AN $L^p$ INEQUALITY FOR ‘SELF-RECIPROCAL’ POLYNOMIALS. II

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ABSTRACT. The main result of this paper is a sharp integral mean inequality for the derivative of a ‘self-reciprocal’ polynomial.

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1. Introduction

Let $\mathcal{P}_n$ be the class of all polynomials of degree at most $n$. We say that a polynomial $f \not\equiv 0$ belongs to $\mathcal{P}_n$ if it is of degree at most $n$ and satisfies the condition $f(z) \equiv z^n f(1/z)$. Frappier, Rahman and Ruscheweyh [3, p. 96] call such a polynomial ‘self-reciprocal’. Such polynomials have been studied for about thirty years (see [1, 2] [3, §7.5], [5, 6, 8, 9] [10, pp. 229–230], [11, pp. 431–432], [12, 13]).

For any entire function $F$, let

$$M_p(F; r) := \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta \right)^{1/p} \quad (0 < p < \infty; r > 0).$$

It is well-known (see for example [7, p. 143]) that for any given $r > 0$, the integral mean $M_p(F; r)$ is a non-decreasing function of $p$ and that

$$M_p(F; r) \to \max_{|z|=r} |F(z)| \quad \text{as} \quad p \to \infty.$$ 

This explains the notation

$$\mathcal{M}_\infty(F; r) := \max_{|z|=r} |F(z)| \quad (r > 0).$$

It is also known (see [7, p. 139]) that for any given $r > 0$,

$$M_p(F; r) \to \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(re^{i\theta})| \, d\theta \right) \quad \text{as} \quad p \to 0.$$ 

So, we set

$$\mathcal{M}_0(F; r) := \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(re^{i\theta})| \, d\theta \right).$$

When $F$ is a polynomial, the associated quantity $\mathcal{M}_0(F; 1)$ is called its logarithmic Mahler measure. Let $F(z) := a_m \prod_{\mu=1}^{m} (z - z_{\mu})$, and suppose that $F(0)$ is not zero. Then by Jensen’s theorem (see [14, p. 124])

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(e^{i\theta})| \, d\theta = \log |F(0)| \prod_{|z_{\mu}|<1} |z_{\mu}|.$$ 

Hence

$$\mathcal{M}_0(F; 1) = \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(e^{i\theta})| \, d\theta \right) = |a_m| \prod_{\mu=1}^{m} \max\{|z_{\mu}|, 1\}.$$ 

Let $F(z) := z^m \Phi(z)$, where $\Phi(0) \neq 0$. Then $\mathcal{M}_0(F; 1) = \mathcal{M}_0(\Phi; 1)$, and (1.4) applies to $\Phi$. Hence, in (1.4) the restriction ‘$F(0) \neq 0$’ may be dropped, making it a very useful formula.

Bernstein’s inequality for polynomials says that if $f$ is a polynomial of degree at most $n$ such that $\mathcal{M}_\infty(f; 1) = 1$, then

$$\mathcal{M}_\infty(f'; 1) \leq n,$$

where the inequality becomes an equality if and only if $f(z) \equiv e^{i\gamma} z^n$, $\gamma \in \mathbb{R}$. Since $\mathcal{M}_p(F; r)$ is a non-decreasing function of $p$, it follows that for any polynomial $f$ of degree at most $n$ such that $\mathcal{M}_\infty(f; 1) = 1$, we have

$$\mathcal{M}_p(f'; 1) \leq n \quad (0 \leq p < \infty).$$
It is interesting that this is also sharp; for any \( p \in [0, \infty) \), the inequality becomes an equality for polynomials of the form \( f(z) \equiv e^{i\gamma} z^n \), \( \gamma \in \mathbb{R} \). Thus
\[
\sup \left\{ \frac{M_p(f'; 1)}{M_{\infty}(f; 1)} : f \in \mathcal{P}_n, f(z) \neq 0 \right\} = n \quad (0 \leq p \leq \infty).
\]

There exists (see [3, p. 97]) a polynomial \( f_\ast \in \mathcal{P}_n \) such that
\[
M_{\infty}(f'_\ast ; 1) \geq M_{\infty}(f_\ast ; 1) (n - 1).
\]
This is surprising since, for any \( p \in [0, \infty) \), the extremals in (1.5) have all their zeros at the origin whereas any polynomial in \( \mathcal{P}_n \) must have at least half of its zeros outside the open unit disk.

If \( f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \) is such that \( a_0 = a_n \) then (see [4, Theorem 2]),
\[
M_{\infty}(f' ; 1) \leq M_{\infty}(f ; 1) \left( n - \frac{1}{2} + \frac{1}{2(n+1)} \right).
\]
In particular, (1.7) holds for any \( f \in \mathcal{P}_n \).

In this paper we consider the following question that was mentioned to me by Professor Q. I. Rahman. It asks for an analogue of (1.5) for the subclass \( \mathcal{P}_n \).

**Question.** What is the value of the constant
\[
\kappa_{n, p} := \sup \left\{ \frac{M_p(f'; 1)}{M_{\infty}(f; 1)} : f \in \mathcal{P}_n \right\}
\]
for any given \( p \in [0, \infty] \)?

From (1.6) and (1.7) we know that
\[
n - 1 \leq \kappa_{n, \infty} \leq n - \frac{1}{2} + \frac{1}{2(n+1)}.
\]
However, the exact value of \( \kappa_{n, \infty} \) remains unknown and elusive.

The following result contains an answer to the question in the case where \( p \) lies in \([0, 2]\).

**Theorem 1.1.** Let \( f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \) be a polynomial of degree at most \( n \) such that \( f(z) \equiv z^n f(1/z) \). Furthermore, let \( M_p(\cdot, \cdot) \) be as in (1.1) and \( M_{\infty}(\cdot, \cdot) \) be as in (1.2). Then,
\[
M_p(f' ; 1) \leq \frac{n}{\sqrt{2}} \sqrt{\frac{1}{n} \sum_{k=0}^{n-1} |f(e^{2k\pi i/n})|^2 - 2|a_0|^2} \quad (0 \leq p \leq 2),
\]
and so a fortiori
\[
M_p(f' ; 1) \leq \frac{n}{\sqrt{2}} \sqrt{(M_{\infty}(f ; 1))^2 - 2|a_0|^2} \quad (0 \leq p \leq 2).
\]
The example \( f(z) := z^n + 1 \) shows that both (1.9) and (1.10) are sharp for every \( p \in [0, 2] \).

It may be added that if \( |f(e^{2k\pi i/n})| \leq 1 \) for \( k = 0, 1, \ldots, n-1 \), then \( \frac{1}{n} \sum_{k=0}^{n-1} |f(e^{2k\pi i/n})|^2 \leq 1 \), and so (1.9) contains the following result, which is stronger than (1.10).
Corollary 1.2. Let \( f(z) := \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) be a polynomial of degree at most \( n \) such that \( f(z) = z^{n} f(1/z) \). Furthermore, let \( |f(e^{2k\pi i/n})| \leq 1 \) for \( k = 0, 1, \ldots, n - 1 \). Then
\[
M_{p}(f'; 1) \leq \frac{n}{\sqrt{2}} \sqrt{1 - 2|a_{0}|^{2}} \quad (0 \leq p \leq 2).
\]
As the example \( f(z) := (z^{n} + 1)/2 \) shows, the estimate is sharp for every \( p \in [0, 2] \).

Remark 1.1. In view of (1.4), the case \( p = 0 \) of (1.10) can be stated as follows:

Let \( f(z) := \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) be a polynomial of degree at most \( n \) such that \( f(z) = z^{n} f(1/z) \), and let \( \zeta_{1}, \ldots, \zeta_{n-1} \) be the zeros of \( f' \). Then
\[
|a_{0}|^{2} + |a_{n}|^{2} \left\{ \prod_{\nu=1}^{n-1} \max\{|\zeta_{\nu}|, 1\} \right\}^{2} \leq \left( M_{\infty}(f; 1)^{2} \right)/2.
\]

Theorem 1.1 implies the following result.

Corollary 1.3. Let \( g(x) := \sum_{\nu=0}^{n} c_{\nu} x^{\nu} \) be a polynomial of degree at most \( n \) with coefficients in \( C \). Furthermore, for any \( p \in (0, \infty) \), let
\[
I_{p, 1} := \frac{1}{\pi} \int_{-1}^{1} \left| n g(x) + i\sqrt{1 - x^{2}} g'(x) \right|^{p} \frac{dx}{\sqrt{1 - x^{2}}}
\]
and
\[
I_{p, 2} := \frac{1}{\pi} \int_{-1}^{1} \left| n g(x) - i\sqrt{1 - x^{2}} g'(x) \right|^{p} \frac{dx}{\sqrt{1 - x^{2}}}.
\]
Then, for any \( p \in [0, 2] \), we have
\[
\left( \frac{I_{p, 1} + I_{p, 2}}{2} \right)^{1/p} \leq n \left\{ \frac{1}{n} \left( |g(1)|^{2} + 2 \sum_{k=1}^{n-1} \left| g\left( \cos \frac{k\pi}{n} \right) \right|^{2} + |g(-1)|^{2} \right) - \left( \frac{|c_{n}|}{2^{n-1}} \right)^{2} \right\}.
\]
In particular, if \( |g(x)| \leq 1 \) at the points \( x_{k} := \cos (k\pi/n) \), \( k = 0, 1, \ldots, n \) then, for any \( p \in (0, 2] \), we have
\[
\left( \frac{I_{p, 1} + I_{p, 2}}{2} \right)^{1/p} \leq n \sqrt{2 - \left( \frac{|c_{n}|}{2^{n-1}} \right)^{2}}.
\]
Both (1.11) and (1.12) become equalities for \( T_{n}(x) := \cos (n \arccos x) \), the Chebyshev polynomial of the first kind of degree \( n \).

Remark 1.2. Since
\[
\frac{1}{\pi} \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^{2}}} = 1,
\]
the quantity
\[
\frac{1}{\pi} \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^{2}}} = 1,
\]
appearing on the left-hand sides of (1.11) and (1.12), is a ‘weighted integral mean’ of
\[
\frac{1}{2} \left\{ \left| n g(x) + i\sqrt{1 - x^{2}} g'(x) \right|^{p} + \left| n g(x) - i\sqrt{1 - x^{2}} g'(x) \right|^{p} \right\}.
\]
2. AN AUXILIARY RESULT

Lemma 2.1. Let \( f(z) := \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) be a polynomial of degree \( n \) such that \( a_0 = a_n \). Then,

\[
\frac{1}{n} \sum_{k=0}^{n-1} |f(e^{2k\pi i/n})|^2 = \sum_{\nu=0}^{n} |a_{\nu}|^2 + 2 |a_0|^2 = \sum_{\nu=0}^{n} |a_{\nu}|^2 + 2 |a_n|^2. \tag{2.1}
\]

Proof. For any real \( \theta \), we have

\[
|f(e^{i\theta})|^2 = f(e^{i\theta}) \overline{f(e^{i\theta})} = \sum_{\nu=0}^{n} a_{\nu} e^{i\nu \theta} \sum_{\nu=0}^{n} \overline{a_{\nu}} e^{-i\nu \theta} = \sum_{m=-n}^{n} b_m e^{im\theta}, \tag{2.2}
\]

where

\[
b_{-n} = a_0 \overline{a_n} = |a_0|^2, \quad b_n = a_n \overline{a_0} = |a_0|^2, \quad b_0 = \sum_{\nu=0}^{n} |a_{\nu}|^2
\]

and

\[
b_m = \sum_{\mu=0}^{n-m} a_{m+\mu} \overline{a_\mu}, \quad b_{-m} = \overline{b_m} \quad (m = 1, \ldots, n-1).
\]

The values of \( b_1, \ldots, b_{n-1} \) and \( b_{-1}, \ldots, b_{-n+1} \) are of little importance.

Let \( \omega := e^{2m\pi i/n} \), where \( m \in \{\pm 1, \ldots, \pm (n-1)\} \). Note that \( \omega^n = 1 \). Since \( \omega \neq 1 \), we see that

\[
\sum_{k=0}^{n-1} \left( e^{2m\pi i/n} \right)^k = \sum_{k=0}^{n-1} \omega^k = \frac{1 - \omega^n}{1 - \omega} = 0 \quad (m = \pm 1, \ldots, \pm (n-1)). \tag{2.4}
\]

From \([2.2], [2.3] \) and \([2.4] \) it follows that

\[
\sum_{k=0}^{n-1} |f(e^{2k\pi i/n})|^2 = b_{-n} \sum_{k=0}^{n-1} e^{-2k\pi i} + b_n \sum_{k=0}^{n-1} e^{2k\pi i} + \sum_{k=0}^{n-1} b_0
\]

\[
+ \sum_{m=-n+1}^{n-1} b_m \sum_{k=0}^{n-1} \left( e^{2m\pi i/n} \right)^k \\
\quad m \neq 0
\]

\[
= nb_{-n} + nb_n + nb_0 = n \sum_{\nu=0}^{n} |a_{\nu}|^2 + 2n |a_0|^2,
\]

which is equivalent to \([2.1] \). \( \blacksquare \)

3. PROOFS OF THEOREM [1.1] AND COROLLARY [1.3]

Proof of Theorem [1.1] We present the proof in three steps.

Step I. First we show that if \( f(z) := \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) is a polynomial of degree at most \( n \), whose coefficients satisfy the condition

\[
|a_{\nu}| = |a_{n-\nu}| \quad (\nu = 0, 1, \ldots, n), \tag{3.1}
\]

then

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(e^{i\theta})|^2 \, d\theta \leq \frac{n^2}{2} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \, d\theta.
\]
Indeed, (3.1) allows us to write
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(e^{i\theta})|^2 \, d\theta = \sum_{\nu=0}^{n} \nu^2 |a_{\nu}|^2 = \frac{1}{2} \left( \sum_{\nu=0}^{n} \nu^2 |a_{\nu}|^2 + \sum_{\nu=0}^{n} \nu^2 |a_{n-\nu}|^2 \right) \\
= \frac{1}{2} \sum_{\nu=0}^{n} \left\{ \nu^2 + (n - \nu)^2 \right\} |a_{\nu}|^2 \\
\leq \frac{n^2}{2} \sum_{\nu=0}^{n} |a_{\nu}|^2 = \frac{n^2}{2} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \, d\theta \right).
\]

Step II. By Lemma 2.1,
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \, d\theta = \sum_{\nu=0}^{n} |a_{\nu}|^2 = \frac{1}{n} \sum_{k=0}^{n-1} \left| f(e^{2k\pi i/n}) \right|^2 - 2 |a_0|^2.
\]
Hence,
\[
(3.2) \quad M_2(f'; 1) \leq \frac{n}{\sqrt{2}} \left( \frac{1}{n} \sum_{k=0}^{n-1} \left| f(e^{2k\pi i/n}) \right|^2 - 2 |a_0|^2 \right)^{1/2}.
\]

Step III. Finally, note that \( M_p(f'; 1) \leq M_2(f'; 1) \) for \( 0 \leq p \leq 2 \) since \( M_p(f'; 1) \) is a nondecreasing function of \( p. \) Hence (3.2) implies (1.9). \( \square \)

**Proof of Corollary 1.3** Consider the polynomial
\[
f(z) := z^n g \left( \frac{z + z^{-1}}{2} \right).
\]
It clearly belongs to \( \wp_{2n}, \) and writing it in the form \( f(z) := \sum_{\nu=0}^{2n} a_{\nu} z^{\nu} \) we see that \( a_{2n} = a_0 = c_n/2^n. \) Since
\[
|f'(e^{i\theta})| = |i n g(\cos \theta) + g'(\cos \theta)(-\sin \theta)|,
\]
Theorem [11] with \( 2n \) in place of \( n, \) may be applied to \( f \) to conclude that for any \( p \in (0, 2], \) we have
\[
\left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |n g(\cos \theta) + i (\sin \theta) g'(\cos \theta)|^p \, d\theta \right)^{1/p} \leq n \left[ \frac{1}{n} \left| g(1) \right|^2 + 2 \sum_{k=1, k \neq n}^{2n-1} \left| g\left( \cos \frac{k\pi}{n} \right) \right|^2 + \left| g(-1) \right|^2 \right]^{1/2},
\]
which is equivalent to (1.11). \( \square \)

**References**


