, \dots, p_n > 0$ such that $\sum_{k=1}^n \frac{1}{p_k} = 1$ and suppose ω_k are weight functions on X. For a positive function f on $(0, \infty)$, we define the operator T by $\int_X K(x, y)f(y)dy$. Let f_k be p-integrable positive function defined on $(0, \infty)$ for $k = 1, \dots, n + 1$. Then, there exist weight functions ν_k, finite μ-almost everywhere on X such that:*

$$\prod_{k=1}^n \left(\int_{X_k} (Tf_k)^{p_k} \omega_k d\mu_k\right)^{\frac{1}{p_k}} \le C(K, p) \prod_{k=1}^n \left(\int_{X_k} f_k^{p_k} \nu_k d\mu_k\right)^{\frac{1}{p_k}} \tag{2.1}$$

holds, if and only if there exist positive functions Φ on X with $\int_X (T\Phi)^{p_k} \omega_k d\mu < \infty$ equivalently to $\Phi^{1-p_k} T^*((T\Phi)^{p_k-1}\omega_k) < \infty$

Proof. We employ hypothesis of induction on n. When $n = 2$ we have $p_1 p_2 > 0$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$ then (2.1) holds for a particular case of $p_1, p_2 > 1$ (see, [31]). Suppose (2.1) holds for $n \geq 2$. We claim that it holds for $n+1$, then we let $p_1, p_2, \ldots p_{n+1} > 0$ be any real numbers with $\sum_k^{n+1} \frac{1}{p_k} = 1$ and also $f_k \in L^{p_k}$, $k = 1, \ldots, n+1$. Then, for $p_k > 1$, we have $p_1 > 0$ and $\frac{p_1}{p_1 - 1} > 0$ therefore, $\prod_{k=1}^{n+1} \int_{X_k} (Tf_k)^{p_k} \omega_k d\mu_k$

$$\begin{aligned}
&= \prod_{k=2}^{n+1} \int_{X_k} T\left(f_1 \times f_k\right)^{p_k} \omega_k d\mu_k \\
&= \prod_{k=2}^{n+1} \int_{X_k} \left(\int_X (K(x,y)^{\frac{1}{p_1}} f_1 \Phi^{-(\frac{p_1-1}{p_1})} K(x,y)^{\frac{p_1-1}{p_1}} \Phi^{\frac{p_1-1}{p_1}} \right. \\
&\qquad \left. \times K(x,y)^{\frac{1}{p_k}} f_k \Phi^{-\frac{1}{p_k}} K(x,y)^{\frac{p_k-1}{p_k}} \Phi^{\frac{1}{p_k}}) d\mu \right)^{p_k} \omega_k d\mu_k \\
&\leq C(K, p_k) \int_{X_1} \left(\int_X (K(x,y)^{\frac{1}{p_1}} f_1 \Phi^{-(\frac{p_1-1}{p_1})} K(x,y)^{\frac{p_1-1}{p_1}} \Phi^{\frac{p_1-1}{p_1}}) d\mu \right)^{p_1} \omega_1 d\mu_1 \\
&\qquad \times \prod_{k=2}^{n+1} \int_{X_k} \left(\int_X \left(K(x,y)^{\frac{1}{p_k}} f_k \Phi^{-\frac{1}{p_k}} K(x,y)^{\frac{p_k-1}{p_k}} \Phi^{\frac{1}{p_k}} \right) d\mu \right)^{p_k} \omega_k d\mu_k \\
&= C(K, p_k)(\Psi \times \Gamma)
\end{aligned}$$

where

$$\begin{aligned}
\Psi &\leq \int_{X_1} \left[\left(\int_X K(x,y) f_1^{p_1} \Phi^{-(p_1-1)} d\mu \right) \left(\int_X K(x,y) \Phi \right)^{p_1-1} d\mu \right] \omega_1 d\mu_1 \\
&= \int_{X_1} f_1^{p_1} \nu_1 d\mu_1
\end{aligned}$$

and

$$\Gamma = \prod_{k=2}^{n+1} \int_{X_k} \left(\int_X \left(K(x,y)^{\frac{1}{p_k}} f_k \Phi^{-\frac{1}{p_k}} K(x,y)^{\frac{p_k-1}{p_k}} \Phi^{\frac{1}{p_k}} \right) d\mu \right)^{p_k} \omega_k d\mu_k.$$

Since, $\frac{p_k(p_1-1)}{p_1} > 0$ and for $k = 2, \ldots, n+1$, we have

$$\sum_{k=2}^{n+1} \frac{1}{p_k(p_1-1)/p_1} = \frac{p_1}{p_1-1} \sum_{n=2}^{n+1} \frac{1}{p_n} = \frac{p_1}{p_1-1}(1 - \frac{1}{p_1}) = 1$$

and by using the same argument $n+1$ times, then

$$\begin{aligned}
\Gamma = \prod_{k=2}^{n+1} \int_{X_k} \left(\left((f_n^{p_1/(p_1-1) \times p_n(p_1-1)/p_1} d\mu)^{\frac{p_1}{p_n(p_1-1)}} \right)^{\frac{(p_1-1)}{p_1}} \right)^{p_n} \nu_n d\mu_n \\
= \prod_{k=2}^{n+1} \int_{X_k} (f_n^{p_n} \nu_n) d\mu_n
\end{aligned}$$

Therefore,

$$\Psi \times \Gamma \leq \left(\int_{X_1} f_1^{p_1} \nu_1 d\mu_1 \right) \left(\int_{X_2} f_2^{p_2} \nu_2 \, d\mu_2 \right) \cdots \left(\int_{X_{n+1}} f_{n+1}^{p_{n+1}} \nu_{n+1} \, d\mu_{n+1} \right)$$

Hence,

$$\prod_{k=1}^{n}\left(\int_{X_k}(Tf_k)^{p_k}\omega_k d\mu_k\right)^{\frac{1}{p_k}} \le C(K,p)\prod_{k=1}^{n}\left(\int_{X_k} f_k^{p_k}\nu_k d\mu_k\right)^{\frac{1}{p_k}}$$

Otherwise, $f \equiv 0$. Inequality (2.1) is reversed with any $f < 0$ or both $\nu < 0$ and $\omega < 0$ and we have equality with $\omega_k \equiv \nu_k$ and $K(x,y) = C(k,p_k) \equiv 1$.

On the other hand, we assume that (2.1) holds for some $\nu_k < \infty$ μ-almost everywhere. By using the σ-finiteness of μ, we can obtain a positive function Φ such that $\int_X \Phi^{p_k}(\nu_k)d\mu < \infty$ then, $\Phi^{1-p_k}T^*((T\Phi)^{p_k-1}\omega_k) < \infty$ holds. If all the above conditions hold then the proof of the converse is easily obtained and this completes the proof of the theorem.

By making the same calculation with $p_k < 0$ on inequality (2.1), we noted that the inequality is reversed, if and only if there exist at least a $p_i \in p_k$ such that $0 < p_i < 1$. Also, if more than one p_i is positive then at least one positive p_i is less than one. The result is also valid for dual operator T^*. ∎

Theorem 2.2. *Let $p_k, q_k > 0$ such that $\sum_{i=1}^{k}\frac{1}{p_i} = 1$ and suppose $u_k = (\omega_k)^{\frac{1}{q}}$ are weight functions on X. Then, there exist weight functions ν_k, finite μ-almost everywhere on X such that the weighted norm inequality:*

$$\prod_{k=1}^{n}\left(\int_{X_k}(Tf_k)^{q_k}\omega_k d\mu_k\right)^{\frac{1}{q_k}} \le C(K,p,q)\prod_{k=1}^{n}\left(\int_{X_k} f_k^{p_k}\nu_k d\mu_k\right)^{\frac{1}{p_k}} \tag{2.2}$$

holds, for each $f_k \ge 0$, if and only if there are positive functions Φ^{p_k} on X satisfying $\Phi(y)^{p_k} \le \nu_k$ and $C(K,p_k,q_k)$ is a constant independent of f.

Proof. We have

$$\begin{aligned}
&\prod_{k=1}^{n}\left(\int_{X_k}(Tf_k)^{q_k}\omega_k d\mu_k\right)^{\frac{1}{q_k}}\\
=\ &\sup_{x<\infty}\left(\prod_{k=1}^{n}u_k(x)\int_X K(x,y)^{\frac{1}{p_1}}f_k(y)K(x,y)^{\frac{p_1-1}{p_1}}dy\right)\\
\le\ &\sup_{z<x} ess\prod_{k=1}^{n}\left(K(x,z)^{\frac{1}{p_1}}u_k(x)\int_X K(x,y)^{\frac{p_1-1}{p_1}}f_k(y)dy\right)\\
=\ &\left(\sup_{z<x} ess K(x,z)^{\frac{1}{p_1}}u_1(x)\int_X K(x,y)^{\frac{p_1-1}{p_1}}\Phi(y)^{-1}f_1(y)\Phi(y)dy\right)\\
&\times\left(\sup_{z<x} ess\prod_{k=2}^{n+1}K(x,z)^{\frac{1}{p_k}}u_k(x)\int_X K(x,y)^{\frac{p_k-1}{p_k}}\Phi(y)^{-1}f_k(y)\Phi(y)dy\right).
\end{aligned}$$

Since, $u(x)$ and $\Phi(x)$ depend on p and q with constant C and also the integral

$$\begin{aligned}
&\sup_{z<x} ess K(x,z)^{\frac{1}{p_1}}u_1(x)(\sup_{t>x} ess K(t,x)^{\frac{1}{p_1}}u_1(t))^{-1}\\
\le\ &\sup_{z<x} ess K(x,z)^{\frac{1}{p_1}}u_1(t)(\sup_{z>x} ess K(x,z)^{\frac{1}{p_1}}u_1(t))^{-1} = 1
\end{aligned}$$

with $K(t,.)$ non-decreasing then, the result follows by Hölder's inequality. ∎

Theorem 2.3. *Let $1 < p_k \le q_k < \infty$ with $p_1,\dots,p_n > 0$ and $q_1,\dots,q_n > 0$ such that $\sum_{k=1}^{n}\frac{1}{p_k} = 1$ and suppose $u_k = (\omega_k)^{\frac{1}{q}}$ are weight functions on X. Then, there exist weight* *functions ν_k, finite μ-almost everywhere on X such that the weighted inequality:*

$$\prod_{k=1}^{n}\left(\int_{X_k}(Tf_k)^{q_k}\omega_k d\mu_k\right)^{\frac{1}{q_k}} \leq C(K,p,q)\prod_{k=1}^{n}\left(\int_{X_k} f_k^{p_k}\nu_k d\mu_k\right)^{\frac{1}{p_k}} \tag{2.3}$$

holds, for each $f_k \geq 0$, if and only if there are positive functions Φ^{p_k} on X satisfying $\Phi(y)^{p_k} \leq \nu_k$ with $s(x) \leq (\int_X K(y,z)\Phi(z)^{-p}dz)^{\frac{1}{p+1}}$ and $C(K,p_k,q_k)$ is a constant independent of f.

Proof. The integral

$$\prod_{k=1}^{n}\int_{X_k}(Tf_k)^{q_k}\,\omega_k d\mu_k$$

$$\leq \prod_{k=1}^{n}\int_{X_k} u(x)^q\left(\left(\int_X K(x,y)(f(y)\Phi(y)s(y))^{p_k}d\mu(y)\right)^{\frac{q_k}{p_k}}\right.$$

$$\left.\times\left(\int_X K(x,y)\Phi(y)^{-p_k'}s(y)^{-p_k'}d\mu(y)\right)^{\frac{q_k}{p_k'}}\right)d\mu_k(x)$$

$$\leq (p_1'+1)^{\frac{q_1}{p_1'}}\left(\int_{X_1}\left(\int_X K(x,y)u(x)^{q_1}(K(y,z)\Phi(z)^{-p_1'}dz)^{\frac{q_1}{p_1'(p_1'+1)}}d\mu(x)\right)^{\frac{p_1}{q_1}}\right.$$

$$\left.\times(f(y)\Phi(y)s(y))^{p_1}d\mu_1(y)\right)^{\frac{q_1}{p_1}}$$

$$\times\prod_{k=2}^{n+1}(p_k'+1)^{\frac{(n-1)q_k}{p_k'}}\left(\int_{X_k}\left(\int_X K(x,y)u(x)^{q_k}(K(y,z)\Phi(z)^{-p_k'}dz)^{\frac{q_k}{p_k'(p_k'+1)}}d\mu(x)\right)^{\frac{p_k}{q_k}}\right.$$

$$\left.\times(f(y)\Phi(y)s(y))^{p_k}d\mu(y)\right)^{\frac{q_k}{p_k}}$$

by Hölder's and Minkowski integral inequalities and since $u(x)$ and $\Phi(x)$ depend on $p's$ and $q's$ with constant C and that

$$\int_X K(x,y)\Phi(x)^{-p'}dx \leq C^{p'}\left(\int_X K(z,y)u(z)^q dz\right)^{\frac{-p}{q}}$$

then, we have

$$\prod_{k=1}^{n}\left(\int_{X_k}(Tf_k)^{q_k}\omega_k d\mu_k\right)^{\frac{1}{q_k}} \leq C(K,p,q)\prod_{k=1}^{n}\left(\int_{X_k} f_k^{p_k}\nu_k d\mu_k\right)^{\frac{1}{p_k}}$$

where, $C(K,p,q) = (p'+1)^{\frac{n(q^2+pp')}{q^2p'}}(p')^{\frac{-np}{q^2}}C^{\frac{n(q+pp')}{q(p'+1)}}$ where the conjugate of q is defined in the same way as conjugate of p. We note that Theorem 2.1 holds for the following corollaries if we define kernel $K(x,y)$ of operator T by $\Phi(x-y)$ and e^{-xy} respectively. ∎

Corollary 2.4. *Let $\Phi, \omega_k \geq 0$ be locally integrable with respect to Lebesque measure on $\mathbb{R}^n$ and suppose $\Phi(x) = \Phi(|x|)$ is non-increasing as a function of $|x|$. Define the convolution operator T by*

$$T\left(\prod_{k=1}^{n} f_k\right)(x) = (\Phi^* f)(x) = \int_{\mathbb{R}^n}\Phi(x-y)\prod_{k=1}^{n} f_k(y)dy$$

for every fixed $p_k \in (1,\infty)$ *and a constant* $C > 0$*, depending on* p_k *and* q_k*. Then, there exist weight functions* $\nu_k < \infty$ *finite* μ*-almost everywhere on* $\mathbb{R}^n$ *such that the weighted inequality:*

$$\prod_{k=1}^{n}\left(\int_{\mathbb{R}^n}(Tf_k)^{q_k}\omega_k d\mu_k(x)\right)^{\frac{1}{q_k}} \leq C(K,p,q)\prod_{k=1}^{n}\left(\int_{\mathbb{R}^n} f_k^{p_k}\nu_k d\mu_k\right)^{\frac{1}{p_k}} \tag{2.4}$$

holds, for each $f_k \geq 0$*, if and only if for all* $y \in \mathbb{R}^n$ *such that for all positive functions* $\Phi(y)^{p_k} \leq \nu_k$ *with* $u_k = \omega^{1/q}$ *and* $C(K, p_k, q_k)$ *a constant independent of* f*.*

Corollary 2.5. *Let* $\omega_k \geq 0$ *be a locally integrable with respect to Lebesque measure on* $\mathbb{R}_+ = (0,\infty)$ *and denote the Laplace transform (T) of* f_k *on* $\mathbb{R}_+$ *by*

$$T\left(\prod_{k=1}^{n} f_k\right)(x) = \int_0^\infty e^{-xy}\prod_{k=1}^{n} f_k(y)dy$$

$x \in \mathbb{R}_+$ *for a fixed* $p_k \in (1,\infty)$ *and a constant* $C > 0$*, depending on* p_k *and* q_k*. Then, there exist weight functions* $\nu_k < \infty$ *finite* μ*-almost everywhere such that:*

$$\prod_{k=1}^{n}\left(\int_{\mathbb{R}^n}(Tf_k)^{q_k}\omega_k d\mu_k(x)\right)^{\frac{1}{q_k}} \leq C(K,p,q)\prod_{k=1}^{n}\left(\int_{\mathbb{R}^n} f_k^{p_k}\nu_k d\mu_k\right)^{\frac{1}{p_k}} \tag{2.5}$$

holds, if and only if $(T\omega)(x) < \infty$*.*

3. MULTI-DIMENSIONAL HARDY-TYPE INEQUALITIES WITH WEIGHTS

In literature, one dimensional Hardy-type inequalities has received rigorous treatment in various directions while the multi-dimensional case has been given little attention. The characterizations for pairs of weights (u, v) such that the operator $T_2 : L^p(\mathbb{R}^2_+, v) \to L^q(\mathbb{R}^2_+, u)$, is bounded in the case when $1 < p \leq q < \infty$ were treated in [37], that is. If $p > 1$ then

$$\int_0^\infty\int_0^\infty |T_2 f(x,y)|^p dxdy \leq \left(\frac{p}{p-1}\right)^{2p}\int_0^\infty\int_0^\infty |f(x,y)|^p dxdy$$

. However, it has recently been pointed out that the proof of classical Hardy integral inequality (1.2) will also work for the corresponding multidimensional L^p-spaces that is the n-dimensional case of classical Hardy operator:

$$\int_0^\infty\cdots\int_0^\infty |T_n f(x_1,\cdots,x_n)|^p dx_1\cdots dx_n \leq C\int_0^\infty\cdots\int_0^\infty |f(x_1,\cdots,x_n)|^p dx_1\cdots dx_n$$

where

$$C = \left(\frac{p}{p-1}\right)^{np}$$

and

$$T_n f(x_1,\cdots,x_n) = \frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\cdots,t_n)dt_1\cdots t_n$$

Also, the corresponding weighted mixed-norm version can be proved similarly see[25].

The multidimensional generalized Hardy-Polya type inequality described by convex functions are discussed in this section. We use the following notation throughout the remaining sections:

$$\int_{\mathbf{t}}^{\mathbf{b}} := \int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}$$

and we have similar expression for

$$\int_0^{b_1}\cdots\int_0^{b_n},\int_0^{x_1}\cdots\int_0^{x_n},\int_0^{\infty}\cdots\int_0^{\infty}$$

and

$$\int_X\cdots\int_X$$

where b_i's, x_i's and v_i's are the components of $\mathbf{b}$, $\mathbf{x}$ and $\mathbf{v}$ for all $i = 1,\dots,n \in \mathbb{Z}_+$ respectively. All functions are measurable and $g(x) = \mathbf{x}^p$ except otherwise stated. Based on the methods in [32], [21] and [28], we further make some new generalizations of multidimensional Hardy-type integral inequalities by introducing real function $g(\mathbf{x})$. Some multidimensional Hardy-type integral inequalities are obtained. Some applications are also considered. First, we give some lemmas which are fundamental to prove certain inequalities in our context.

Lemma A. *Let $0 < b_i \le \infty$, $i = 1, 2\dots,n \in \mathbb{Z}_+$, $-\infty \le a < c \le \infty$ and let Φ be a positive function $[a,c]$.*

(A) If Φ is convex, then,

$$\int_{\mathbf{0}}^{\mathbf{b}}\Phi\left(\frac{1}{x_1\dots x_n}\int_{\mathbf{0}}^{\mathbf{x}}f(\mathbf{t})d\mathbf{t}\right)\frac{d\mathbf{x}}{x_1\dots x_n}$$
$$\le\int_{\mathbf{0}}^{\mathbf{b}}\Phi(f(\mathbf{x}))\left(1-\frac{x_1}{b_1}\right)\dots\left(1-\frac{x_n}{b_n}\right)\frac{d\mathbf{x}}{x_1\dots x_n}$$

for every function f on $(\mathbf{0},\mathbf{b})$ such that $a < f(\mathbf{x}) < c$.

(B) If Φ is concave, then,

$$\int_{\mathbf{0}}^{\mathbf{b}}\Phi\left(\frac{1}{x_1\dots x_n}\int_{\mathbf{0}}^{\mathbf{x}}f(\mathbf{t})d\mathbf{t}\right)\frac{d\mathbf{x}}{x_1\dots x_n}$$
$$\ge\int_{\mathbf{0}}^{\mathbf{b}}\Phi(f(\mathbf{x}))\left(1-\frac{x_1}{b_1}\right)\dots\left(1-\frac{x_n}{b_n}\right)\frac{d\mathbf{x}}{x_1\dots x_n}$$

for every function f on $(\mathbf{0},\mathbf{b})$ such that $a < f(\mathbf{x}) < c$.

Lemma B. *Let $\mathbf{b} \in (\mathbf{0},\infty]$, $-\infty \le a < c \le \infty$ and Φ be a positive function on $[a,c]$. Suppose that the weight function u defined on $(\mathbf{0},\mathbf{b})$ is nonnegative such that $\frac{u(\mathbf{x})}{x_1^2\dots x_n^2}$ is locally integrable on $(\mathbf{0},\mathbf{b})$ and the weight function v is defined by*

$$v(\mathbf{t}) = t_1\dots t_n\int_{\mathbf{t}}^{\mathbf{b}}\frac{u(\mathbf{x})}{x_1^2\dots x_n^2}d\mathbf{x},\ \mathbf{t}\in(\mathbf{0},\mathbf{b}).$$

(C) If Φ is convex, then,

$$\int_{\mathbf{0}}^{\mathbf{b}}u(\mathbf{x})\Phi\left(\frac{1}{x_1\dots x_n}\int_{\mathbf{0}}^{\mathbf{x}}f(\mathbf{t})d\mathbf{t}\right)\frac{d\mathbf{x}}{x_1\dots x_n}$$
$$\le\int_{\mathbf{0}}^{\mathbf{b}}v(\mathbf{x})\Phi(f(\mathbf{x}))\frac{d\mathbf{x}}{x_1\dots x_n}$$

holds for every function f on $(\mathbf{0},\mathbf{b})$ such that $a < f(x_1,\dots,x_1) < c$.

(D) If Φ is concave, then,

$$\int_{\mathbf{0}}^{\mathbf{b}} u(\mathbf{x})\Phi\left(\frac{1}{x_1\dots x_n}\int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{t}\right)\frac{d\mathbf{x}}{x_1\dots x_n}$$

$$\geq \int_{\mathbf{0}}^{\mathbf{b}} v(\mathbf{x})\Phi(f(\mathbf{x}))\frac{d\mathbf{x}}{x_1\dots x_n}$$

holds for every function f on $(\mathbf{0},\mathbf{b})$ such that $a < f(x_1,\dots,x_1) < c$.

Lemma 3.1. *If Φ is positive and continuous on $[\mathbf{0},\infty)$, f is a non-negative function on $[\mathbf{0},\mathbf{b}]$, $0 < x_i < b_i \leq \infty$ $(i = 1\dots n \in \mathbb{Z}_+)$ and λ is non-decreasing on $[\mathbf{0},\infty]$, assume*

$$0 < \int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi(f(\mathbf{x}))dg(x_1)\dots dg(x_n) < \infty$$

for each continuous and non-decreasing function g on $[\mathbf{0},\infty)$ and $\mathbf{v}\in\mathbb{R}^n$ such that $0 < v_i < \infty$ with,

(E) Φ convex, then,

$$\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi\left(L^{-1}\int_{\mathbf{0}}^{\mathbf{x}} f(v_1,\dots,v_n)d\lambda(v_1)\dots d\lambda(v_n)\right)dg(x_1)\dots dg(x_n)$$

$$\leq \int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi(f(\mathbf{x}))dg(x_1)\dots dg(x_n)$$

(F) Φ concave, then,

$$\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi\left(L^{-1}\int_{\mathbf{0}}^{\mathbf{x}} f(v_1,\dots,v_n)d\lambda(v_1)\dots d\lambda(v_n)\right)dg(x_1)\dots dg(x_n)$$

$$\geq \int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi(f(\mathbf{x}))dg(x_1)\dots dg(x_n)$$

where $L = \int_{\mathbf{0}}^{\infty} d\lambda(v_1)\dots d\lambda(v_n)$

Proof. Applying the iterative integrals of functions g and f on measurable set X with measure λ and Y with measure μ (σ-finite) then,

$$\int_{\mathbf{X}} g(y_1,\dots,y_n)\left(\int_{\mathbf{Y}} f(\mathbf{x})d\lambda_1\dots d\lambda_n\right)d\mu_1\dots d\mu_n$$

$$= \int_{(X_1\dots X_n)\times(Y_1\dots Y_n)} f(\mathbf{x})\times g(y_1,\dots,y_n)(d\mu_1\dots d\mu_n\times d\lambda_1\dots d\lambda_n)$$

$$= \int_{(Y_1\dots Y_n)} g(\mathbf{x})\left(\int_{(X_1\dots X_n)} f(y_1,\dots,y_n)d\mu_1\dots d\mu_n\right)d\lambda_1\dots d\lambda_n$$

Since Φ is convex in (E) above, taking $L = \int_{\mathbf{0}}^{\infty}(d\lambda(v_1)\dots d\lambda(v_n))$ and by imploring Fubini's theorem, then,

$$\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi\left(L^{-1}\int_{\mathbf{0}}^{\mathbf{x}} f(v_1,\dots,v_n)d\lambda(v_1)\dots d\lambda(v_n)\right)dg(x_1)\dots dg(x_n)$$

$$\leq L^{-1}\int_{\mathbf{0}}^{\infty} g(\mathbf{x})^{-p}dg(x_1)\dots dg(x_n)\int_{\mathbf{0}}^{\mathbf{x}} \Phi(f(v_1,\dots,v_n))d\lambda(v_1)\dots d\lambda(v_n)$$

$$\leq L^{-1}\int_{\mathbf{0}}^{\infty} d\lambda(v_1)\dots d\lambda(v_n)\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi(f(\mathbf{x}))dg(x_1)\dots dg(x_n)$$

$$= \int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{x}))g(\mathbf{x})^{-p}dg(x_1)\dots dg(x_n)$$

When Φ is concave, then the proof is easily obtained by reversing inequality (E). ∎

Corollary 3.2. *If $p > 1$, $f \geq 0$, $g(\mathbf{x}) = \mathbf{x}$ is continuous, non-decreasing on $[\mathbf{0},\infty)$. Let Φ be positive and continuous on $[\mathbf{0},\infty)$, and define $d\lambda(v_1)\dots d\lambda(v_n)$ by $(v_1^{\alpha-1}\dots v_n^{\alpha-1})dv_1\dots dv_n$ on $[0,1]$ and 0 for $v > 1$, $1 < \alpha \leq n$ and $n \in \mathbb{Z}_+$. Assume*

$$\int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{x})^p)d\mathbf{x} < \infty$$

with,

(G) Φ convex, then,

$$\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi\left(\left(\int_{\mathbf{0}}^{\mathbf{1}} f(v_1,\dots,v_n)d\lambda(v_1)\dots d\lambda(v_n)\right)^p\right)d\mathbf{x}$$

$$\leq \prod_{i=1}^{n}(\alpha_i-1)^{-p}\int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{x})^p)d\mathbf{x}$$

(H) Φ concave, then,

$$\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi\left(\left(\int_{\mathbf{0}}^{\mathbf{1}} f(v_1,\dots,v_n)d\lambda(v_1)\dots d\lambda(v_n)\right)^p\right)d\mathbf{x}$$

$$\geq \prod_{i=1}^{n}(\alpha_i-1)^{-p}\int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{x})^p)d\mathbf{x}$$

Proof. Since the integral of two or more variables of a summable functions can be obtained by successive integrations with respect to each variable separately or by pairs that is an iterative integral and with Φ convex on (G), we have

$$\int_{\mathbf{0}}^{\mathbf{b}} (g(\mathbf{x})^{-p})\Phi\left(\left(\int_{\mathbf{0}}^{\mathbf{1}} (v_1^{\alpha-1}\dots v_n^{\alpha-1})f(v_1,\dots,v_n)dv_1\dots dv_n\right)^p\right)d\mathbf{x}$$

$$\leq \int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{x})^p)\left(\int_{\mathbf{0}}^{\mathbf{1}} g(v_1,\dots,v_n)^{-1}(v_1^{\alpha-1}\dots v_n^{\alpha-1})dv_1\dots dv_n\right)^p d\mathbf{x}$$

result follows by substituting and integrating the inner integral on $[0,1]$ by single step of integration by part. Also, if there exist a continuous inverse which is necessarily concave on function Φ then (H) is proved using similar method with inequalities reversed. The results also hold if we assume $g(\mathbf{x}) = \mathbf{x}^k$ whenever $1 < k < \alpha \leq n \in \mathbb{Z}_+$. ∎

Corollary 3.3. *If $p > 1$, f is continuous, non-decreasing on $[\mathbf{0},\mathbf{b}]$. Let Φ be positive and continuous on $[\mathbf{0},\infty)$, and define $d\lambda(v_1)\dots d\lambda(v_n)$ by $(v_1^{\alpha-1}\dots v_n^{\alpha-1})dv_1\dots dv_n$ on $[0,1]$, $\alpha \in \mathbb{R}$ and λ is non-decreasing on $[\mathbf{0},\mathbf{1}]$. Assume*

$$0 < \int_{\mathbf{0}}^{\infty} \Phi(f(\mathbf{x})^p)d\mathbf{x} < \infty$$

if Φ is convex, then,

$$\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi\left(\left(\int_{\mathbf{0}}^{\mathbf{1}} f(v_1,\dots,v_n)d\lambda(v_1)\dots d\lambda(v_n)\right)^p\right)d\mathbf{x}$$

$$\leq (\alpha(1-k))^{-np}\int_{\mathbf{0}}^{\mathbf{b}}\Phi(f(\mathbf{x})^p)d\mathbf{x}$$

whenever $g(\mathbf{x})=\mathbf{x}^k$ is a decreasing function over $[0,1]$ and $1<k\in\mathbb{Z}_+$.

Proof. Using the same method in the proof of Corollary 3.2 and since g is decreasing on $[0,1]$, then we obtain, on using Chebyshev's integral inequality on

$$\int_{\mathbf{0}}^{\mathbf{b}}\Phi(f(\mathbf{x})^p)\left(\int_{\mathbf{0}}^{\mathbf{1}} g(v_1,\dots,v_n)^{-1}(v_1^{\alpha-1}\dots v_n^{\alpha-1})dv_1\dots dv_n\right)^p d\mathbf{x}$$

$$\leq \int_{\mathbf{0}}^{\mathbf{b}}\Phi(f(\mathbf{x})^p)\left(\int_{\mathbf{0}}^{\mathbf{1}} g(v_1,\dots,v_n)^{-1}dv_1\dots dv_n\right)^p$$

$$\times\left(\int_{\mathbf{0}}^{\mathbf{1}}(v_1^{\alpha-1}\dots v_n^{\alpha-1})dv_1\dots dv_n\right)^p d\mathbf{x}$$

and the result follows since g and λ are not similarly ordered, otherwise the inequality is reversed. Also, the inequality is reversed if they are not similarly ordered and p lies between $0<p<1$. See ([17], Theorem 43, see also section 5.8 page 123).

We distinguished two cases for this inequality:

(1) holds for p odd if $\alpha<0$ for any chosen k and

(2) holds for p even if $\alpha>0$ for any chosen k.

We obtain the corresponding reverse inequality if Φ has a continuous inverse which is necessarily concave. ∎

Theorem 3.4. *If $0<b_i\leq\infty$, $i=1,2\dots,n\in\mathbb{Z}_+$, $-\infty\leq a<c\leq\infty$ and let Φ be a positive function $[a,c]$. Let function f be defined on $(\mathbf{0},\mathbf{b})$ such that $a<f(\mathbf{x})<c$.*

(I) If Φ is convex, then,

$$\int_{\mathbf{0}}^{\mathbf{b}}\Phi\left(\frac{1}{(x_1\dots x_n)^\Gamma}\int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{t}\right)\frac{d\mathbf{x}}{x_1\dots x_n}$$

$$\leq\frac{b_i^{p+1}}{p+1}\int_{\mathbf{0}}^{\mathbf{b}}\Phi(f(\mathbf{x}))\left(1-\frac{x_1^{p+1}}{b_1^{p+1}}\right)\dots\left(1-\frac{x_n^{p+1}}{b_n^{p+1}}\right)d\mathbf{x}$$

for every positive function g on $(\mathbf{0},\mathbf{b})$ with $\Gamma=-(p+1)$ and $0\leq p<\infty$.

(J) If Φ is concave, then,

$$\int_{\mathbf{0}}^{\mathbf{b}}\Phi\left(\frac{1}{(x_1\dots x_n)^\Gamma}\int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{t}\right)\frac{d\mathbf{x}}{x_1\dots x_n}$$

$$\geq\frac{b_i^{p+1}}{p+1}\int_{\mathbf{0}}^{\mathbf{b}}\Phi(f(\mathbf{x}))\left(1-\frac{x_1^{p+1}}{b_1^{p+1}}\right)\dots\left(1-\frac{x_n^{p+1}}{b_n^{p+1}}\right)d\mathbf{x}$$

for every positive function g on $(\mathbf{0},\mathbf{b})$ with $\Gamma=-(p+1)$ and $0\leq p<\infty$.

Proof. The proof of the inequality (J) is practically the same as (I) by making the same calculation with Φ concave and the inequality is reversed, we give that of (I). If Φ is convex. Then, by Jensen's inequality and the Fubini theorem, (I) yields

$$\begin{aligned}
&\int_{\mathbf{0}}^{\mathbf{b}} \Phi\left(\frac{1}{(x_1 \ldots x_n)^{\Gamma}} \int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t}) d\mathbf{t}\right) \frac{d\mathbf{x}}{x_1 \ldots x_n} \\
&\leq \int_{\mathbf{0}}^{\mathbf{b}} \left(\int_{\mathbf{0}}^{\mathbf{x}} \Phi(f(\mathbf{t})) d\mathbf{t}\right) x^{-(\Gamma+1)} \ldots x^{-(\Gamma+1)} d\mathbf{x} \\
&= \int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{t})) \left(\int_{\mathbf{t}}^{\mathbf{b}} x^{-(\Gamma+1)} \ldots x^{-(\Gamma+1)} d\mathbf{x}\right) d\mathbf{t} \\
&= \frac{b_i^{p+1}}{p+1} \int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{t})) \left(1 - \frac{t_1^{p+1}}{b_1^{p+1}}\right) \ldots \left(1 - \frac{t_n^{p+1}}{b_n^{p+1}}\right) d\mathbf{t}
\end{aligned}$$

∎

Theorem 3.5. *If* $0 < b_i \leq \infty$, $i = 1, 2 \ldots, n \in \mathbb{Z}_+$, $-\infty \leq a < c \leq \infty$ *and let* Φ *be a positive function* $[a, c]$. *Let* $p \geq 1$ *and function* f *be defined on* $(\mathbf{0}, \mathbf{b})$ *such that* $a < f(\mathbf{x}) < c$.

(K) If Φ *is convex, then,*

$$\begin{aligned}
&\int_{\mathbf{0}}^{\mathbf{b}} \Phi\left(\frac{1}{(x_1 \ldots x_n)^{p}} \int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t}) d\mathbf{t}\right) \frac{d\mathbf{x}}{x_1 \ldots x_n} \\
&\leq p^{-n} \int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{x})) \frac{1}{\mathbf{x}^p} \left(1 - \frac{x_1^p}{b_1^p}\right) \ldots \left(1 - \frac{x_n^p}{b_n^p}\right) d\mathbf{x}
\end{aligned}$$

for every positive function g *on* $(\mathbf{0}, \mathbf{b})$.

(L) If Φ *is concave, then,*

$$\begin{aligned}
&\int_{\mathbf{0}}^{\mathbf{b}} \Phi\left(\frac{1}{(x_1 \ldots x_n)^{p}} \int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t}) d\mathbf{t}\right) \frac{d\mathbf{x}}{x_1 \ldots x_n} \\
&\geq p^{-n} \int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{x})) \frac{1}{\mathbf{x}^p} \left(1 - \frac{x_1^p}{b_1^p}\right) \ldots \left(1 - \frac{x_n^p}{b_n^p}\right) d\mathbf{x}
\end{aligned}$$

for every positive function g *on* $(\mathbf{0}, \mathbf{b})$.

Proof. The proof are completely similar to that of Theorem 3.4 and hence the details are omitted. The result also holds for $0 < p \leq 1$ and in fact if $p = 1$, we have lemma 2.1 in [21] and lemma 2.1 in [28] by defining weight function

$$v = t_1 \ldots t_n \int_{\mathbf{t}}^{\mathbf{b}} \frac{u(\mathbf{x})}{(x_1 \ldots x_n)^{p+1}} d\mathbf{x}$$

with weight function $u(\mathbf{x}) = 1$ and also holds whenever $u(\mathbf{x})$ is a power function whose power not equal to the value of p except possibly for power zero.

If $\Phi(t)$ is assuming t^p in Theorem 3.5, we obtain the following natural multidimensional Hardy-type inequalities and the reverse inequality. ∎

Corollary 3.6. *If* $0 < b_i \leq \infty$, $i = 1, 2 \ldots, n \in \mathbb{Z}_+$, $-\infty \leq a < c \leq \infty$ *and let* Φ *be a positive function* $[a, c]$. *Let function* f *be defined on* $(\mathbf{0}, \mathbf{b})$ *such that* $a < f(\mathbf{x}) < c$. *If*

$$0 < \int_{\mathbf{0}}^{\mathbf{b}} (x_1 \ldots x_n) f^p(\mathbf{x}) d\mathbf{x} < \infty$$

then,

(M) With $p > 1$ or $p < 0$ such that n is even, then

$$\int_{\mathbf{0}}^{\mathbf{b}} \Big(\frac{1}{(x_1 \dots x_n)^p} \int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t}) d\mathbf{t}\Big)^p \frac{d\mathbf{x}}{x_1 \dots x_n}$$
$$\leq \frac{1}{p^n} \left(\frac{p}{1-p}\right)^{np} \int_{\mathbf{0}}^{\mathbf{b}} \left(1 - \frac{x_1^{p-1}}{b_1^{p-1}}\right) \dots \left(1 - \frac{x_n^{p-1}}{b_n^{p-1}}\right) (f^p(\mathbf{x})) x_1 \dots x_n d\mathbf{x}$$

(N) With $0 < p < 1$, then

$$\int_{\mathbf{0}}^{\mathbf{b}} \Big(\frac{1}{(x_1 \dots x_n)^p} \int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t}) d\mathbf{t}\Big)^p \frac{d\mathbf{x}}{x_1 \dots x_n}$$
$$\geq \frac{1}{p^n} \left(\frac{p}{1-p}\right)^{np} \int_{\mathbf{0}}^{\mathbf{b}} \left(1 - \frac{x_1^{p-1}}{b_1^{p-1}}\right) \dots \left(1 - \frac{x_n^{p-1}}{b_n^{p-1}}\right) (f^p(\mathbf{x})) x_1 \dots x_n d\mathbf{x}$$

for every positive function g on $(\mathbf{0}, \mathbf{b})$.

Proof. Inequality (M) is proved for $p > 1$ using inequality (K) by assuming $\Phi(t) = t^p$, $t_i = s_i^{\frac{p-1}{p}}$, $x_i = y_i^{\frac{p-1}{p}}$, $b_i^{\frac{p}{p-1}} = a_i$ whenever $i = 1, \dots, n$ and the function

$$f(x_1, ..., x_n) = f(x_1^{\frac{p}{p-1}}, ..., x_n^{\frac{p}{p-1}}) x_1^{\frac{p}{p-1}-1} ... x_n^{\frac{p}{p-1}-1}$$

then, we have

$$\int_{\mathbf{0}}^{\mathbf{a}} \left(\frac{1}{(x_1 \dots x_n)^p} \int_{\mathbf{0}}^{\mathbf{x}} f(t_1^{\frac{p}{p-1}}, \dots, t_n^{\frac{p}{p-1}}) t_1^{\frac{p}{p-1}-1} \dots t_n^{\frac{p}{p-1}-1} d\mathbf{t}\right)^p \frac{d\mathbf{x}}{x_1 \dots x_n}$$
$$= \left(\frac{p-1}{p}\right)^{np} \int_{\mathbf{0}}^{\mathbf{a}} \left(\frac{1}{x_1^p \dots x_n^p} \int_0^{x_1^{\frac{p}{p-1}}} \dots \int_0^{x_n^{\frac{p}{p-1}}} f(s_1, \dots, s_n) ds_1 \dots ds_n\right)^p \frac{d\mathbf{x}}{x_1 \dots x_n}$$
$$= \left(\frac{p-1}{p}\right)^{np+n} \int_{\mathbf{0}}^{\mathbf{b}} \left(\int_{\mathbf{0}}^{\mathbf{y}} f(s_1, \dots, s_n) ds_1 \dots ds_n\right)^p y_1^{-p(p-1)} \dots y_n^{-p(p-1)} \frac{dy_1 \dots dy_n}{y_1 \dots y_n}$$
$$= \left(\frac{p-1}{p}\right)^{np+n} \int_{\mathbf{0}}^{\mathbf{b}} \left(\int_{\mathbf{0}}^{\mathbf{y}} f(s_1, \dots, s_n) ds_1 \dots ds_n\right)^p y_1^{-p(p-1)-1} \dots y_n^{-p(p-1)-1} dy_1 \dots dy_n$$

Also, from the left hand side of (K), we have

$$p^{-n} \int_{\mathbf{0}}^{\mathbf{a}} \frac{1}{x_1^p \dots x_n^p} f^p(x_1^{\frac{p}{p-1}}, \dots, x_n^{\frac{p}{p-1}}) x_1^{p(\frac{p}{p-1}-1)} \dots x_n^{p(\frac{p}{p-1}-1)} \left(1 - \frac{x_1^p}{a_1^p}\right) \dots \left(1 - \frac{x_n^p}{a_n^p}\right) d\mathbf{x}$$
$$= \left(\frac{p-1}{pp}\right)^n \int_{\mathbf{0}}^{\mathbf{a}} \left(1 - \frac{y_1^{\frac{p-1}{p}p}}{b_1^{\frac{p-1}{p}p}}\right) \dots \left(1 - \frac{y_1^{\frac{p-1}{p}p}}{b_1^{\frac{p-1}{p}p}}\right) (f^p(y_1, \dots, y_n)) y_1^{2-p} \dots y_n^{2-p} d\mathbf{x}$$
$$= \left(\frac{p-1}{pp}\right)^n \int_{\mathbf{0}}^{\mathbf{b}} \left(1 - \frac{y_1^{p-1}}{b_1^{p-1}}\right) \dots \left(1 - \frac{y_n^{p-1}}{b_n^{p-1}}\right) f^p(y_1, \dots, y_n) y_1^{\frac{-(p-1)^2}{p}} \dots y_n^{\frac{-(p-1)^2}{p}} dy_1 \dots dy_n$$

The parameter p involved in the proof of inequality (M) is greater than 1. However, they have analogous with a p less than 0 when n is even while $0 < p < 1$ resulted in a reversal of the sign of the inequality and (N) is proved. ∎

Remark 3.1. If $\Phi(t) = \exp(t)$ the analogue of Theorem 3.5 whenever $f \to \ln f^p$ in one dimensional case can be found in ([17], theorem 335, page 250). This type of inequality is called Knopp's inequality with reference ([23], cf. Remark 3.2) it was originally discovered by G. Polya (see Remark 2.3) and it is therefore refer to as Polya-Knopp's inequality. In our own case, we obtain the following natural multidimensional Hardy-type inequalities.

Corollary 3.7. *If* $0 < b_i \leq \infty$, $i = 1, 2 \ldots, n \in \mathbb{Z}_+$, $-\infty \leq a < c \leq \infty$ *and let* Φ *be a positive function* $[a, c]$. *Let function* f *be defined on* $(\mathbf{0}, \mathbf{b})$ *such that* $a < f(\mathbf{x}) < c$. *If*

$$0 < \int_{\mathbf{0}}^{\mathbf{b}} f(\mathbf{x}) d\mathbf{x} < \infty$$

then we write inequality (M) as:

$$\left(\int_{\mathbf{0}}^{\mathbf{b}} \left(\exp\left(\frac{1}{(x_1 \ldots x_n)^p} \int_{\mathbf{0}}^{\mathbf{x}} \ln f(\mathbf{t}) d\mathbf{t}\right)\right)^p d\mathbf{x}\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{e}{(4p^2-1)^{\frac{1}{p}}}\right)^n \left(\int_{\mathbf{0}}^{\mathbf{b}} t_i^{p(1-4p)+1} f^p(\mathbf{t}) \prod_{i=1}^{n} \left(1 - \frac{(t_i)^{4p^2-1}}{(b_i)^{4p^2-1}}\right) d\mathbf{t}\right)^{\frac{1}{p}}$$

for every positive function g *on* $(\mathbf{0}, \mathbf{b})$.

Proof. Our proof is partially related to Knopp's original idea see([23], page 211). Since the sum of a convex function is also a convex function then we have,

$$\left(\int_{\mathbf{0}}^{\mathbf{b}} \left(\exp\left(\frac{1}{(x_1 \ldots x_n)^p} \int_{\mathbf{0}}^{\mathbf{x}} \ln f(\mathbf{t}) d\mathbf{t}\right)\right)^p d\mathbf{x}\right)^{\frac{1}{p}}$$

$$= (\int_{\mathbf{0}}^{\mathbf{b}} (\exp\left(\frac{1}{(x_1 \ldots x_n)^p} \int_{\mathbf{0}}^{\mathbf{x}} \ln(t_1 \ldots t_n) f(\mathbf{t}) d\mathbf{t}\right)$$

$$\times \exp\left(\frac{-1}{(x_1 \ldots x_n)^p} \int_{\mathbf{0}}^{\mathbf{x}} \ln(t_1 \ldots t_n) d\mathbf{t}\right))^p d\mathbf{x})^{\frac{1}{p}}$$

Since $f(\mathbf{x}) = e^{\mathbf{x}}$ is convex and it is apparent that the local minimum of any convex function is also a global minimum then either by using Jensen's inequality or the Arithmetic-Geometric mean inequality(AG-inequality) we have,

$$\exp\left(\frac{1}{(x_1 \ldots x_n)^p} \int_{\mathbf{0}}^{\mathbf{x}} \ln(t_1 \ldots t_n) f(\mathbf{t}) d\mathbf{t}\right)$$

$$\leq \frac{1}{(x_1 \ldots x_n)^p} \int_{\mathbf{0}}^{\mathbf{x}} (t_1 \ldots t_n) f(\mathbf{t}) d\mathbf{t}$$

and

$$\frac{-1}{(x_1 \ldots x_n)^p} \int_{\mathbf{0}}^{\mathbf{x}} \ln(t_1 \ldots t_n) d\mathbf{t} = \frac{1}{(x_1 \ldots x_n)^{p-1}} (-\ln(x_1 \ldots x_n) + (\mathbf{1}))$$

Hence,

$$\int_{\mathbf{0}}^{\mathbf{b}} (\exp\left(\frac{1}{(x_1 \ldots x_n)^p} \int_{\mathbf{0}}^{\mathbf{x}} \ln(t_1 \ldots t_n) f(\mathbf{t}) d\mathbf{t}\right)$$

$$\times \exp\left(\frac{-1}{(x_1 \ldots x_n)^p} \int_0^{x_1} \ldots \int_0^{x_n} \ln(t_1 \ldots t_n) d\mathbf{t}\right))^p d\mathbf{x}$$

$$\leq \int_{\mathbf{0}}^{\mathbf{b}} \left(\frac{1}{(x_1 \dots x_n)^{p-1}} \exp(-\ln(x_1 \dots x_n) + (1 \dots 1)) \right)^p$$

$$\times \left(\frac{1}{(x_1 \dots x_n)^{pp}} \int_{\mathbf{0}}^{\mathbf{x}} (t_1 \dots t_n)^p f^p(\mathbf{t}) d\mathbf{t} \right) d\mathbf{x}$$

$$= e^{np} \int_{\mathbf{0}}^{\mathbf{b}} \frac{1}{(x_1 \dots x_n)^{2pp}} \left(\int_{\mathbf{0}}^{\mathbf{x}} (t_1 \dots t_n)^p f^p(\mathbf{t}) d\mathbf{t} \right) d\mathbf{x}$$

$$= e^{np} \left(\int_{\mathbf{0}}^{\mathbf{b}} (t_1 \dots t_n)^p f^p(\mathbf{t}) d\mathbf{t} \right) \int_{t_1}^{b_1} \dots \int_{t_1}^{b_n} \frac{1}{(x_1 \dots x_n)^{(2p)^2}} d\mathbf{x}$$

On using the analysis of Theorem 3.5 then,

$$\left(e^{np} \left(\int_{\mathbf{0}}^{\mathbf{b}} (t_1 \dots t_n)^p f^p(\mathbf{t}) d\mathbf{t} \right) \left(\int_{t_1}^{b_1} \dots \int_{t_1}^{b_n} \frac{1}{(x_1 \dots x_n)^{(2p)^2}} d\mathbf{x} \right) \right)^{\frac{1}{p}}$$

$$= \left(\frac{e}{(4p^2-1)^{\frac{1}{p}}} \right)^n \left(\int_{\mathbf{0}}^{\mathbf{b}} t_i^{p(1-4p)+1} f^p(\mathbf{t}) \prod_{i=1}^{n} \left(1 - \frac{(t_i)^{4p^2-1}}{(b_i)^{4p^2-1}} \right) d\mathbf{t} \right)^{\frac{1}{p}}$$

∎

Finally, if $\Phi(t) = \ln(t)$ whenever $f \to \exp f$, we obtain the following reversed multidimensional Hardy-type inequalities.

Corollary 3.8. *If* $0 < b_i \leq \infty$, $i = 1, 2 \dots, n \in \mathbb{Z}_+$, $-\infty \leq a < c \leq \infty$ *and let* Φ *be a positive function* $[a, c]$. *Let function* f *be defined on* $(\mathbf{0}, \mathbf{b})$ *such that* $a < f(\mathbf{x}) < c$. *If*

$$0 < \int_{\mathbf{0}}^{\mathbf{b}} f(\mathbf{x}) d\mathbf{x} < \infty$$

then we write inequality (M) as:

$$\left(\int_{\mathbf{0}}^{\mathbf{b}} \left(\ln \left(\frac{1}{(x_1 \dots x_n)^p} \int_{\mathbf{0}}^{\mathbf{x}} \exp f(\mathbf{t}) d\mathbf{t} \right) \right)^p d\mathbf{x} \right)^{\frac{1}{p}}$$

$$\geq \left(\int_{\mathbf{0}}^{\mathbf{b}} f(\mathbf{x}) \prod_{i=1}^{n} \left(1 - \frac{x_i}{b_i} \right) \frac{d\mathbf{x}}{x_1 \dots x_n} \right)$$

for every positive function g *on* $(\mathbf{0}, \mathbf{b})$.

Proof. Since, the natural logarithmic functions are strictly increasing and the exponential functions which is necessarily the inverse of natural logarithmic function is strictly convex and therefore the proof of Corollary 3.8 follows inversely from that of Corollary 3.7 resulted in reversing the inequalities. ∎

4. MULTI-DIMENSIONAL HARDY-TYPE INEQUALITIES WITH $1 < p \leq q < \infty$

In this section, we discussed some weighted case of multi-dimensional Hardy integral inequality.

Theorem 4.1. *If* $0 < p \leq q < \infty$ *with* $1/p + 1/q = 1$ *and* $0 < b_i \leq \infty$, $i = 1, \ldots, n \in \mathbb{Z}_+$ *and let* Φ *be a positive and convex function on* (a, c) *with* $-\infty \leq a < c \leq \infty$ *and let* $\omega(x)$ *be a weight function defined on* $(0, b)$ *for any non-negative function* $f(x)$ *such that* $a < f(x) < c$. *If the weight function* ν *is defined by*

$$C = \sup_{0<t_i\leq b_i} \left(\frac{t_1 \ldots t_n}{\nu(t_1 \ldots t_n)}\right)^{1/p} \left(\int_{\mathbf{t}}^{\mathbf{b}} \omega(\mathbf{x}) x_1^{-(q+1)} \cdots x_n^{-(q+1)} d\mathbf{x}\right)^{1/q}$$

Then,

$$\left(\int_{\mathbf{0}}^{\mathbf{b}} \left(\Phi\left(\frac{1}{g(\mathbf{x})} \int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t}) d\mathbf{t}\right)\right)^{q/p} \omega(\mathbf{x}) \frac{d\mathbf{x}}{x_1 \ldots x_n}\right)^{1/q} \leq K \left(\int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{x})) \nu(\mathbf{x})\, d\mathbf{x}\right)^{1/p}$$

holds, whenever $g(\mathbf{x}) \equiv \mathbf{x}^p$.

Proof. By using Jensen and Minkowski integral inequalities, we obtain

$$\begin{aligned}
&\left(\int_{\mathbf{0}}^{\mathbf{b}} \left(\Phi\left(\frac{1}{g(\mathbf{x})} \int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t}) d\mathbf{t}\right)\right)^{q/p} \omega(\mathbf{x}) \frac{d\mathbf{x}}{x_1 \ldots x_n}\right)^{1/q} \\
\leq\ & \left(\int_{\mathbf{0}}^{\mathbf{b}} \left(\frac{1}{x_1^p \ldots x_n^p} \int_{\mathbf{0}}^{\mathbf{x}} \Phi(f(\mathbf{t})) d\mathbf{t}\right)^{q/p} \omega(\mathbf{x}) \frac{d\mathbf{x}}{x_1 \ldots x_n}\right)^{1/q} \\
\leq\ & \left(\int_{\mathbf{0}}^{\mathbf{b}} \frac{t_1 \ldots t_n}{\nu(\mathbf{t})} \Phi(f(\mathbf{t})) \left(\int_{\mathbf{t}}^{\mathbf{b}} \omega(\mathbf{x}) x_1^{-(q+1)} \ldots x_n^{-(q+1)} d\mathbf{x}\right)^{p/q} \nu(\mathbf{t}) \frac{d\mathbf{t}}{\mathbf{t}}\right)^{1/p} \\
\leq\ & C \left(\int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{t})) \nu \frac{d\mathbf{t}}{\mathbf{t}}\right)^{1/p}
\end{aligned}$$

where, $\nu(x)$ is a weight function on $(0, b)$ and $K = C$ but if K is sharp then we have $K \leq C$ which concludes the proof. In view of a comparison with Theorem 4.1, we noted that, the Theorem holds in the reverse direction if Φ is concave by defining

$$C = \inf_{0<t_i\leq b_i} \frac{t_1 \cdots t_n}{\nu(\mathbf{t})} \left(\int_{\mathbf{t}}^{\mathbf{b}} \omega(\mathbf{x}) x_1^{-(q+1)} \cdots x_n^{-(q+1)} d\mathbf{x}\right)^{1/q}$$

∎

Corollary 4.2. *If* $0 < p = q < \infty$, $0 < b_i \leq \infty$, $i = 1, \ldots, n \in \mathbb{Z}_+$ *and let* Φ *be a positive and convex function on* (a, c) *with* $-\infty \leq a < c \leq \infty$ *and* $\omega(x) \equiv 1$. *Also, let* $f(x)$ *be non-negative on* $a < f(x) < c$ *and*

$$\nu = \frac{(1 - t_1^p/b_1^p) \cdots (1 - t_n^p/b_n^p)}{(t_1 \ldots t_n)^p}$$

Then,

$$\int_{\mathbf{0}}^{\mathbf{b}} \Phi\left(\frac{1}{g(\mathbf{x})}\int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t})\,d\mathbf{t}\right)\frac{\omega(\mathbf{x})}{\mathbf{x}}\,d\mathbf{x} \le K\int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{x}))\nu(\mathbf{x})d\mathbf{x}$$

holds, whenever $g(\mathbf{x}) \equiv \mathbf{x}^p$.

Proof. It is easy to obtain that $C = p^{-n}$ and the remaining proof follow directly from Theorem 4.1 from which we obtain Theorem 3.5. In addition, if $p = q = 1$ then we obtain the following useful multidimensional version of classical Hardy integral inequality:

$$\int_{\mathbf{0}}^{\mathbf{b}} \Phi\left(\frac{1}{x_1\ldots x_n}\int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{t}\right)\frac{d\mathbf{x}}{x_1\cdots x_n} \le \int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{x}))\nu(\mathbf{x})d\mathbf{x}.$$

∎

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