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A NEW PROPERTY OF GENERAL MEANS OF ORDER p WITH AN APPLICATION TO THE THEORY OF ECONOMIC GROWTH

OLIVIER DE LA GRANDVILLE

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STANFORD UNIVERSITY, DEPARTMENT OF MANAGEMENT SCIENCE AND ENGINEERING, HUANG ENGINEERING CENTER, 475 VIA ORTEGA, STANFORD, CALIFORNIA 94305
lagrandvil@aol.com

ABSTRACT. The purpose of this note is to demonstrate a new property of the general mean of order p of m ordered positive numbers $M_p(x_1, ..., x_j, ..., x_m) = \left[\sum_{j=1}^{m}(f_j x_j^p)\right]^{1/p}$. If $p < 0$ and if $\lim_{t\to\infty} x_m(t)/x_j(t) = \infty$, $j = 1, ..., m-1$, the elasticity of $M_p(x_1(t), ..., x_m(t))$ with respect to x_m, defined by $\varepsilon_{M,x_m} \equiv (\partial f/f)/(\partial x_m/x_m)$, tends toward zero, and therefore $\lim_{t\to\infty} \sum_{j=1}^{m-1} \varepsilon_{M,x_j} = 1$. This property is then applied to optimal growth theory.

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1. INTRODUCTION

Since the seminal work of Hardy, Littlewood and Polya *Inequalities* [5], the properties of the general mean (or power mean) of order p have been well understood and documented. A wealth of new properties and extensions have been added in the second part of the 20th century: they are in the classic treatise by Bullen, Mitrinović and Vasić, *Means and Their Inequalities* [3], and a remarkable outgrowth of results has since risen; proof is the new version of this work by Peter S. Bullen, under the title *Handbook of Means and Their Inequalities* [2] .

In this note we offer a property of general means which, to the best of our knowledge, has not yet been uncovered. Simple as its proof may be, the property is far from intuitive, and has far-reaching implications in the theory of optimal economic growth, as we will see.

2. DEFINITION

Let $f(x_1, ..., x_j, ..., x_m)$ be a positive, differentiable function of m positive numbers $x_1, ..., x_j, ..., x_m$. The elasticity of f with respect to x_j is defined by

$$\lim_{\Delta x_j \to 0} (\Delta f/f)/(\Delta x_j/x_j) \equiv (\partial f/f)/(\partial x_j/x_j) = \partial \ln f/\partial \ln x_j.$$

Its geometric interpretation is the relative increase of the tangent to the surface f in the direction of x_j divided by the corresponding relative increase in x_j.

3. LEMMA

For $p < 0$, if lim$_{t\to\infty} x_m(t)/x_j(t) = \infty$, $j = 1, ..., m-1$, the elasticity of $M_p(x_1(t), ..., x_m(t))$ with respect to x_m, denoted ε_{M,x_m}, tends toward zero, and $\lim_{t\to\infty} \sum_{j=1}^{m-1} \varepsilon_{M,x_j} = 1$.

Proof. With

$$M_p(x_1(t), ..., x_m(t)) = \left[\sum\nolimits_{j=1}^{m} (f_j x_j^p)\right]^{1/p},$$

$$\lim_{t\to\infty} \partial \ln M_p/\partial \ln x_m = \lim_{t\to\infty} \left[1 + \sum\nolimits_{j=1}^{m-1} \frac{f_j}{f_m} (\frac{x_j}{x_m})^p\right]^{-1} = 0.$$

Furthermore, M_p being homogeneous of degree 1 in $x_1, ..., x_m$, Euler's identity applies and $\lim_{t\to\infty} \sum_{j=1}^{m-1} \varepsilon_{M,x_j} = 1$. ∎

4. APPLICATION

It turns out that general (or power) means find extensive applications in economics, in particular in the theory of economic growth. The seminal paper on the subject is due to Arrow, Chenery, Minhas and Solow [1]. The authors were looking for the family of production functions $Y = F(K, L)$ (where $Y \equiv$ production; $K \equiv$ physical capital; $L \equiv$ labor) that would meet two conditions: a) it should be homogeneous of degree one, conforming usual observations – therefore production, or income per person $Y/L \equiv y$ would become a function of the variable $K/L \equiv r$ only; $Y/L \equiv y \equiv F(K/L, 1) = f(r)$; and b) it should correspond to the empirical relationship observed between income per person and the partial derivative $\partial F/\partial L$, equal to the wage rate w; this observed relationship was of the form $y = aw^\sigma$, where $a > 0$ and $0 < \sigma < 1$; since $w = \partial F/\partial L \equiv f(r) - rf'(r) \equiv y - ry'$, the above-mentioned authors were led to solve the differential equation $y = a(y - ry')^\sigma$, and to the conclusion that $Y = F(K, L)$ was a linear transformation of the mean of order $p \equiv 1 - 1/\sigma$ of K and L (K and L are assumed to be unitless, i.e. index numbers). Denoting $q \equiv \partial F/\partial K$, and identifying the relevant constant of integration[1], one ends up in fact with the power mean

[1] On this identification see La Grandville (2009, pp. 85-86, [6])

$$Y_t/Y_0 = [\delta\,(K_t/K_0)^p + (1-\delta)(L_t/L_0)^p]^{1/p}$$

where $\delta \equiv q_0K_0/Y_0$. In order to simplify notation, this mean will now be written as

$$Y_t = [\delta K_t^p + (1-\delta)L_t^p]^{1/p}\,;$$

Y_t, K_t, and L_t stand for the indices Y_t/Y_0, K_t/K_0, and L_t/L_0 respectively[2].

Suppose now that technological progress applies in such a way that at any time t existing capital and labour are enhanced by factors $G_K(t)$ and $G_L(t)$, two increasing functions of time t.The production function is thus written

$$Y = \{\delta[K_tG_K(t)]^p + (1-\delta)[L_tG_L(t)]\}^{1/p}. \tag{4.1}$$

Until now it was thought that whatever the differential equations governing the time paths of K, L, G_K and G_L, in the long term the growth rate of income per person $\dot{y}/y$ could tend asymptotically toward the growth rate of $G_L(t)$ only if technological progress was purely labour-augmenting – equivalently only if G_K is constant. We will now show, thanks to the above lemma, that this is not so, and that $\dot{y}/y$ may very well tend toward the growth rate of G$_L$, denoted $\dot{g}_L/g_L$, even in the presence of capital augmenting technical progress.

Denote the augmented variables $G_K(t)\ K(t) \equiv U(t)$ and $G_L(t)\ L(t) \equiv V(t)$. The production function is a mean of order p of U_t and V_t, written $M_p(U_t, V_t)$:

$$Y_t = F(K_tG_K(t), L_tG_L(t)) = M_p(U_t, V_t) = [\delta U_t^p + (1-\delta)V_t^p]^{1/p}, p \neq 0.$$

Its growth rate is

$$\begin{aligned} \frac{\dot{Y}_t}{Y_t} &= \frac{\dot{M}_p(t)}{M_p(t)} = \frac{\partial M}{\partial U}(t)\frac{U_t}{M_t}\left[\frac{\dot{U}_t}{U_t}\right] + \frac{\partial M}{\partial V}(t)\frac{V_t}{M_t}\left[\frac{\dot{V}_t}{V_t}\right] = \\ & \varepsilon_{M,U}\left[\frac{\dot{K}_t}{K_t} + g_K(t)\right] + \varepsilon_{M,V}\left[\frac{\dot{L}_t}{L_t} + g_L(t)\right]. \end{aligned}$$

Without any loss in generality, suppose that $g_K(t)$ and $g_L(t)$ are bounded. Applying the lemma, it suffices that inequality

$$\lim_{t\to\infty} \dot{V}_t/V_t > \lim_{t\to\infty} \dot{U}_t/U_t$$

applies for $\lim_{t\to\infty}\varepsilon_{M,U} = 0, \lim_{t\to\infty}\varepsilon_{M,V} = 1$, and therefore $\lim_{t\to\infty}\dot{y}/y = \lim_{t\to\infty}(\dot{Y}/Y - \dot{L}/L) = g_L(\infty)$ .We will now show that there are important circumstances under which this inequality is fulfilled.

[2]From the above equation it can be immediately seen that income per person is a power mean of K/L and 1 :

$$y_t = [\delta(K_t/L_t)^p + 1 - \delta]^{1/p} = M_p(r, 1).$$

Not only is M_p is an increasing function of p but, as a mean of two numbers the curve M_p in (M, p) space has one and only one inflection point. Due to the extreme complexity of the second derivative $\partial^2M/\partial p^2$, an analytical proof looked unreacheable, which led Robert Solow and the present author to offer this property as a conjecture (JIPAM, 2006, Vol. 7, no. 1, article 3, [7]). Thanh Nam and Nguyet Minh finally succeeded in demonstrating the conjecture in a 5-page, 3-step, highly skilful proof (2008, Vol. 9, no. 3, article 86, [8]). The importance of this property is the following: if σ is between 0.5 and 1, optimal trajectories of the economy are such that income per person is in the vicinity of this inflection point, and developed economies seem to be precisely in that vicinity (on this, see La Grandville, 2009, [6]).

Suppose an economy is in competitive equilibrium. Such an equilibrium is characterized by the equality between the marginal productivity of capital $\partial F/\partial K$ and the rate of interest, denoted $i(t)$, always assumed to be smaller than δ.We thus have

$$\frac{\partial F}{\partial K}(K_t G_K(t), L_t G_L(t)) = i\,(t)\ . \tag{4.2}$$

In a first step, let us examine what this equality implies in terms of optimality for our future. Innocuous as equation (4.2) may seem, it conveys two pleasant surprises. First, its implementation maximizes, from any initial time to infinity, the integral sum of all discounted consumption flows society may acquire: indeed, defining consumption C as the difference between production (or income) Y and investment $I \equiv \dot{K}$, this integral is

$$W = \int_0^\infty [F(K_t G_K(t), L_t G_L(t)) - \dot{K}_t] e^{-\int_0^t i(z)dz} dt. \tag{4.3}$$

Denoting $H(K, \dot{K}, t)$ the integrand of (4.3) and applying the Euler equation

$$\frac{\partial H}{\partial K}(K, \dot{K}, t) - \frac{d}{dt}\frac{\partial H}{\partial \dot{K}}(K, \dot{K}, t) = 0$$

yields (4.2). Due to the concavity of H in $(K, \dot{K})$, (4.2) constitutes both a necessary and a sufficient condition for (4.3) to be maximized. Note that if (4.2) is not a differential equation, as is generally the case of Euler equations, but simply an ordinary equation, the reason is that the integrand of the functional (4.3) is an affine function of $\dot{K}$.

There is even more to equation (4.2). We will now make use of the beautiful idea Robert Dorfman (1969, [3]) had when he introduced a "modified Hamiltonian" as follows – in order to honor Professor Dorfman's memory, we choose to call this new Hamiltonian a Dorfmanian, and designate it by D. Let λ_t denote the discounted valuation of one unit of capital received by society at time t, equal to $\lambda_t = \frac{\partial W}{\partial K_t} = \frac{\partial}{\partial K_t}\{\int_t^\infty [F(K_\tau, \tau) - \dot{K}_\tau] e^{-\int_t^\tau i(z)dz} d\tau\}$. The Dorfmanian is equal to the traditional Hamiltonian H plus $\dot{\lambda}_t K_t$, or $D = [F(K_t, t) - \dot{K}_t] e^{-\int_0^t i(z)dz} + \lambda_t \dot{K}_t + \dot{\lambda}_t K_t$. In our case we have

$$D(K_t, \dot{K}_t, t) = C_t e^{-\int_0^t i(z)dz} + \frac{d}{dt}(\lambda_t K_t).$$

D thus represents the discounted valuation of society's activity at any point of time t, since it is equal to consumption plus the rate of increase in the value of capital at that time, in present value. Noting the concavity of D with respect to K_t and $\dot{K}_t$ and setting its gradient to zero, it can be verified that (4.2) maximizes D as well. We therefore conclude that equation (4.2) constitutes a necessary and sufficient condition for maximising not only one, but *two* fundamental quantities: first, the sum over an infinite time horizon of all discounted consumption flows at society's diposal; and secondly, at *any* point of time, the value of society's activity.

To derive the optimal time paths for the economy, we start by looking for a solution K_t^* to system (4.1),(4.2). For the time being, let us consider $i(t)$ as a constant i, with $i < \delta$ (naturally, i could be set at different values in comparative dynamics as long as inequality $i < \delta$ is enforced). We obtain

$$K_t^* = \left(\frac{1-\delta}{\delta}\right)^{1/p} \frac{L_t G_L(t)/G_K(t)}{[\delta^{-\sigma} i^{\sigma-1} G_K(t)^{1-\sigma} - 1]^{1/p}},\ p \neq 0,\ \sigma \neq 1\,. \tag{4.4}$$

A unique, positive solution is defined by (4.4) if and only if the following condition on σ, i, δ, and $G_K(t)$ holds:

$$\sigma < \frac{\ln i - \ln G_K(t)}{\ln(i/\delta) - \ln G_K(t)}.$$

If $\sigma < 1$ is smaller than one (our case since $p = 1 - 1/\sigma < 0$), this condition is fulfilled because the right-hand side of the above inequality is always larger than one.

We now show that U_t plays the role of $x_m(t)$ in the lemma. Let $\dot{L}(t)/L(t)$ be denoted as $n(t)$; from (4.4) the growth rate of K_t is

$$\frac{\dot{K}_t^*}{K_t^*} = g_L(t) + n(t) - g_K(t)\left[1 - \frac{\sigma}{1 - \delta^\sigma i^{1-\sigma} G_K(t)^{-(1-\sigma)}}\right];$$

using the fact that $\dot{V}_t/V_t = n(t) + g_L(t)$, this implies

$$\frac{\dot{U}_t}{U_t} = \frac{\dot{K}_t^*}{K_t^*} + g_K(t) = \dot{V}_t/V_t + bg_K(t)\left[\frac{1}{1 - \delta^\sigma i^{1-\sigma} G_K(t)^{-(1-\sigma)}}\right].$$

The difference between $\dot{U}_t/U_t$ and $\dot{V}_t/V_t$ is bounded away from zero; therefore $\lim_{t\to\infty} U_t/V_t = \infty$, and the lemma applies to U_t: $\lim_{t\to\infty} e_{M,U_t} = 0$, $\lim_{t\to\infty} e_{M,V_t} = 1$, from which we conclude that $\lim_{t\to\infty} \dot{Y}_t/Y_t = n(\infty) + g_L(\infty)$ and $\lim_{t\to\infty} \dot{y}_t/y_t = g(\infty)$, as was to be proved.

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