The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 8, Issue 1, Article 3, pp. 1-7, 2011

AN L^p INEQUALITY FOR 'SELF-RECIPROCAL' POLYNOMIALS. II

M. A. QAZI

Received 24 November, 2008; accepted 27 March, 2009; published 8 September, 2011.

DEPARTMENT OF MATHEMATICS, TUSKEGEE UNIVERSITY, TUSKEGEE, ALABAMA 36088 USA,
qazima@aol.com

ABSTRACT. The main result of this paper is a sharp integral mean inequality for the derivative of a 'self-reciprocal' polynomial.

Key words and phrases: Polynomials, Restricted zeros, Growth, Inequalities.

2000 *Mathematics Subject Classification.* Primary 41A17. Secondary 30A10, 41A10.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Let $\mathcal{P}_n$ be the class of all polynomials of degree at most n. We say that a polynomial $f \not\equiv 0$ belongs to $\wp_n$ if it is of degree at most n and satisfies the condition $f(z) \equiv z^n f(1/z)$. Frappier, Rahman and Ruscheweyh [3, p. 96] call such a polynomial 'self-reciprocal'. Such polynomials have been studied for about thirty years (see [1, 2] [3, §7.5], [5, 6, 8, 9] [10, pp. 229–230], [11, pp. 431–432], [12, 13]).

For any entire function F, let

$$\mathcal{M}_p(F\,;\,r) := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|F\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|^p\,\mathrm{d}\theta\right)^{1/p} \qquad (0<p<\infty\,;\, r>0)\,. \tag{1.1}$$

It is well-known (see for example [7, p. 143]) that for any given $r > 0$, the integral mean $\mathcal{M}_p(F\,;\,r)$ is a non-decreasing function of p and that

$$\mathcal{M}_p(F\,;\,r) \to \max_{|z|=r}|F(z)| \quad \text{as} \quad p\to\infty\,.$$

This explains the notation

$$\mathcal{M}_\infty(F\,;\,r) := \max_{|z|=r}|F(z)| \qquad (r>0)\,. \tag{1.2}$$

It is also known (see [7, p. 139]) that for any given $r > 0$,

$$\mathcal{M}_p(F\,;\,r) \to \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|F\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|\,\mathrm{d}\theta\right) \quad \text{as} \quad p\to 0\,.$$

So, we set

$$\mathcal{M}_0(F\,;\,r) := \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|F\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|\,\mathrm{d}\theta\right)\,. \tag{1.3}$$

When F is a polynomial, the associated quantity $\mathcal{M}_0(F\,;\,1)$ is called its *logarithmic Mahler measure*. Let $F(z) := a_m\prod_{\mu=1}^{m}(z-z_\mu)$, and suppose that $F(0)$ is not zero. Then by Jensen's theorem (see [14, p. 124])

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|F\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|\,\mathrm{d}\theta = \log\frac{|F(0)|}{\prod_{|z_\mu|<1}|z_\mu|}\,.$$

Hence

$$\mathcal{M}_0(F\,;\,1)=\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|F\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|\,\mathrm{d}\theta\right)=|a_m|\prod_{\mu=1}^{m}\max\{|z_\mu|\,,\,1\}\,. \tag{1.4}$$

Let $F(z) := z^m\Phi(z)$, where $\Phi(0) \neq 0$. Then $\mathcal{M}_0(F\,;\,1) = \mathcal{M}_0(\Phi\,;\,1)$, and (1.4) applies to Φ. Hence, in (1.4) the restriction '$F(0) \neq 0$' may be dropped, making it a very useful formula.

Bernstein's inequality for polynomials says that if f is a polynomial of degree at most n such that $\mathcal{M}_\infty(f\,;\,1) = 1$, then

$$\mathcal{M}_\infty(f'\,;\,1) \le n\,,$$

where the inequality becomes an equality if and only if $f(z) \equiv \mathrm{e}^{\mathrm{i}\gamma} z^n$, $\gamma \in \mathcal{R}$. Since $\mathcal{M}_p(F\,;\,r)$ is a non-decreasing function of p, it follows that for any polynomial f of degree at most n such that $\mathcal{M}_\infty(f\,;\,1) = 1$, we have

$$\mathcal{M}_p(f'\,;\,1) \le n \qquad (0\le p<\infty)\,.$$

It is interesting that this is also sharp; for any $p \in [0\,,\,\infty)$, the inequality becomes an equality for polynomials of the form $f(z) \equiv \mathrm{e}^{\mathrm{i}\gamma}\, z^n,\ \gamma \in \mathcal{R}$. Thus

$$\sup\left\{\frac{\mathcal{M}_p(f'\,;\,1)}{\mathcal{M}_\infty(f\,;\,1)}\ :\ f \in \mathcal{P}_n\,,\ f(z) \not\equiv 0\right\} = n \qquad (0 \le p \le \infty)\,. \tag{1.5}$$

There exists (see [3, p. 97]) a polynomial $f_* \in \wp_n$ such that

$$\mathcal{M}_\infty(f_*'\,;\,1) \ge \mathcal{M}_\infty(f_*\,;\,1)\,(n-1)\,. \tag{1.6}$$

This is surprising since, for any $p \in [0\,,\,\infty]$, the extremals in (1.5) have all their zeros at the origin whereas any polynomial in $\wp_n$ must have at least half of its zeros outside the open unit disk.

If $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ is such that $a_0 = a_n$ then (see [4, Theorem 2]),

$$\mathcal{M}_\infty(f'\,;\,1) \le \mathcal{M}_\infty(f\,;\,1)\,\left(n - \frac{1}{2} + \frac{1}{2\,(n+1)}\right)\,. \tag{1.7}$$

In particular, (1.7) holds for any $f \in \wp_n$.

In this paper we consider the following question that was mentioned to me by Professor Q. I. Rahman. It asks for an analogue of (1.5) for the subclass $\wp_n$.

Question. What is the value of the constant

$$\kappa_{n,p} := \sup\left\{\frac{\mathcal{M}_p(f'\,;\,1)}{\mathcal{M}_\infty(f\,;\,1)}\ :\ f \in \wp_n\right\} \tag{1.8}$$

for any given $p \in [0\,,\,\infty]$?

From (1.6) and (1.7) we know that

$$n - 1 \le \kappa_{n,\infty} \le n - \frac{1}{2} + \frac{1}{2\,(n+1)}\,.$$

However, the exact value of $\kappa_{n,\infty}$ remains unknown and elusive.

The following result contains an answer to the question in the case where p lies in $[0\,,\,2]$.

Theorem 1.1. *Let $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree at most n such that $f(z) \equiv z^n f(1/z)$. Furthermore, let $\mathcal{M}_p(.\,;\,.)$ be as in (1.1) and $\mathcal{M}_\infty(.\,;\,.)$ be as in (1.2). Then,*

$$\mathcal{M}_p(f'\,;\,1) \le \frac{n}{\sqrt{2}}\sqrt{\frac{1}{n}\sum_{k=0}^{n-1}\left|f\left(\mathrm{e}^{2k\pi\mathrm{i}/n}\right)\right|^2 - 2|a_0|^2} \qquad (0 \le p \le 2)\,, \tag{1.9}$$

and so a fortiori

$$\mathcal{M}_p(f'\,;\,1) \le \frac{n}{\sqrt{2}}\sqrt{\left(\mathcal{M}_\infty(f\,;\,1)\right)^2 - 2|a_0|^2} \qquad (0 \le p \le 2)\,. \tag{1.10}$$

The example $f(z) := z^n + 1$ shows that both (1.9) and (1.10) are sharp for every $p \in [0\,,\,2]$.

It may be added that if $\left|f\left(\mathrm{e}^{2k\pi\mathrm{i}/n}\right)\right| \le 1$ for $k = 0, 1, \ldots, n-1$, then $\frac{1}{n}\sum_{k=0}^{n-1}\left|f\left(\mathrm{e}^{2k\pi\mathrm{i}/n}\right)\right|^2 \le 1$, and so (1.9) contains the following result, which is stronger than (1.10).

Corollary 1.2. *Let $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree at most n such that $f(z) \equiv z^n f(1/z)$. Furthermore, let $|f\left(\mathrm{e}^{2k\pi \mathrm{i}/n}\right)| \le 1$ for $k = 0, 1, \dots, n-1$. Then*

$$\mathcal{M}_p(f'\,;\,1) \le \frac{n}{\sqrt{2}}\sqrt{1 - 2|a_0|^2} \qquad (0 \le p \le 2)\,.$$

As the example $f(z) := (z^n + 1)/2$ shows, the estimate is sharp for every $p \in [0\,,\,2]$.

Remark 1.1. In view of (1.4), the case $p = 0$ of (1.10) can be stated as follows:

Let $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree at most n such that $f(z) \equiv z^n f(1/z)$, and let $\zeta_1, \dots, \zeta_{n-1}$ be the zeros of f'. Then

$$|a_0|^2 + |a_n|^2\left\{\prod_{\nu=1}^{n-1}\max\{|\zeta_\nu|\,,\,1\}\right\}^2 \le \frac{\left(\mathcal{M}_\infty(f\,;\,1)\right)^2}{2}\,.$$

Theorem 1.1 implies the following result.

Corollary 1.3. *Let $g(x) := \sum_{\nu=0}^{n} c_\nu x^\nu$ be a polynomial of degree at most n with coefficients in $\mathbb{C}$. Furthermore, for any $p \in (0\,,\,\infty)$, let*

$$I_{p\,,1} := \frac{1}{\pi}\int_{-1}^{1}\left|n\,g(x) + \mathrm{i}\sqrt{1-x^2}g'(x)\right|^p \frac{\mathrm{d}x}{\sqrt{1-x^2}}$$

and

$$I_{p\,,2} := \frac{1}{\pi}\int_{-1}^{1}\left|n\,g(x) - \mathrm{i}\sqrt{1-x^2}g'(x)\right|^p \frac{\mathrm{d}x}{\sqrt{1-x^2}}\,.$$

Then, for any $p \in [0\,,\,2]$, we have

$$\left(\frac{I_{p\,,1} + I_{p\,,2}}{2}\right)^{1/p}$$

$$\le n\sqrt{\frac{1}{n}\left\{|g(1)|^2 + 2\sum_{k=1}^{n-1}\left|g\left(\cos\frac{k\pi}{n}\right)\right|^2 + |g(-1)|^2\right\} - \left(\frac{|c_n|}{2^{n-1}}\right)^2}\,. \tag{1.11}$$

In particular, if $|g(x)| \le 1$ at the points $x_k := \cos(k\pi/n)$, $k = 0, 1, \dots, n$ then, for any $p \in (0\,,\,2]$, we have

$$\left(\frac{I_{p\,,1} + I_{p\,,2}}{2}\right)^{1/p} \le n\sqrt{2 - \left(\frac{|c_n|}{2^{n-1}}\right)^2}\,. \tag{1.12}$$

Both (1.11) and (1.12) become equalities for $T_n(x) := \cos(n\arccos x)$, the Chebyshev polynomial of the first kind of degree n.

Remark 1.2. Since

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{1-x^2}} = 1\,,$$

the quantity

$$\left(\frac{I_{p\,,1} + I_{p\,,2}}{2}\right)^{1/p}$$

appearing on the left-hand sides of (1.11) and (1.12), is a ‘weighted integral mean’ of

$$\frac{1}{2}\left\{\left|n\,g(x) + \mathrm{i}\sqrt{1-x^2}g'(x)\right|^p + \left|n\,g(x) - \mathrm{i}\sqrt{1-x^2}g'(x)\right|^p\right\}\,.$$

2. AN AUXILIARY RESULT

Lemma 2.1. *Let* $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree* n *such that* $a_0 = a_n$. *Then,*

$$\frac{1}{n}\sum_{k=0}^{n-1}\left|f\left(\mathrm{e}^{2k\pi\mathrm{i}/n}\right)\right|^2 = \sum_{\nu=0}^{n}|a_\nu|^2 + 2\,|a_0|^2 = \sum_{\nu=0}^{n}|a_\nu|^2 + 2\,|a_n|^2\,. \tag{2.1}$$

Proof. For any real θ, we have

$$\left|f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^2 = f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\overline{f\left(\mathrm{e}^{\mathrm{i}\theta}\right)} = \sum_{\nu=0}^{n} a_\nu \mathrm{e}^{\mathrm{i}\nu\theta}\sum_{\nu=0}^{n}\overline{a}_\nu \mathrm{e}^{-\mathrm{i}\nu\theta} = \sum_{m=-n}^{n} b_m \mathrm{e}^{\mathrm{i}m\theta}\,, \tag{2.2}$$

where

$$b_{-n} = a_0\,\overline{a}_n = |a_0|^2,\; b_n = a_n\,\overline{a}_0 = |a_0|^2,\; b_0 = \sum_{\nu=0}^{n}|a_\nu|^2 \tag{2.3}$$

and

$$b_m = \sum_{\mu=0}^{n-m} a_{m+\mu}\,\overline{a}_\mu,\quad b_{-m} = \overline{b}_m \qquad (m = 1,\dots,n-1)\,.$$

The values of $b_1,\dots,b_{n-1}$ and $b_{-1},\dots,b_{-n+1}$ are of little importance.

Let $\omega := \mathrm{e}^{2m\pi\mathrm{i}/n}$, where $m \in \{\pm1,\dots,\pm(n-1)\}$. Note that $\omega^n = 1$. Since $\omega \neq 1$, we see that

$$\sum_{k=0}^{n-1}\left(\mathrm{e}^{2m\pi\mathrm{i}/n}\right)^k = \sum_{k=0}^{n-1}\omega^k = \frac{1-\omega^n}{1-\omega} = 0 \qquad (m = \pm1,\dots,\pm(n-1))\,. \tag{2.4}$$

From (2.2), (2.3) and (2.4) it follows that

$$\begin{aligned}\sum_{k=0}^{n-1}\left|f\left(\mathrm{e}^{2k\pi\mathrm{i}/n}\right)\right|^2 &= b_{-n}\sum_{k=0}^{n-1}\mathrm{e}^{-2k\pi\mathrm{i}} + b_n\sum_{k=0}^{n-1}\mathrm{e}^{2k\pi\mathrm{i}} + \sum_{k=0}^{n-1} b_0\\ &\quad + \sum_{\substack{m=-n+1,\\ m\neq 0}}^{n-1} b_m\sum_{k=0}^{n-1}\left(\mathrm{e}^{2m\pi\mathrm{i}/n}\right)^k\\ &= nb_{-n} + nb_n + nb_0 = n\sum_{\nu=0}^{n}|a_\nu|^2 + 2n|a_0|^2\,,\end{aligned}$$

which is equivalent to (2.1). ∎

3. PROOFS OF THEOREM 1.1 AND COROLLARY 1.3

Proof of Theorem 1.1. We present the proof in three steps.

Step I. First we show that if $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree at most n, whose coefficients satisfy the condition

$$|a_\nu| = |a_{n-\nu}| \qquad (\nu = 0,1,\dots,n)\,, \tag{3.1}$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f'\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^2\,\mathrm{d}\theta \le \frac{n^2}{2}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^2\,\mathrm{d}\theta\,.$$

Indeed, (3.1) allows us to write

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f'\left(e^{i\theta}\right)\right|^2\,d\theta &= \sum_{\nu=0}^{n}\nu^2|a_\nu|^2 = \frac{1}{2}\left(\sum_{\nu=0}^{n}\nu^2|a_\nu|^2 + \sum_{\nu=0}^{n}\nu^2|a_{n-\nu}|^2\right)\\ &= \frac{1}{2}\sum_{\nu=0}^{n}\left\{\nu^2+(n-\nu)^2\right\}|a_\nu|^2\\ &\le \frac{n^2}{2}\sum_{\nu=0}^{n}|a_\nu|^2 = \frac{n^2}{2}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f\left(e^{i\theta}\right)\right|^2\,d\theta\,.\end{aligned}$$

Step II. By Lemma 2.1,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f\left(e^{i\theta}\right)\right|^2\,d\theta = \sum_{\nu=0}^{n}|a_\nu|^2 = \frac{1}{n}\sum_{k=0}^{n-1}\left|f\left(e^{2k\pi i/n}\right)\right|^2 - 2\,|a_0|^2\,.$$

Hence,

$$\mathcal{M}_2(f'\,;\,1) \le \frac{n}{\sqrt{2}}\sqrt{\frac{1}{n}\sum_{k=0}^{n-1}|f\left(e^{2k\pi i/n}\right)|^2 - 2|a_0|^2}\,. \tag{3.2}$$

Step III. Finally, note that $\mathcal{M}_p(f'\,;\,1) \le \mathcal{M}_2(f'\,;\,1)$ for $0 \le p \le 2$ since $\mathcal{M}_p(f'\,;\,1)$ is a nondecreasing function of p. Hence (3.2) implies (1.9). ∎

Proof of Corollary 1.3. Consider the polynomial

$$f(z) := z^n g\left(\frac{z+z^{-1}}{2}\right)\,.$$

It clearly belongs to $\wp_{2n}$, and writing it in the form $f(z) := \sum_{\nu=0}^{2n} a_\nu z^\nu$ we see that $a_{2n} = a_0 = c_n/2^n$. Since

$$\left|f'\left(e^{i\theta}\right)\right| = |i\,n\,g(\cos\theta) + g'(\cos\theta)(-\sin\theta)|\,,$$

Theorem 1.1, with $2n$ in place of n, may be applied to f to conclude that for any $p \in (0\,,\,2]$, we have

$$\begin{aligned}&\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|n\,g(\cos\theta) + i\,(\sin\theta)g'(\cos\theta)|^p\,d\theta\right)^{1/p}\\ &\le n\left[\frac{1}{n}\left\{|g(1)|^2 + 2\sum_{\substack{k=1,\\ k\neq n}}^{2n-1}\left|g\left(\cos\frac{k\pi}{n}\right)\right|^2 + |g(-1)|^2\right\} - \left(\frac{|c_n|}{2^{n-1}}\right)^2\right]^{1/2}\,,\end{aligned}$$

which is equivalent to (1.11). ∎

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