The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 8, Issue 1, Article 6, pp. 1-12, 2011

SHARP L^p IMPROVING RESULTS FOR SINGULAR MEASURES ON $\mathbb{C}^{n+1}$

E. FERREYRA, M. URCIUOLO

Received 18 November, 2009; accepted 25 February, 2010; published 12 October, 2011.

FAMAF-CIEM, UNIVERSIDAD NACIONAL DE CÓRDOBA-CONICET, CIUDAD UNIVERSITARIA, 5000 CÓRDOBA, ARGENTINA
eferrey@famaf.unc.edu.ar
urciuolo@famaf.unc.edu.ar

ABSTRACT. For $j = 1, ..., n,$ let Ω_j be open sets of the complex plane and let φ_j be holomorphic functions on Ω_j such that φ_j'' does not vanish identically on Ω_j. We consider $\varphi(z_1, ..., z_n) = \varphi_1(z_1) + ... + \varphi_n(z_n)$. We characterize the pairs (p, q) such that the convolution operator with the surface measure supported on a compact subset of the graph of φ is $p - q$ bounded.

Key words and phrases: L^p improving measures, convolution operators.

2000 *Mathematics Subject Classification.* Primary 42B20. Secondary 42B25.

ISSN (electronic): 1449-5910

Partially supported by ANPCyT and SECyT-UNC.

Acknowledgement: The authors are deeply indebted with Prof. Fulvio Ricci for his generous comments.

1. INTRODUCTION

For $1 \leq j \leq n$, let Ω_j be open sets of the complex plane and let $\varphi_j : \Omega_j \to \mathbb{C}$ be holomorphic functions on Ω_j such that φ_j'' does not vanish identically on Ω_j. We take $\varphi : \Omega_1 \times ... \times \Omega_n \to \mathbb{C}$ given by

$$\varphi(z_1, ..., z_n) = \varphi_1(z_1) + ... + \varphi_n(z_n).$$

Let us consider the canonical identification $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ given by $(x_1, y_1, ..., x_n, y_n) \to (x_1 + iy_1, ..., x_n + iy_n)$. Let D_j be bounded open sets such that $\overline{D_j} \subset \Omega_j$ and such that $\varphi_j'' \neq 0$ on ∂D_j. Let $D = D_1 \times ... \times D_n$ and let μ be the Borel measure on $\mathbb{R}^{2n+2}$ given by

$$\mu(E) = \int_D \chi_E(\mathbf{z}, \varphi(\mathbf{z}))\, d\sigma(\mathbf{z}), \tag{1.1}$$

where $\mathbf{z} = (x_1 + iy_1, ..., x_n + iy_n)$ and $d\sigma(\mathbf{z}) = dx_1 dy_1 ... dx_n dy_n$ denotes the Lebesgue measure on $\mathbb{R}^{2n}$. We consider the convolution operator given by $Tf = \mu * f$, for $f \in S(\mathbb{R}^{2n+2})$, and the type set

$$E_\mu = \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in [0,1] \times [0,1] : \|T\|_{p,q} < \infty \right\},$$

where the $L^p(\mathbb{R}^{2n+2})$ spaces are taken with the Lebesgue measure. Our aim is to determine this set. In the case that E_μ does not reduce to the diagonal $\frac{1}{p} = \frac{1}{q}$, we say that the measure μ is L^p improving. A well known result asserts that a necessary condition for a measure μ to be L^p improving is that its support is not contained in any affine submanifold of $\mathbb{R}^{2n+2}$ (see Proposition 1.1 in [7]), so we will only consider the case when φ_j'' does not vanish identically on Ω_j for all $1 \leq j \leq n$.

The case of real hypersurfaces in $\mathbb{R}^n$ has been widely studied (see for example [2], [4], [6], [7], [8]). When the codimension of the surface is greater than one, this matter becomes more complicated.

If for all $1 \leq j \leq n$, $\varphi_j''(z)$ does not vanish on D_j, with standard techniques we obtain that E_μ is the closed triangle with vertices $(0,0)$ $(1,1)$ and $\left(\frac{n+1}{n+2}, \frac{1}{n+2}\right)$. On the other case, if for some $1 \leq j \leq n$, $\left\{ z \in D_j : \varphi_j''(z) = 0 \right\}$ is a finite set $z_{j,1}, ..., z_{j,l_j}$, we will prove that E_μ is a closed polygonal region whose vertices depend on the order of each $z_{j,i}$, $1 \leq j \leq n$, $1 \leq i \leq l_j$, as zero of the function

$$\omega_{j,z_{j,i}}(z) = \varphi_j(z) - \varphi_j(z_{j,i}) - (z - z_{j,i})\, \varphi_j'(z_{j,i}). \tag{1.2}$$

In a first step, we study the case $\varphi_j(z) = z^{m_j} g_j(z)$, $m_j \geq 2$, g_j being holomorphic in a neighborhood of the origin and $g_j(0) \neq 0$. We obtain that there exists a neighborhood V of the origin in $\mathbb{C}^n$ such that the associated type set is a closed polygonal region with vertices depending on $m_1, ..., m_n$. Our proof will be based on a suitable adaptation of the argument due to M. Christ, developed in [1], where the author studies the type set associated to the two dimensional measure supported on the parabola. We will derive the general case from this one, with classical arguments.

Throughout this paper c will denote a positive constant not necessarily the same at each occurrence.

2. THE CASE $\varphi_j(z) = z^{m_j} g_j(z)$, $1 \leq j \leq n$.

For $r > 0$, we set $B_r = \{ z \in \mathbb{C} : |z| \leq r \}$. Let $\varphi(z_1, ..., z_n) = \sum_{j=1}^{n} \varphi_j(z_j)$, where $\varphi_j(z) = z^{m_j} g_j(z)$, $2 \leq m_1 \leq ... \leq m_n$, and g_j are holomorphic functions in B_{r_j} for some $r_j > 0$, with

$g_j(0) \neq 0$. We also assume that $\varphi_j, \varphi_j', ..., \varphi_j^{(m_j)}$ are different from zero on $B_{r_j} - \{0\}$. Let μ be defined by (1.1) with $D = \prod_{1\leq j\leq n} B_{r_j}$. To study E_μ, without loss of generality we suppose $r_j = 1$ for all $1 \leq j \leq n$, so we take $D = B_1^n$. The Riesz Thorin interpolation theorem implies that E_μ is a convex subset of $[0,1] \times [0,1]$. It is well known that if $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\mu$ then $p \leq q$. (See [10] p.33). Also, by duality, E_μ is symmetric with respect to the non principal diagonal.

For $1 \leq J \leq n$, we set $S_J = \sum_{j=J}^{n} m_j^{-1}$. Also we set $S_{n+1} = 0$.

Lemma 2.1. *If* $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\mu$ *and* $0 \leq J \leq n$ *then* $\frac{1}{q} \geq \frac{J+1+S_{J+1}}{1+S_{J+1}}\frac{1}{p} - \frac{J+S_{J+1}}{1+S_{J+1}}$.

Proof. We set $\mathbf{z} = (z_1, ..., z_n)$. For $0 < \delta < 1$, we set $f = \chi_{Q_\delta}$ where $Q_\delta \subset \mathbb{C}^{n+1}$ is given by

$$Q_\delta = \{(\mathbf{z}, w) : |z_j| \leq \delta, 1 \leq j \leq J;\ \ |z_j| \leq \delta^{1/m_j},\ J+1 \leq j \leq n;\ \ |w| \leq b\delta\}$$

with $b = \sum_{j=1}^{n}\left(\sup_{B_1}\left|\varphi_j'\right| + 2\sup_{B_1}|g_j|\right)$. We define $A_\delta \subset \mathbb{C}^{n+1}$ by $A_\delta =$

$$\{(\mathbf{z}, w) : |z_j| \leq 1,\ 1 \leq j \leq J;\ |z_j| \leq \delta^{1/m_j},\ J+1 \leq j \leq n;\ |\ w - \varphi(z_1, ..., z_n)\ | \leq \delta\}.$$

We first show that there exists a constant $c > 0$ such that for $(\mathbf{z}, w) \in A_\delta$

$$|(\mu * f)(\mathbf{z}, w)| \geq c\delta^{2J+2S_{J+1}}. \tag{2.1}$$

To see (2.1) we take a fix $(\mathbf{z}, w) \in A_\delta$. For $\boldsymbol{\varsigma} = (\varsigma_1, ..., \varsigma_n) \in \mathbf{z} + \left(\Pi_{j=1}^J B_\delta \times \Pi_{j=J+1}^n B_{\delta^{1/m_j}}\right)$ we have that

$$(\boldsymbol{\varsigma}, \varphi(\boldsymbol{\varsigma})) - (\mathbf{z}, w) \in Q_\delta,$$

indeed, we have $|\varsigma_j - z_j| \leq \delta$, for $1 \leq j \leq J$, and $|\varsigma_j - z_j| \leq \delta^{1/m_j}$, for $J+1 \leq j \leq n$. We also have

$$|\varphi(\boldsymbol{\varsigma}) - w| \leq |\ \varphi(\boldsymbol{\varsigma}) - \varphi(\mathbf{z})\ | + |\varphi(\mathbf{z}) - w|.$$

The mean value theorem gives us, for $1 \leq j \leq J$,

$$|\ \varphi_j(z_j) - \varphi_j(\varsigma_j)\ | \leq \delta \sup_{B_1}\left|\varphi_j'\right|$$

and for $J+1 \leq j \leq n$

$$|\ \varphi_j(z_j) - \varphi_j(\varsigma_j)\ | \leq |\ \varphi_j(z_j)\ | + |\ \varphi_j(\varsigma_j)\ | \leq 2\delta \sup_{B_1}|g_j|.$$

So

$$|\varphi(\boldsymbol{\varsigma}) - w| \leq \delta \sum_{j=1}^{n}\left(\sup_{B_1}\left|\varphi_j'\right| + 2\sup_{B_1}|g_j|\right).$$

Then (2.1) follows. Now,

$$\|\mu * f\|_q \geq \left(\int_{A_\delta} |\mu * f|^q\right)^{\frac{1}{q}} \geq c\delta^{2J+2S_{J+1}}\,|A_\delta|^{\frac{1}{q}} =$$

$$= c\delta^{2J+2S_{J+1}+(2+2S_{J+1})\frac{1}{q}}.$$

But $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\mu$ implies $\|\mu * f\|_q \leq c\,\|f\|_p = c\delta^{(2J+2+2S_{J+1})\frac{1}{p}}$. So, for all $\delta > 0$ small enough,

$$\delta^{J+S_{J+1}+(1+S_{J+1})\frac{1}{q}} \leq c\delta^{(J+1+S_{J+1})\frac{1}{p}}$$

then

$$\frac{1}{q} \geq \frac{J+1+S_{J+1}}{1+S_{J+1}}\frac{1}{p} - \frac{J+S_{J+1}}{1+S_{J+1}}$$

and the lemma follows. ∎

We denote by L_J, $0 \leq J \leq n$, the lines given by

$$\frac{1}{q} = \frac{J+1+S_{J+1}}{1+S_{J+1}}\frac{1}{p} - \frac{J+S_{J+1}}{1+S_{J+1}}.$$

Also we denote by A_J, $0 \leq J \leq n$, and by B_J, $1 \leq J \leq n$ the intersection of L_J with the non principal diagonal $\left\{\left(\frac{1}{p}, 1-\frac{1}{p}\right) : 0 \leq \frac{1}{p} \leq 1\right\}$ and the intersection of L_{J-1} with L_J respectively. A computation shows that, for $0 \leq J \leq n$,

$$A_J = \left(\frac{J+1+2S_{J+1}}{J+2+2S_{J+1}}, \frac{1}{J+2+2S_{J+1}}\right) \tag{2.2}$$

and for $1 \leq J \leq n$

$$B_J = \left(\frac{1+S_{J+1}+(J-1)\,m_J^{-1}}{1+Jm_J^{-1}+S_{J+1}}, \frac{1-m_J^{-1}}{1+Jm_J^{-1}+S_{J+1}}\right). \tag{2.3}$$

Let ψ be a $C_0^\infty(\mathbb{R})$ function supported in the interval $\left[\frac{1}{2}, 4\right]$ such that $\psi \equiv 1$ on $[1,2]$, and $0 \leq \psi \leq 1$. We observe that $1 \leq \sum\limits_{k\in\mathbb{N}\cup\{0\}} \psi\left(2^k x\right) \leq 3$ for $x \in (0,2)$. For each $k_1, ..., k_n \in \mathbb{N}\cup\{0\}$ we set

$$\mu_{k_1,...,k_n}(E) = \int_D \chi_E\left(\mathbf{z}, \varphi(\mathbf{z})\right)\psi\left(2^{k_1}|z_1|\right)...\psi\left(2^{k_n}|z_n|\right)d\sigma(\mathbf{z}).$$

So $\mu \leq \sum\limits_{k_1,...,k_n\in\mathbb{N}} \mu_{k_1,...,k_n}$. We also denote by $T_{k_1,...,k_n}$ the convolution operator given, for $f \in S(\mathbb{R}^{2n+2})$, by

$$T_{k_1,...,k_n} f = \mu_{k_1,...,k_n} * f. \tag{2.4}$$

Proposition 2.2. *If* $\boldsymbol{\xi} = (s_1, t_1, ..., s_{n+1}, t_{n+1}) \in \mathbb{R}^{2n+2}$ *then*

$i)$

$$\left|\left(\mu_{k_1,...,k_n}\right)^\wedge(\boldsymbol{\xi})\right| \leq c\frac{\prod\limits_{j=1}^{n} 2^{k_j(m_j-2)}}{\left(1+|(s_{n+1},t_{n+1})|\right)^n},$$

$ii)$ *for* $0 \leq J \leq n-1$

$$\left|\left(\sum_{k_{J+1},...,k_n\in\mathbb{N}} \mu_{k_1,...,k_n}\right)^\wedge(\boldsymbol{\xi})\right| \leq c\frac{\prod\limits_{j=1}^{J} 2^{k_j(m_j-2)}}{\left(1+|(s_{n+1},t_{n+1})|\right)^{J+2S_{J+1}}},$$

$iii)$ *for* $1 \leq J \leq n$

$$\left|\left(\sum_{k_J\in\mathbb{N}} \mu_{k_1,...,k_n}\right)^\wedge(\boldsymbol{\xi})\right| \leq c\frac{\prod\limits_{j=1}^{J-1} 2^{k_j(m_j-2)} \prod\limits_{j=J+1}^{n} 2^{k_j(m_J-2)}}{\left(1+|(s_{n+1},t_{n+1})|\right)^{J-1+2m_J^{-1}+m_J S_{J+1}}}.$$

Proof. We set

$$I_{j,k_j}(s,t,s_{n+1},t_{n+1}) = \int e^{-i\left(sx+ty+\left\langle(s_{n+1},t_{n+1}),\varphi_j(x,y)\right\rangle\right)}\psi\left(2^{k_j}|(x,y)|\right)dxdy,$$

thus

$$\left(\mu_{k_1,...,k_n}\right)^\wedge(\boldsymbol{\xi}) = \prod_{j=1}^{n} I_{j,k_j}(s_j,t_j,s_{n+1},t_{n+1})$$

and

$$\left(\sum_{k_{J+1},\dots,k_n\in\mathbb{N}}\mu_{k_1,\dots,k_n}\right)^{\wedge}(\boldsymbol{\xi})$$
$$=\prod_{j=1}^{J}I_{j,k_j}\left(s_j,t_j,s_{n+1},t_{n+1}\right)\prod_{j=J+1}^{n}\sum_{k_j\in\mathbb{N}}I_{j,k_j}\left(s_j,t_j,s_{n+1},t_{n+1}\right).$$

Since φ_j is a holomorphic function a computation shows that for (x,y) such that $2^{k_j}\left|(x,y)\right|\in\operatorname{supp}\psi$

$$\left|Hess_{x,y}\left(sx+ty+\left\langle\left(s_{n+1},t_{n+1}\right),\varphi_j\left(x,y\right)\right\rangle\right)\right|=\left|\varphi_j''\left(x+iy\right)\right|^2\left|\left(s_{n+1},t_{n+1}\right)\right|^2$$
$$\geq c2^{-2k_j(m_j-2)}\left|\left(s_{n+1},t_{n+1}\right)\right|^2,$$

then using the method of the stationary phase (see proposition 6, p. 344 in [9]) we obtain

$$\left|I_{j,k_j}\left(s,t,s_{n+1},t_{n+1}\right)\right|\leq\frac{c2^{k_j(m_j-2)}}{1+\left|\left(s_{n+1},t_{n+1}\right)\right|},\tag{2.5}$$

thus $i)$ follows. Now a change of variables shows that

$$I_{j,k_j}\left(s,t,s_{n+1},t_{n+1}\right)=2^{-2k_j}I_{j,0}^{k_j}\left(2^{-k_j}s,2^{-k_j}t,2^{-k_jm_j}s_{n+1},2^{-k_jm_j}t_{n+1}\right),$$

where

$$I_{j,0}^{k_j}\left(s,t,\widetilde{s},\widetilde{t}\right)=\int e^{-i\left(sx+ty+\left\langle\left(\widetilde{s},\widetilde{t}\right),(x+iy)^{m_j}g_j\left(2^{-k_j}x,2^{-k_j}y\right)\right\rangle\right)}\psi\left(\left|(x,y)\right|\right)dxdy.$$

We note that for (x,y) such that $\left|(x,y)\right|\in\operatorname{supp}\psi$

$$\left|Hess_{x,y}\left(sx+ty+\left\langle\left(\widetilde{s},\widetilde{t}\right),(x+iy)^{m_j}g_j\left(2^{-k_j}x,2^{-k_j}y\right)\right\rangle\right)\right|$$
$$=\left|\frac{d^2}{dz^2}z^{m_j}g_j\left(2^{-k_j}z\right)\right|^2\left|\left(\widetilde{s},\widetilde{t}\right)\right|^2\geq c\left|\left(\widetilde{s},\widetilde{t}\right)\right|^2$$

with c independent of k_j. Indeed, since $g_j(0)\neq 0$, there exists k_0 such that for $k\geq k_0$,

$$\left|\frac{d^2}{dz^2}z^{m_j}g_j\left(2^{-k_j}z\right)\right|$$
$$=\left|m_jm_{j-1}z^{m_j-2}g_j\left(2^{-k_j}z\right)+2m_j2^{-k_j}z^{m_j-1}g_j'\left(2^{-k_j}z\right)+2^{-2k_j}z^{m_j}g_j''\left(2^{-k_j}z\right)\right|\geq c,$$

and since φ_j'' does not vanish on $B_1-\{0\}$, if $k\leq k_0$,

$$\left|\frac{d^2}{dz^2}z^{m_j}g_j\left(2^{-k_j}z\right)\right|=\left|\frac{d^2}{dz^2}2^{k_jm_j}\varphi_j\left(2^{-k_j}z\right)\right|=\left|2^{k_j(m_j-2)}\varphi_j''\left(2^{-k_j}z\right)\right|\geq c.$$

Then

$$\left|I_{j,0}^{k_j}\left(s,t,\widetilde{s},\widetilde{t}\right)\right|\leq\frac{c}{1+\left|\left(\widetilde{s},\widetilde{t}\right)\right|}.\tag{2.6}$$

Now, as in the proof of Lemma 1 in [5],

$$\left|\sum_{k_j\in\mathbb{N}}I_{j,k_j}\left(s,t,s_{n+1},t_{n+1}\right)\right|$$
$$=\left|\sum_{k_j\in\mathbb{N}}2^{-2k_j}I_{j,0}^{k_j}\left(2^{-k_j}s,2^{-k_j}t,2^{-k_jm_j}s_{n+1},2^{-k_jm_j}t_{n+1}\right)\right|$$

$$= \left| \sum_{2^{m_j k_j} \le 1 + |(s_{n+1}, t_{n+1})|} \right| + \left| \sum_{2^{m_j k_j} \ge 1 + |(s_{n+1}, t_{n+1})|} \right| .$$

To estimate the first sum we use (2.6) to obtain that the sum is bounded by

$$\frac{c}{1 + |(s_{n+1}, t_{n+1})|} \sum_{2^{m_j k_j} \le 1 + |(s_{n+1}, t_{n+1})|} 2^{k_j (m_j - 2)} \le \frac{c}{(1 + |(s_{n+1}, t_{n+1})|)^{\frac{2}{m_j}}},$$

and in the second sum, we use that

$$\left| I_{j,0}^{k_j} \left(2^{-k_j} s, 2^{-k_j} t, 2^{-k_j m_j} s_{n+1}, 2^{-k_j m_j} t_{n+1} \right) \right| \le \int |\psi| = c$$

and we obtain

$$\left| \sum_{2^{m_j k_j} \ge 1 + |(s_{n+1}, t_{n+1})|} \right| \le \frac{c}{(1 + |(s_{n+1}, t_{n+1})|)^{\frac{2}{m_j}}},$$

so

$$\left| \sum_{k_j \in \mathbb{N}} I_{j,k_j} (s, t, s_{n+1}, t_{n+1}) \right| \le \frac{c}{(1 + |(s_{n+1}, t_{n+1})|)^{\frac{2}{m_j}}}. \tag{2.7}$$

Thus $ii)$ follows from (2.5) and (2.7). To prove $iii)$ we use (2.5) and the estimate

$$\left| I_{j,k_j} (s, t, s_{n+1}, t_{n+1}) \right| \le c 2^{-2k_j},$$

to obtain

$$\left| I_{j,k_j} (s, t, s_{n+1}, t_{n+1}) \right| \le c \frac{2^{\theta_j k_j (m_j - 2)}}{(1 + |(s_{n+1}, t_{n+1})|)^{\theta_j}} 2^{-2(1 - \theta_j) k_j}.$$

To estimate $\left| \left(\sum_{k_J \in N} \mu_{k_1, \dots, k_n} \right)^{\wedge} (\boldsymbol{\xi}) \right|$, we use this last estimate for $j > J$ with $\theta_j = \frac{m_J}{m_j}$, (2.5) for $j < J$ and (2.7) for $j = J$.

For $B = \left(\frac{1}{p}, \frac{1}{q} \right) \in [0, 1] \times [0, 1]$ and $T : L^p \to L^q$ we write, to simplify the notation, $\|T\|_B$ instead of $\|T\|_{p,q}$. We also set, for $1 \le J \le n$,

$$C_J = \left(\frac{2m_J^{-1} + J + m_J S_{J+1}}{1 + J + 2m_J^{-1} + m_J S_{J+1}}, \frac{1}{1 + J + 2m_J^{-1} + m_J S_{J+1}} \right). \tag{2.8}$$

Lemma 2.3. *Let $T_{k_1, \dots, k_n}$ be defined by (2.4) and let A_J and C_J be defined by (2.2) and (2.8) respectively. Then*

$i)$

$$\|T_{k_1, \dots, k_n}\|_{A_n} \le c \prod_{j=1}^{n} 2^{2k_j \frac{(m_j - 2)}{n+2}},$$

$ii)$ *for* $0 \le J \le n - 1$

$$\left\| \sum_{k_{J+1}, \dots, k_n \in \mathbb{N}} T_{k_1, \dots, k_n} \right\|_{A_J} \le c \prod_{j=1}^{J} 2^{2k_j \frac{m_j - 2}{J + 2 + 2S_{J+1}}},$$

$iii)$ *for* $1 \le J \le n$

$$\left\| \sum_{k_J \in \mathbb{N}} T_{k_1,\dots,k_n} \right\|_{C_J} \le \left(\prod_{j=1}^{J-1} 2^{k_j(m_j-2)} \prod_{j=J+1}^{n} 2^{k_j(m_J-2)} \right)^{\frac{2}{J+1+2m_J^{-1}+m_J S_{J+1}}}.$$

Proof. To prove $i)$ we use the complex interpolation theorem. For $Re\,(z) > 0$ and $(s,t) \in \mathbb{R}^2$ we consider the fractional integration kernel

$$I_z\,(s,t) = \frac{2^{-\frac{z}{2}}}{\Gamma\left(\frac{z}{2}\right)} \left|(s,t)\right|^{z-2}$$

and its analytic extension to $z \in \mathbb{C}$. In particular we have $\widehat{I_z} = cI_{2-z}$, also $I_0 = c\delta$ where δ denotes the Dirac distribution at the origin. We also define J_z as the distribution on $\mathbb{R}^{2n+2}$ given by the tensor product $J_z = \delta \otimes ... \otimes \delta \otimes I_z$. For z such that $-n \le Re\,(z) \le 2$ we consider the analytic family of operators

$$U_z f = e^{z^2} \mu_{k_1,\dots,k_n} * J_z * f.$$

Taking account of Proposition 2.2 $i)$ we obtain that

$$\left\|U_{-n+i\gamma}\right\|_{2,2} \le c \prod_{j=1}^{n} 2^{k_j(m_j-2)},$$

also it is easy to check that

$$\left\|U_{2+i\gamma}\right\|_{1,\infty} \le ce^{-\gamma^2} \left| \Gamma\left(\frac{2+i\gamma}{2}\right)\right|^{-1} \le c,$$

so by interpolation,

$$\left\|T_{k_1,\dots,k_n}\right\|_{A_n} = c \left\|U_0\right\|_{\frac{n+2}{n+1},n+2} \le c \prod_{j=1}^{n} 2^{2k_j \frac{(m_j-2)}{n+2}}.$$

Now $ii)$ follows similarly, applying the complex interpolation theorem to the operators $U_z f = e^{z^2} \sum_{k_{J+1},\dots,k_n \in N} \mu_{k_1,\dots,k_n} * J_z * f$, on the strip $-J - 2S_{J+1} \le Re\,(z) \le 2$ and using Proposition 2.2 $ii)$. Also, $iii)$ follows in analogous way, applying the complex interpolation theorem to the operators $U_z f = e^{z^2} \sum_{k_J \in N} \mu_{k_1,\dots,k_n} * J_z * f$, on the strip $-\left(J - 1 + 2m_J^{-1} + m_J S_{J+1}\right) \le Re\,(z) \le 2$ and then using Proposition 2.2 $iii)$. ∎

Following the approach in [1], we recall that for $k_J \in \mathbb{N}$

$$I_{J,0}^{k_J}\left(s,t,\widetilde{s},\widetilde{t}\right) = \int e^{-i\left(sx+ty+\left\langle\left(\widetilde{s},\widetilde{t}\right),(x+iy)^{m_J} g_J\left(2^{-k_J}x,2^{-k_J}y\right)\right\rangle\right)} \psi\left(\left|(x,y)\right|\right) dxdy.$$

If $(x+iy)^{m_J} g_J\left(2^{-k_J}x, 2^{-k_J}y\right) = u\,(x,y) + iv\,(x,y)$,

$$\frac{\partial}{\partial x}\left(sx+ty+\left\langle\left(\widetilde{s},\widetilde{t}\right),(x+iy)^{m_J} g_J\left(2^{-k_J}x,2^{-k_J}y\right)\right\rangle\right)$$

$$= s + \widetilde{s}u_x\,(x,y) + \widetilde{t}v_x\,(x,y)$$

and

$$\frac{\partial}{\partial y}\left(sx+ty+\left\langle\left(\widetilde{s},\widetilde{t}\right),(x+iy)^{m_J} g_J\left(2^{-k_J}x,2^{-k_J}y\right)\right\rangle\right)$$

$$= t + \widetilde{s}u_y\,(x,y) + \widetilde{t}v_y\,(x,y)$$

and so if the gradient of the phase function vanishes at some (x,y) with $\left|(x,y)\right| \in \operatorname{supp}\psi$ then

$$s+it=-\left(\widetilde{s}+i\widetilde{t}\right)\overline{\left(u+iv\right)'\left(x+iy\right)}.$$

Now,

$$\left(u+iv\right)'(z)=m_J z^{m_J-1}g_J\left(2^{-k_J}z\right)+z^{m_J}2^{-k_J}g_J'\left(2^{-k_J}z\right)=2^{k_J(m_J-1)}\varphi_J'\left(2^{-k_J}z\right).$$

so from the first equality we obtain that there exists k_0 such that for $k_J \geq k_0$, $\left|(u+iv)'\right|$ is bounded from above and from below uniformly on k_J, also, since φ_J' does not vanish on $B_1-\{0\}$, from the second equality we obtain the same assertion for $1\leq k_J<k_0$ and so there exist constants $c_1^J, c_2^J>0$ such that $\left(s,t,\widetilde{s},\widetilde{t}\right)$ belongs to the cone

$$\Gamma_0^J=\left\{\left(s,t,\widetilde{s},\widetilde{t}\right): c_1^J\left|(s,t)\right|\leq\left|(\widetilde{s},\widetilde{t})\right|\leq c_2^J\left|(s,t)\right|\right\}.$$

We define

$$\Gamma_0=\left\{\left(s,t,\widetilde{s},\widetilde{t}\right): c_1\left|(s,t)\right|\leq\left|(\widetilde{s},\widetilde{t})\right|\leq c_2\left|(s,t)\right|\right\}$$

with $c_1=\min_{1\leq J\leq n}\left\{c_1^J\right\}$ and $c_2=\max_{1\leq J\leq n}\left\{c_2^J,2c_1\right\}$.

Let M be a function belonging to $C^{\infty}\left(\mathbb{R}^4-\{0\}\right)$ homogeneous of degree zero with respect to the euclidean dilations on $\mathbb{R}^4$ such that $\operatorname{supp} M\subset\Gamma_0$ and for $1\leq J\leq n$ and $k\in\mathbb{Z}$ let $M_{J,k}(z,w)=M\left(2^{-k}z,2^{-km_J}w\right)$. Moreover, we choose M such that $\{M_{J,k}\}_{k\in\mathbb{Z}}$ is a C^{∞} partition of unity in $\{(z,w)\in\mathbb{R}^4: z\neq 0 \text{ and } w\neq 0\}$. Let c_0 be a constant such that $\widetilde{M}_{J,k}=\sum_{|i-k|\leq c_0}M_{J,i}$ be identically one on $\operatorname{supp} M_{J,k}$. Also, for $\left(\xi_1,...,\xi_{n+1}\right)\in\mathbb{C}^{n+1}$, we set $\mathcal{M}_{J,k}\left(\xi_1,...,\xi_{n+1}\right)=M_{J,k}\left(\xi_J,\xi_{n+1}\right)$ and $\widetilde{\mathcal{M}}_{J,k}\left(\xi_1,...,\xi_{n+1}\right)=\widetilde{M}_{J,k}\left(\xi_J,\xi_{n+1}\right)$. Let $\widetilde{Q}_{J,k}$ be the operator with multiplier $\widetilde{\mathcal{M}}_{J,k}$.

Let $H\in C_c^{\infty}\left(\mathbb{R}^4\right)$ be identically one in a neighborhood of the origin, and for $\left(\xi_1,...,\xi_{n+1}\right)\in\mathbb{C}^{n+1}$ let $\mathcal{H}_{J,k}\left(\xi_1,...,\xi_{n+1}\right)=H\left(2^{-k}\xi_J,2^{-km_J}\xi_{n+1}\right)$ and let $P_{J,k}$ be the Fourier multiplier operator with symbol $\mathcal{H}_{J,k}$.

The following lemma is the key argument contained in [1], adapted to our $2n$ dimensional setting. The proof is in [2], p. 37, for the case n dimensional, but it can be straightforward adapted to this case.

Lemma 2.4. *Let $\{\sigma_k\}_{k\in\mathbb{N}}$ be a sequence of positive measures on $\mathbb{R}^{2n+2}$, and let $T_kf=\sigma_k*f$, for $f\in S\left(\mathbb{R}^{2n+2}\right)$. Suppose $1\leq J\leq n$, $1<p\leq 2$ and $p\leq q<\infty$. If there exists $A>0$ such that $\sup_{k\in\mathbb{N}}\left\|T_k\right\|_{p,q}\leq A$, $\left\|\sum_{1\leq k\leq N}T_kP_{J,k}\right\|_{p,q}\leq A$ and $\left\|\sum_{1\leq k\leq N}T_k\left(I-P_{J,k}\right)\left(I-\widetilde{Q}_{J,k}\right)\right\|_{p,q}\leq A$ for all $N\in\mathbb{N}$, then there exists $c>0$, c independent of A, N and $\{\sigma_k\}_{k\in\mathbb{N}}$, such that*

$$\left\|\sum_{1\leq k\leq N}T_k\right\|_{p,q}\leq cA.$$

Our next aim is to study the operators $\sum_{1\leq k_J\leq N}T_{k_1,...,k_n}\left(I-P_{J,k_J}\right)\left(I-\widetilde{Q}_{J,k_J}\right)$ and $\sum_{1\leq k_J\leq N}T_{k_1,...,k_n}P_{J,k_J}$. As in [2] we obtain the following result

Lemma 2.5. *For $1<p,q<\infty$ and $N\in\mathbb{N}$ there exists $c>0$ independent of N such that*

$a)$

$$\left\|\sum_{1\leq k_J\leq N}T_{k_1,...,k_n}\left(I-P_{J,k_J}\right)\left(I-\widetilde{Q}_{J,k_J}\right)\right\|_{p,q}\leq c\left\|\sum_{1\leq k_J\leq N}T_{k_1,...,k_n}\right\|_{p,q}$$

and

$b)$

$$\left\| \sum_{1\leq k_J \leq N} T_{k_1,\ldots,k_n} P_{J,k_J} \right\|_{p,q} \leq c \left\| \sum_{1\leq k_J\leq N} T_{k_1,\ldots,k_n} \right\|_{p,q}.$$

Lemma 2.6. *If $N \in \mathbb{N}$ then*

$a)$ *the kernel of the convolution operator*

$$\sum_{1\leq k_J\leq N} T_{k_1,\ldots,k_n} \left(I - P_{J,k_J}\right)\left(I - \widetilde{Q}_{J,k_J}\right)$$

belongs to weak- $L^{1+m_J^{-1}}$ *and its norm is less than* $c2^{-\sum_{j\neq J} 2k_j}$, *with* c *independent of* N,

$b)$ *the kernel of the convolution operator*

$$\sum_{1\leq k_J\leq N} T_{k_1,\ldots,k_n} P_{J,k_J}$$

belongs to weak- $L^{1+m_J^{-1}}$ *and its norm is less than* $c2^{-\sum_{j\neq J} 2k_j}$, *with* c *independent of* N.

Proof. $a)$ A computation shows that the kernel $K_{k_1,\ldots,k_n}$ of the convolution operator $T_{k_1,\ldots,k_n}\left(I - P_{J,k_J}\right)\left(I - \widetilde{Q}_{J,k_J}\right)$ is the function given by

$$K_{k_1,\ldots,k_n}\left(z_1,\ldots,z_{n+1}\right)$$

$$= 2^{k_J m_J} G_J \left(-2^{k_J} z_J, 2^{k_J m_J}\left(-z_{n+1} + \sum_{j\neq J} \varphi_j\left(-z_j\right)\right)\right) \prod_{j\neq J} \psi\left(2^{k_j}\left|z_j\right|\right) \tag{2.9}$$

where $G_J = \left(I_{J,0}^{k_J}\left(1-H\right)\left(1-\widetilde{M_{J,0}}\right)\right)^{\wedge}$. Now, as in the proof of (2.3) in [3] we obtain that the functions G_J belong to $S\left(\mathbb{R}^4\right)$ and that they are uniformly (with respect to k_J) rapidly decreasing at infinity. So, as in the proof of Lemma 2.6 in [2] we get $a)$. Now $b)$ follows similarly after noting that the kernel of the operator $T_{k_1,\ldots,k_n} P_{J,k_J}$ is of the form (2.9) with $G_J = \left(I_{J,0}^{k_J} H\right)^{\wedge}$. ∎

Let J_0 be defined by $J_0 = 0$ if $m_1 > 2$ and $J_0 = \max\{j : 1 \leq j \leq n,\ m_j = 2\}$ if $m_1 = 2$. These previous lemmas allows us to prove the following result

Proposition 2.7. *If $J > J_0$ then there exists $c > 0$, independent of $k_1,\ldots,k_{J-1}$, such that for $N \in \mathbb{N}$*

$a)$

$$\left\| \sum_{1\leq k_J,\ldots,k_n\leq N} T_{k_1,\ldots,k_n}\left(I - P_{J,k_J}\right)\left(I - \widetilde{Q}_{J,k_J}\right) \right\|_{B_J} \leq c2^{\left(-\sum_{j=1}^{J-1} 2k_j \frac{m_j\left(m_j^{-1} - m_J^{-1}\right)}{\left(1+S_{J+1}+Jm_J^{-1}\right)}\right)}$$

and

$b)$

$$\left\| \sum_{1\leq k_J,\ldots,k_n\leq N} T_{k_1,\ldots,k_n} P_{J,k_J} \right\|_{B_J} \leq c2^{\left(-\sum_{j=1}^{J-1} 2k_j \frac{m_j\left(m_j^{-1} - m_J^{-1}\right)}{\left(1+S_{J+1}+Jm_J^{-1}\right)}\right)}.$$

Proof. We denote by $E_J = \left(1, \frac{1}{1+m_J^{-1}}\right)$. Since $B_J = tC_J + (1-t)\, E_J$ with $t = \frac{m_J + Jm_J + m_J^2 S_{J+1} + 2}{Jm_J + m_J^2 + m_J^2 S_{J+1}}$, Lemma 2.3 $iii)$, Lemma 2.6 $a)$ and the Marcinkiewicz interpolation theorem imply that

$$\left\| \sum_{1 \le k_J \le N} T_{k_1,\dots,k_n} \left(I - P_{J,k_J}\right) \left(I - \widetilde{Q}_{J,k_J}\right) \right\|_{B_J}$$

$$\le c \left(\prod_{j=1}^{J-1} 2^{k_j (m_j - 2)} \prod_{j=J+1}^{n} 2^{k_j (m_J - 2)} \right)^{\frac{2t}{J+1+2m_J^{-1} + m_J S_{J+1}}} 2^{-\sum_{j \ne J} (2k_j)(1-t)}.$$

Now if t is defined as above,

$$t \frac{2\,(m_j - 2)}{J + 1 + 2m_J^{-1} + m_J S_{J+1}} - 2\,(1-t)$$
$$= -\frac{2\,(m_J + m_J^2 - m_J m_j - 2)}{m_J\,(J + m_J + m_J S_{J+1})},$$

so $a)$ follows. Analogously, $b)$ follows. ∎

At this point we have already proved all the results needed to follow straightforward the proof of Theorem 3.12 in [2] to obtain the next

Theorem 2.8. *E_μ is the closed convex polygonal region with vertices $(1,1)$, $B_n, \dots, B_{J_0+1}, A_{J_0}$ and the symmetric points with respect to the non principal diagonal $\left(\frac{1}{p}, \frac{1}{p'}\right)$.*

Remark 2.1. We observe that E_μ is the closed convex polygonal region with vertices $(1,1)$, $B_n, \dots, B_1$ and the symmetric points with respect to the non principal diagonal $\left(\frac{1}{p}, \frac{1}{p'}\right)$.

3. THE GENERAL CASE

For $1 \le j \le n$, let Ω_j be open sets of the complex plane and let $\varphi_j : \Omega_j \to \mathbb{C}$ be holomorphic functions on Ω_j such that φ_j'' does not vanish identically on Ω_j. We take $\varphi : \Omega_1 \times \dots \times \Omega_n \to \mathbb{C}$ given by

$$\varphi(z_1, \dots, z_n) = \varphi_1(z_1) + \dots + \varphi_n(z_n).$$

Let D_j be bounded open sets such that $\overline{D_j} \subset \Omega_j$ and such that $\varphi_j'' \ne 0$ on ∂D_j. Let $D = D_1 \times \dots \times D_n$ and let μ be the Borel measure on $\mathbb{R}^{2n+2}$ given by (1.1). If φ_j'' does not vanish on D_j, let $l_j = 0$. On the other case, let $\{z_{j,i}\}_{1 \le i \le l_j}$ be the zeros of φ_j'' in D_j and let $m_{j,i}$ be the order of $z_{j,i}$ as a zero of

$$\omega_{j,i}(z) = \varphi_j(z) - \varphi_j(z_{j,i}) - (z - z_{j,i})\, \varphi_j'(z_{j,i}).$$

In any case, let $m_{j,0} = 2$. Let

$$\mathcal{M} = \{(m_{1,i_1}, \dots, m_{n,i_n}) : \ 0 \le i_j \le l_j, \ 1 \le j \le n\}.$$

For $\mathbf{i} = (i_1, \dots, i_n)$ we denote $\mathbf{m_i} = (m_{1,i_1}, \dots, m_{n,i_n})$. If $\mathbf{m_i} \in \mathcal{M}$ we take the multiindex $\sigma(\mathbf{m_i}) = (\sigma(m_{1,i_1}), \dots, \sigma(m_{n,i_n}))$ where σ is a permutation of the set $\{m_{1,i_1}, \dots, m_{n,i_n}\}$ such that $\sigma(m_{1,i_1}) \le \dots \le \sigma(m_{n,i_n})$. We denote with $E_{\mathbf{m_i}}$ the closed convex poligonal region with vertices $(1,1)$,

$$B_{J,i_J} = \left(\frac{1 + S_{J+1}^{\mathbf{i}} + (J-1)\,(\sigma(m_{J,i_J}))^{-1}}{1 + J\,(\sigma(m_{J,i_J}))^{-1} + S_{J+1}^{\mathbf{i}}}, \frac{1 - (\sigma(m_{J,i_J}))^{-1}}{1 + J\,(\sigma(m_{J,i_J}))^{-1} + S_{J+1}^{\mathbf{i}}} \right),$$

$1 \leq J \leq n$ and its symetrics with respect to the non principal diagonal, where $S_J^{\mathbf{i}} = \sum_{j=J}^{n} \left(\sigma\left(m_{j,i_j}\right)\right)^{-1}$.

Theorem 3.1. *E_μ is the closed convex polygonal region given by*

$$E_\mu = \bigcap_{\mathbf{m_i} \in \mathcal{M}} E_{\mathbf{m_i}}.$$

Proof. For each $z_j \in \overline{D_j}$ we have a ball $B_{r(z_j)}(z_j) \subset \Omega_j$ such that for $z \in B_{r(z_j)}(z_j)$,

$$\omega_{j,z_j}(z) = \varphi_j(z) - \varphi_j(z_j) - (z - z_j)\varphi_j'(z_j) = (z - z_j)^{m_{j,z_j}} g_{j,z_j}(z)$$

with $g_{j,z_j}(z_j) \neq 0$, $m_{j,z_j} \geq 2$ and ω_{j,z_j}, $\left(\omega_{j,z_j}\right)'$, ...,$\left(\omega_{j,z_j}\right)^{\left(m_{j,z_j}\right)}$ different from zero on $B_{r(z_j)}(z_j) - \{z_j\}$. We note that if $z_j = z_{j,i}$ for some $1 \leq i \leq l_j$ then $m_{j,z_j} = m_{j,i} > 2$. On the other case $m_{j,z_j} = 2$. Since $\overline{D_j}$ is a compact set, there exists a finite set $F \subset \prod_{1 \leq j \leq n} \overline{D}_j$ such that D can be covered with a finite collection of sets of the form

$$\mathcal{D}_{z_1,...,z_n} = \prod_{1 \leq j \leq n} B_{r(z_j)}(z_j),$$

$(z_1, ..., z_n) \in F$. We denote by $T_{\mathcal{D}_{z_1,...,z_n}}$ the operator of convolution with $\mu_{\mathcal{D}_{z_1,...,z_n}}$ defined by (1.1) with D replaced by $\mathcal{D}_{z_1,...,z_n}$.

Now,

$$\|T\|_{p,q} \leq \sum_{(z_1,...,z_n) \in F} \left\|T_{\mathcal{D}_{z_1,...,z_n}}\right\|_{p,q}.$$

We note that $(m_{1,z_1}, ..., m_{n,z_n}) \in \mathcal{M}$, thus $(m_{1,z_1}, ..., m_{n,z_n}) = \mathbf{m_i}$ for some $\mathbf{i} = (i_1, ..., i_n)$, $0 \leq i_j \leq l_j$, $1 \leq j \leq n$. After a linear change of variables (if necessary) we can apply the results of the previous paragraph to obtain that the type set associated to $T_{\mathcal{D}_{z_1,...,z_n}}$ is $E_{\mathbf{m_i}}$. So

$$\bigcap_{\mathbf{m_i} \in \mathcal{M}} E_{\mathbf{m_i}} \subseteq E_\mu.$$

Now we take $\mathbf{m_i} \in \mathcal{M}$. If $m_{j,i_j} > 2$ for every $1 \leq i_j \leq l_j$, $1 \leq j \leq n$, we observe that since φ_j'' does not vanish on ∂D_j, we can take $B_{r\left(z_{j,i_j}\right)}\left(z_{j,i_j}\right) \subset D_j$ so

$$\mathcal{D}_{\mathbf{i}} = B_{r\left(z_{1,i_1}\right)}(z_{1,i_1}) \times ... \times B_{r(z_{n,i_n})}(z_{n,i_n}) \subset D,$$

and then

$$\left\|T_{\mathcal{D}_{\mathbf{i}}}\right\|_{p,q} \leq \|T\|_{p,q}.$$

Now the type set associated to $\mathcal{D}_{\mathbf{i}}$ is $E_{\mathbf{m_i}}$, so $E_\mu \subseteq E_{\mathbf{m_i}}$. Finally, if some $m_{j,i_j} = 2$, we take any point $\widetilde{z}_j \in D_j$ and a ball B_j with center $\widetilde{z}_j$, contained in D_j such that $\omega_{j,\widetilde{z}_j}$, $\omega'_{j,\widetilde{z}_j}$ and $\omega''_{j,\widetilde{z}_j}$ be different from zero on $B - \{\widetilde{z}_j\}$. For the other $j's$ we take $B_{r\left(z_{j,i_j}\right)}\left(z_{j,i_j}\right)$. Since $E_{\mathbf{m_i}}$ is the type set associated to the cartesian product of these balls, we proceed as before. ∎

REFERENCES

[1] M. CHRIST, *Endpoint Bounds for Singular Fractional Integral Operators*, UCLA Preprint 1988.

[2] E. FERREYRA, T. GODOY and M. URCIUOLO, Endpoint bounds for convolution operators with singular measures, *Coll. Math.*, 76, 1 (1998), pp. 35-47.

[3] E. FERREYRA, T. GODOY and M. URCIUOLO, Convolution operators with fractional measures associated to holomorphic functions, *Acta Math. Hungar.*, 92 (1-2) (2001), pp. 27-38.

[4] E. FERREYRA, T. GODOY and M. URCIUOLO, The type set for homogeneous singular measures on R^3 of polynomial type, *Coll. Math.*, 106 (2006), pp. 161-175.

[5] L. DE CARLI and A. IOSEVICH, Some sharp restriction theorems for homogeneous manifolds, *The Journal of Fourier Analysis and Appl.*, 4, 1 (1996), pp. 105-128.

[6] A. IOSEVICH and E. SAWYER, Sharp $L^p - L^q$ estimates for a class of averaging operators, *Ann. Inst. Fourier*, 46, 5 (1996), pp. 1359-1384.

[7] F. RICCI, Limitatezza $L^p - L^q$ per operatori di convoluzione definiti da misure singolari in R^n, *Bollettino U.M.I.*, (7) 11-A (1997), pp. 237-252.

[8] F. RICCI and E. M. STEIN, Harmonic Analysis on Nilpotent Groups and Singular integrals. III. Fractional Integration along Manifolds, *J. Funct. Anal.*, 86 (1989), pp. 360-389.

[9] E. M. STEIN, *Harmonic Analysis. Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, 1993.

[10] E. M. STEIN and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1975.