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ULAM STABILITY OF RECIPROCAL DIFFERENCE AND ADJOINT FUNCTIONAL EQUATIONS

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ABSTRACT. In this paper, the reciprocal difference functional equation (or RDF equation) and the reciprocal adjoint functional equation (or RAF equation) are introduced. Then the pertinent Ulam stability problem for these functional equations is solved, together with the extended Ulam (or Rassias) stability problem and the generalized Ulam (or Ulam-Gavruta-Rassias) stability problem for the same equations.

Key words and phrases: Reciprocal Difference Functional equation;Reciprocal Adjoint Functional equation; Ulam stability; Extended Ulam stability; Generalized Ulam stability.

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1. INTRODUCTION

In 1940, Ulam [51] proposed the Ulam stability problem of additive mappings. In 1941, D. H. Hyers [13] considered the case of approximately additive mappings $f : E \to E'$ where E and E' are Banach spaces and f satisfies the inequality $\|f(x+y) - f(x) - f(y)\| \leq \epsilon$ for all $x, y \in E$. It was shown that the limit $L(x) = lim_{n\to\infty} 2^{-n} f(2^{-n}x)$ exists for all $x \in E$ and that L is the unique additive mapping satisfying $\|f(x) - L(x)\| \leq \epsilon$. In 1978, Th. M. Rassias [48] generalized the above result of an approximation involving a sum of powers of norms. In 1982-1994, a generalization of the result of D. H. Hyers was proved via theorems [29]-[31], [34], [35], using weaker conditions controlled by a product of different powers of norms.

Theorem 1.1 (J. M. Rassias) [29] Let $f : E \to E'$ be a mapping where E is a real-normed space and E' is a Banach space. Assume that there exists $\theta > 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \, \|x\|^p \, \|y\|^q$$

for all $x, y \in E$ where $r = p + q \neq 1$. Then there exists a unique additive mapping $L : E \to E'$ such that

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2 - 2^r|} \, \|x\|^r$$

for all $x \in E$.

However, the case $r = 1$ in the above inequality is singular. A clever counter-example has been given by Gavruta [11]. A pertinent interesting paper about the stability of additive mappings was presented by Gavruta [10]. The above-mentioned stability involving a product of different powers of norms is called Ulam-Gavruta-Rassias stability by Bouikhalene and Elqorachi [3], Elqorachi and Sibaha [47] and Nakmahachalasint [24]. Besides, J. M. Rassias [33] introduced and investigated also the Euler-Lagrange quadratic mappings.

Very recently, K. Ravi and B. V. Senthil Kumar [46] proved some interesting results on Ulam-Gavruta-Rassias stability of Rassias reciprocal functional equation

$$r(x+y) = \frac{r(x)r(y)}{r(x) + r(y)}. \tag{1.1}$$

The reciprocal function $r(x) = \frac{1}{x}$ is the solution of the functional equation (1.1).
In this paper, J. M. Rassias introduces the Reciprocal Difference Functional equation (or RDF equation)

$$r\left(\frac{x+y}{2}\right) - r(x+y) = \frac{r(x)r(y)}{r(x) + r(y)} \tag{1.2}$$

and the Reciprocal Adjoint Functional equation (or RAF equation)

$$r\left(\frac{x+y}{2}\right) + r(x+y) = \frac{3r(x)r(y)}{r(x) + r(y)} \tag{1.3}$$

and then we investigate these equations (1.2) and (1.3) controlled by the "Product and the mixed product-sum of powers of norms" introduced by J. M. Rassias.

2. SOLUTION OF (1.2) AND (1.3)

Theorem 2.1. *Let X and Y be sets of non-zero real numbers. A function $r : X \to Y$ satisfies the functional equation (1.1) if and only if $r : X \to Y$ satisfies the functional equation (1.2) if and only if $r : X \to Y$ satisfies the functional equation (1.3). Therefore, every solution of functional equations (1.2) and (1.3) is also a reciprocal function.*

Proof. Let $r : X \to Y$ satisfy the functional equation (1.1). Letting $y = x$ in (1.1), we get

$$r(2x) = \frac{1}{2}r(x). \tag{2.1}$$

Replacing x by $\frac{x}{2}$ in (2.1), we obtain

$$r\left(\frac{x}{2}\right) = 2r(x). \tag{2.2}$$

Now, replacing (x, y) by $\left(\frac{x}{2}, \frac{x}{2}\right)$ in (1.1) and using (2.2), we arrive

$$r\left(\frac{x+y}{2}\right) = \frac{2r(x)r(y)}{r(x)+r(y)}. \tag{2.3}$$

Subtracting (1.1) from (2.3), we lead to (1.2). Next, let $r : X \to Y$ satisfy the functional equation (1.2). Letting $y = x$ in (1.2), we arrive

$$r(2x) = \frac{1}{2}r(x). \tag{2.4}$$

Replacing x by $\frac{x}{2}$ in (2.4), we obtain

$$r\left(\frac{x}{2}\right) = 2r(x). \tag{2.5}$$

Applying the result (2.5) in (1.2), we get

$$r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)}. \tag{2.6}$$

Now, adding (2.6) with (1.2), we obtain (1.3). Finally, let $r : X \to Y$ satisfy the functional equation (1.3). Putting $y = x$ in (1.3), we obtain

$$r(2x) = \frac{1}{2}r(x). \tag{2.7}$$

Replacing x by $\frac{x}{2}$ in (2.7), we obtain

$$r\left(\frac{x}{2}\right) = 2r(x). \tag{2.8}$$

Using (2.8) in (1.3), we obtain (1.1). This completes the proof of Theorem 2.1. ∎

3. HYERS-ULAM STABILITY OF RDF EQUATION (1.2)

Theorem 3.1. *Let X and Y be spaces of non-zero real numbers. Assume in addition that $f{:}X{\to}Y$ is a mapping for which there exists a constant c (independent of x,y)≥ 0 such that the functional inequality*

$$\left\| f\left(\frac{x+y}{2}\right) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)} \right\| \leq \frac{\epsilon}{2} \tag{3.1}$$

holds for all $(x, y) \in X^2$. Then the limit

$$r(x) =^{lim}_{n\to\infty} 2^{-n} f(2^{-n}x) \tag{3.2}$$

exists for all $x \in X$, $n \in \mathbb{N}$ and $r{:}X{\to}Y$ is the unique mapping satisfying the functional equation (1.2), such that

$$\|f(x) - r(x)\| \leq \epsilon \tag{3.3}$$

for all $x \in X$. Moreover, functional identity $r(x) = 2^{-n}r(2^{-n}x)$ holds for all $x \in X$ and $n \in \mathbb{N}$.

Proof. Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (3.1), we obtain

$$\left\| \frac{1}{2} f\left(\frac{x}{2}\right) - f(x) \right\| \leq \frac{\epsilon}{2}. \tag{3.4}$$

Replacing x by $\frac{x}{2}$ in (3.4), dividing by 2 and summing the resulting inequality with (3.4), we get $\left\| \frac{1}{2^2} f\left(\frac{x}{2^2}\right) - f(x) \right\| \leq \epsilon\left(1 - \frac{1}{2^2}\right)$. Proceeding further and using induction on a positive integer n, we obtain

$$\left\| \frac{1}{2^n} f\left(\frac{x}{2^n}\right) - f(x) \right\| \leq \epsilon\left(1 - \frac{1}{2^n}\right). \tag{3.5}$$

In order to prove the convergence of the sequence $\{2^{-n} f(2^{-n}x)\}$, we have if $n > p > 0$, then

$$\left\| 2^{-n} f(2^{-n}x) - 2^{-p} f(2^{-p}x) \right\| = 2^{-p} \left\| 2^{-(n-p)} f(2^{-n}x) - f(2^{-p}x) \right\|$$

holds for all $x \in X$ and $n, p \in \mathbb{N}$. Setting $2^{-p}x = y$ in this relation and using (3.5), we obtain

$$\begin{aligned} \left\| 2^{-n} f(2^{-n}x) - 2^{-p} f(2^{-p}x) \right\| &= 2^{-p} \left\| 2^{-(n-p)} f(2^{-n+p}y) - f(y) \right\| \\ &\leq 2^{-p}\epsilon\left(1 - \frac{1}{2^{n-p}}\right) \end{aligned} \tag{3.6}$$

or

$$\left\| 2^{-n} f(2^{-n}x) - 2^{-p} f(2^{-p}x) \right\| \leq \epsilon(2^{-p} - 2^{-n}) < \epsilon 2^{-p}$$

or

$$\lim_{p\to\infty} \left\| 2^{-n} f(2^{-n}x) - 2^{-p} f(2^{-p}x) \right\| = 0.$$

This shows that the sequence $\{2^{-n} f(2^{-n}x)\}$ is a Cauchy sequence and hence the limit (3.2) exists for all $x \in X$. To show that r satisfies (1.2), replacing (x, y) by $(2^{-n}x, 2^{-n}y)$ in (3.1) and dividing by 2^n, we obtain

$$2^{-n} \left\| f\left(2^{-n}\left(\frac{x+y}{2}\right)\right) - f(2^{-n}(x+y)) - \frac{f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x) + f(2^{-n}y)} \right\| \leq 2^{-n}\frac{\epsilon}{2}. \tag{3.7}$$

Allowing $n \to \infty$ in (3.7), we see that r satisfies (1.2) for all $(x, y) \in X^2$. To prove r is a unique reciprocal function satisfying (1.2) subject to (3.3), let us consider an $s : X \to Y$ to be another reciprocal function which satisfies (1.2) and the inequality (3.3). Clearly $s(2^{-n}x) = 2^n s(x)$, $r(2^{-n}x) = 2^n r(x)$ and using (3.3), we arrive

$$\begin{aligned} \|s(x) - r(x)\| &= 2^{-n} \left\| s(2^{-n}x) - r(2^{-n}x) \right\| \\ &\leq 2^{-n} \left(\left\| s(2^{-n}x) - f(2^{-n}x) \right\| + \left\| f(2^{-n}x) - r(2^{-n}x) \right\| \right) \\ &\leq 2^{-n} 2\epsilon = 2^{1-n}\epsilon \end{aligned} \tag{3.8}$$

for all $x \in X$. Allowing $n \to \infty$ in (3.8), we find that r is unique. Applying (3.2) in (3.5), we arrive the result (3.3). This completes the proof of Theorem 3.1. ∎

4. HYERS-ULAM STABILITY OF RAF EQUATION (1.3)

Theorem 4.1. *Let X and Y be spaces of non-zero real numbers. Assume in addition that f:X→Y is a mapping for which there exists a constant c (independent of x,y)≥ 0 such that the functional inequality*

$$\left\| f\left(\frac{x+y}{2}\right) + f(x+y) - \frac{3f(x)f(y)}{f(x) + f(y)} \right\| \leq \frac{\epsilon}{2} \tag{4.1}$$

holds for all $(x, y) \in X^2$. *Then the limit*

$$r(x) =_{n\to\infty}^{lim} 2^{-n} f(2^{-n}x) \tag{4.2}$$

exists for all $x \in X$, $n \in \mathbb{N}$ *and* $r{:}X\to Y$ *is the unique mapping satisfying the functional equation (1.3), such that*

$$\|f(x) - r(x)\| \leq \epsilon \tag{4.3}$$

for all $x \in X$. *Moreover, functional identity* $r(x) = 2^{-n}r(2^{-n}x)$ *holds for all* $x \in X$ *and* $n \in \mathbb{N}$.

Proof. Replacing (x, y) by $\left(\frac{x}{2}, \frac{x}{2}\right)$ in (4.1), we obtain

$$\left\| f(x) - \frac{1}{2} f\left(\frac{x}{2}\right) \right\| \leq \frac{\epsilon}{2}. \tag{4.4}$$

Now replacing x by $\frac{x}{2}$ in (4.4), dividing by 2 and summing the resulting inequality with (4.4), we get $\left\| f(x) - \frac{1}{2^2} f\left(\frac{x}{2^2}\right) \right\| \leq \epsilon\left(1 - \frac{1}{2^2}\right)$. Proceeding further and using induction on a positive integer n, we obtain

$$\left\| f(x) - \frac{1}{2^n} f\left(\frac{x}{2^n}\right) \right\| \leq \epsilon\left(1 - \frac{1}{2^n}\right). \tag{4.5}$$

The proof of the rest of Theorem 4.1 is similar to that of Theorem 3.1. This completes the proof of Theorem 4.1. ∎

5. GENERALIZED ULAM STABILITY OF RDF EQUATION (1.2)

The generalized Ulam (or Ulam-Gavruta-Rassias) stability introduced by J. M. Rassias, concerns functional equations controlled by the product of powers of norms.

Theorem 5.1. *Let* $f{:}X\to Y$ *be a mapping on the spaces of non-zero real numbers. If there exist* $a, b : \rho = a + b > -1$ *and* $c_1 \geq 0$ *such that*

$$\left\| f\left(\frac{x+y}{2}\right) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)} \right\| \leq c_1 \|x\|^a \|y\|^b \tag{5.1}$$

for all $x, y \in X$, *then there exists a unique reciprocal mapping* $r{:}X\to Y$ *such that*

$$\|r(x) - f(x)\| \leq c\,\|x\|^\rho \tag{5.2}$$

holds and r *satisfies (1.2), for all* $x, y \in X$ *where* $c = \frac{2c_1}{2^{\rho+1}-1}$.

Proof. Replacing (x, y) by $\left(\frac{x}{2}, \frac{x}{2}\right)$ in (5.1), we obtain

$$\left\| \frac{1}{2} f\left(\frac{x}{2}\right) - f(x) \right\| \leq \frac{c_1}{2^\rho} \|x\|^\rho. \tag{5.3}$$

Now replacing x by $\frac{x}{2}$ in (5.3), dividing by 2 and summing the resulting inequality with (5.3), we get $\left\| \frac{1}{2^2} f\left(\frac{x}{2^2}\right) - f(x) \right\| \leq \frac{c_1}{2^\rho} \sum_{i=0}^{1} \frac{1}{2^{i(\rho+1)}} \|x\|^\rho$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned}
\left\| \frac{1}{2^n} f\left(\frac{x}{2^n}\right) - f(x) \right\| &\leq \frac{c_1}{2^\rho} \sum_{i=0}^{n-1} \frac{1}{2^{i(\rho+1)}} \|x\|^\rho \\
&\leq \frac{c_1}{2^\rho} \sum_{i=0}^{\infty} \frac{1}{2^{i(\rho+1)}} \|x\|^\rho \\
&\leq \frac{2c_1}{2^{\rho+1}-1} \|x\|^\rho . \qquad (5.4)
\end{aligned}$$

Setting $c = \frac{2c_1}{2^{\rho+1}-1}$ then the equation (5.4) reduces to

$$\left\| \frac{1}{2^n} f\left(\frac{x}{2^n}\right) - f(x) \right\| \leq c \|x\|^\rho . \qquad (5.5)$$

In order to prove the convergence of the sequence $\{2^{-n} f(2^{-n}x)\}$, we have if
$n > p > 0$, then

$$\left\|2^{-n} f(2^{-n}x) - 2^{-p} f(2^{-p}x)\right\| = 2^{-p} \left\|2^{-n+p} f(2^{-n}x) - f(2^{-p}x)\right\|$$

holds for all $x \in X$ and $n, p \in \mathbb{N}$. Setting $2^{-p}x = y$ and using (5.5), we obtain

$$\begin{aligned}
\left\|2^{-n} f(2^{-n}x) - 2^{-p} f(2^{-p}x)\right\| &= 2^{-p} \left\|2^{-n+p} f(2^{-n+p}y) - f(y)\right\| \\
&\leq 2^{-p(\rho+1)} c \|x\|^\rho . \qquad (5.6)
\end{aligned}$$

As $\rho > -1$, the right-hand side of (5.6) tends to 0 as $p \to \infty$. This shows that the sequence $\{2^{-n} f(2^{-n}x)\}$ is a Cauchy sequence. To show that r satisfies (1.2), settting $r(x) = lim_{n\to\infty} 2^{-n} f(2^{-n}x)$, replacing (x, y) by $(2^{-n}x, 2^{-n}y)$ in (5.1) and dividing by 2^n, we obtain

$$2^{-n} \left\| f\left(2^{-n}\left(\frac{x+y}{2}\right)\right) - f(2^{-n}(x+y)) - \frac{f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x) + f(2^{-n}y)} \right\| \leq c_1 2^{-n(\rho+1)} \|x\|^a \|y\|^b . \qquad (5.7)$$

Allowing $n \to \infty$ in (5.7), we see that r satisfies (1.2) for all $(x, y) \in X^2$. To prove r is a unique reciprocal function satisfying (1.2) subject to (5.2), let us consider an $s : X \to Y$ to be another reciprocal function which satisfies (1.2) and the inequality (5.2). Clearly $s(2^{-n}x) = 2^n s(x)$, $r(2^{-n}x) = 2^n r(x)$ and using (5.2), we arrive

$$\begin{aligned}
\|s(x) - r(x)\| &= 2^{-n} \left\|s(2^{-n}x) - r(2^{-n}x)\right\| \\
&\leq 2^{-n} \left(\left\|s(2^{-n}x) - f(2^{-n}x)\right\| + \left\|f(2^{-n}x) - r(2^{-n}x)\right\|\right) \\
&\leq 2^{-n(\rho+1)+1} c \|x\|^\rho \qquad (5.8)
\end{aligned}$$

for all $x \in X$. Allowing $n \to \infty$ in (5.8), we find that r is unique. This completes the proof of the Theorem 5.1. ∎

Theorem 5.2. *Let $f : X \to Y$ be a mapping on the spaces of non-zero real numbers. If there exist $a, b : \rho = a + b < -1$ and $c_1 \geq 0$ such that*

$$\left\| f\left(\frac{x+y}{2}\right) - f(x+y) - \frac{f(x)f(y)}{f(x) + f(y)} \right\| \leq c_1 \|x\|^a \|y\|^b \qquad (5.9)$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \to Y$ such that

$$\|f(x) - r(x)\| \leq c \|x\|^\rho \qquad (5.10)$$

and r satisfies (1.2), for all $x, y \in X$ where $c = \frac{2c_1}{1-2^{\rho+1}}$.

Proof. Replacing (x, y) by (x, x) in (5.9) and multiplying by 2, we obtain

$$\|f(x) - 2f(2x)\| \leq 2c_1 \|x\|^\rho . \tag{5.11}$$

Now replacing x by $2x$ in (5.11), multiplying by 2 and summing the resulting inequality with (5.11), we get $\|f(x) - 2^2 f(2^2 x)\| \leq 2c_1 \sum_{i=0}^{1} 2^{i(\rho+1)} \|x\|^\rho$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned} \|f(x) - 2^n f(2^n x)\| &\leq 2c_1 \sum_{i=0}^{n-1} 2^{i(\rho+1)} \|x\|^\rho \\ &\leq 2c_1 \sum_{i=0}^{\infty} 2^{i(\rho+1)} \|x\|^\rho \\ &\leq \frac{2c_1}{1 - 2^{\rho+1}} \|x\|^\rho . \end{aligned} \tag{5.12}$$

Setting $c = \frac{2c_1}{1-2^{\rho+1}}$ then the equation (5.12) reduces to

$$\|f(x) - 2^n f(2^n x)\| \leq c \|x\|^\rho . \tag{5.13}$$

In order to prove the convergence of the sequence $\{2^n f(2^n x)\}$, we have if $n > p > 0$, then

$$\|2^n f(2^n x) - 2^p f(2^p x)\| = 2^p \left\|2^{n-p} f(2^n x) - f(2^p x)\right\|$$

holds for all $x \in X$ and $n, p \in \mathbb{N}$. Setting $2^p x = y$ in this relation and using (5.13), we obtain

$$\begin{aligned} \|2^n f(2^n x) - 2^p f(2^p x)\| &= 2^p \left\|2^{n-p} f(2^{n-p} y) - f(y)\right\| \\ &\leq 2^{p(\rho+1)} c \|x\|^\rho . \end{aligned} \tag{5.14}$$

As $\rho < -1$, the right-hand side of (5.14) tends to 0 as $p \to \infty$. This shows that the sequence $\{2^n f(2^n x)\}$ is a Cauchy sequence. To prove r satisfies (1.2) and it is unique, the proof is similar to that of Theorem 5.1. This completes the proof of the Theorem 5.2. ∎

Theorem 5.3. *Let $f : X \to Y$ be a mapping for which there exists a constant $\theta > 0$ and f satisfies*

$$\left\| f\left(\frac{x+y}{2}\right) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)} \right\| \leq \theta H(x, y) \tag{5.15}$$

where $H : X^2 \to Y$ be a function such that

$$\phi(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} H\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) \tag{5.16}$$

with the condition

$$\lim_{n\to\infty} \frac{1}{2^n} H\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}\right) = 0 \tag{5.17}$$

holds. Then there exists a unique reciprocal mapping $A : X \to Y$ which satisfies (1.2) and the inequality

$$\|f(x) - A(x)\| \leq \theta\phi(x) \tag{5.18}$$

for all $x \in X$.

Proof. Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (5.15), we obtain

$$\left\| \frac{1}{2} f\left(\frac{x}{2}\right) - f(x) \right\| \leq \theta H\left(\frac{x}{2}, \frac{x}{2}\right) . \tag{5.19}$$

Again replacing x by $\frac{x}{2}$ in (5.19), dividing by 2 and summing the resulting inequality with (5.19), we get $\left\|\frac{1}{2^2}f\left(\frac{x}{2^2}\right)-f(x)\right\| \leq \theta\sum_{i=0}^{1}\frac{1}{2^i}H\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\right)$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned}\left\|\frac{1}{2^n}f\left(\frac{x}{2^n}\right)-f(x)\right\| &\leq \theta\sum_{i=0}^{n-1}\frac{1}{2^i}H\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\right)\\ &\leq \theta\sum_{i=0}^{\infty}\frac{1}{2^i}H\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\right)\end{aligned} \tag{5.20}$$

for all $x \in X$. In order to prove the convergence of the sequence $\{2^{-n}f(2^{-n}x)\}$, replace x by $2^{-p}x$ in (5.20) and divide by 2^p,we find that for $n > p > 0$

$$\begin{aligned}\left\|2^{-p}f(2^{-p}x)-2^{-n-p}f(2^{-n-p}x)\right\| &= 2^{-p}\left\|f(2^{-p}x)-2^{-n}f(2^{-n-p}x)\right\|\\ &\leq \theta\sum_{i=0}^{\infty}\frac{1}{2^{p+i}}H\left(\frac{x}{2^{p+i+1}},\frac{x}{2^{p+i+1}}\right).\end{aligned} \tag{5.21}$$

Allow $p \to \infty$ and using (5.17), the right-hand side of the inequality (5.21) tends to 0. Thus the sequence $\{2^{-n}f(2^{-n}x)\}$ is a Cauchy sequence. Allowing $n \to \infty$ in (5.20), we arrive (5.18). To show that A satisfies (1.2), setting $A(x) =_{n\to\infty}^{lim} 2^{-n}f(2^{-n}x)$, replacing (x,y) by $(2^{-n}x, 2^{-n}y)$ in (5.15) and dividing by 2^n, we obtain

$$2^{-n}\left\|f\left(2^{-n}\left(\frac{x+y}{2}\right)\right)-f(2^{-n}(x+y))-\frac{f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}\right\| \leq 2^{-n}\theta H(2^{-n}x,2^{-n}y). \tag{5.22}$$

Allowing $n \to \infty$ in (5.22), we see that A satisfies (1.2) for all $(x,y) \in X^2$. To prove A is a unique reciprocal function satisfying (1.2). Let $B : X \to Y$ be another reciprocal function which satisfies (1.2) and the inequality (5.18). Clearly $B(2^{-n}x) = 2^nB(x)$, $A(2^{-n}x) = 2^nA(x)$ and using (5.18), we arrive

$$\begin{aligned}\|B(x)-A(x)\| &= 2^{-n}\left\|B(2^{-n}x)-A(2^{-n}x)\right\|\\ &\leq 2^{-n}\left(\left\|B(2^{-n}x)-f(2^{-n}x)\right\|+\left\|f(2^{-n}x)-A(2^{-n}x)\right\|\right)\\ &\leq 2\theta\sum_{i=0}^{\infty}\frac{1}{2^{n+i}}H\left(\frac{x}{2^{n+i+1}},\frac{x}{2^{n+i+1}}\right)\end{aligned} \tag{5.23}$$

for all $x \in X$. Allowing $n \to \infty$ in (5.23) and using (5.17), we find that A is unique. This completes the proof of the Theorem 5.3. ∎

Theorem 5.4. *Let $f{:}X{\to}Y$ be a mapping for which there exists a constant $\theta > 0$ and f satisfies*

$$\left\|f\left(\frac{x+y}{2}\right)-f(x+y)-\frac{f(x)f(y)}{f(x)+f(y)}\right\| \leq \theta H(x,y) \tag{5.24}$$

where $H : X^2 \to Y$ be a function such that

$$\phi(x) = \sum_{i=0}^{\infty}2^iH(2^{i+1}x, 2^{i+1}x) \tag{5.25}$$

with the condition

$$_{n\to\infty}^{lim}2^nH(2^{n+1}x,2^{n+1}x) = 0 \tag{5.26}$$

holds. Then there exists a unique reciprocal mapping $A : X \to Y$ which satisfies (1.2) and the inequality

$$\|f(x) - A(x)\| \leq \theta\phi(x) \tag{5.27}$$

for all $x \in X$.

Proof. Replacing (x, y) by (x, x) in (5.24) and multiplying by 2, we obtain

$$\|f(x) - 2f(2x)\| \leq \theta H(x, x). \tag{5.28}$$

Again replacing x by $2x$ in (5.28), multiplying by 2 and summing the resulting inequality with (5.28), we get $\|f(x) - 2^2 f(2^2 x)\| \leq \theta \sum_{i=0}^{1} 2^i H(2^{i+1}x, 2^{i+1}x)$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned} \|f(x) - 2^n f(2^n x)\| &\leq \theta \sum_{i=0}^{n-1} 2^i H(2^{i+1}x, 2^{i+1}x) \\ &\leq \theta \sum_{i=0}^{\infty} 2^i H(2^{i+1}x, 2^{i+1}x) \end{aligned} \tag{5.29}$$

for all $x \in X$. In order to prove the convergence of the sequence $\{2^n f(2^n x)\}$, replace x by $2^p x$ in (5.29) and multiply by 2^p, we find that for $n > p > 0$

$$\begin{aligned} \left\|2^p f(2^p x) - 2^{n+p} f(2^{n+p} x)\right\| &= 2^p \left\|f(2^p x) - 2^n f(2^{n+p} x)\right\| \\ &\leq \theta \sum_{i=0}^{\infty} 2^{p+i} H(2^{p+i+1}x, 2^{p+i+1}x). \end{aligned} \tag{5.30}$$

Allowing $p \to \infty$ and using (5.26), the right-hand side of the inequality (5.30) tends to 0. Thus the sequence $\{2^n f(2^n x)\}$ is a Cauchy sequence. Allowing $n \to \infty$ in (5.29), we arrive (5.27). To prove A satisfies (1.2) and it is unique, the proof is similar to that of Theorem 5.3. This completes the proof of the Theorem 5.4. ∎

6. EXTENDED ULAM STABILITY OF RDF EQUATION (1.2)

The extended Ulam(or Rassias) stability introduced by J. M. Rassias, concerns functional equations controlled by the mixed product-sum of powers of norms.

Theorem 6.1. *Let $f{:}X{\to}Y$ be a mapping on the spaces of non-zero real numbers. If there exist k and α with $k > 0$ and $\alpha > -\frac{1}{2}$ such that*

$$\left\| f\left(\frac{x+y}{2}\right) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)} \right\| \leq k\left(\|x\|^{\alpha}\|y\|^{\alpha} + \left(\|x\|^{2\alpha} + \|y\|^{2\alpha}\right)\right) \tag{6.1}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r{:}X{\to}Y$ such that

$$\|r(x) - f(x)\| \leq c\,\|x\|^{2\alpha} \tag{6.2}$$

and r satisfies (1.2), for all $x, y \in X$ where $c = \frac{6k}{2^{2\alpha+1}-1}$.

Proof. Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (6.1), we obtain

$$\left\| \frac{1}{2} f\left(\frac{x}{2}\right) - f(x) \right\| \leq \frac{3k}{2^{2\alpha}}\,\|x\|^{2\alpha}. \tag{6.3}$$

Now replacing x by $\frac{x}{2}$ in (6.3), dividing by 2 and summing the resulting inequality with (6.3), we get $\left\| \frac{1}{2^2} f\left(\frac{x}{2^2}\right) - f(x) \right\| \leq \frac{3k}{2^{2\alpha}} \sum_{i=0}^{1} \frac{1}{2^{i(2\alpha+1)}} \|x\|^{2\alpha}$. Using induction on a positive integer n, we get

$$\begin{aligned} \left\| \frac{1}{2^n} f\left(\frac{x}{2^n}\right) - f(x) \right\| &\leq \frac{3k}{2^{2\alpha}} \sum_{i=0}^{n-1} \frac{1}{2^{i(2\alpha+1)}} \|x\|^{2\alpha} \\ &\leq \frac{3k}{2^{2\alpha}} \sum_{i=0}^{\infty} \frac{1}{2^{i(2\alpha+1)}} \|x\|^{2\alpha} \\ &\leq \frac{6k}{2^{2\alpha+1}-1} \|x\|^{2\alpha}. \end{aligned} \tag{6.4}$$

Setting $c = \frac{6k}{2^{2\alpha+1}-1}$ then the equation (6.4) reduces to

$$\left\| \frac{1}{2^n} f\left(\frac{x}{2^n}\right) - f(x) \right\| \leq c \|x\|^{2\alpha}. \tag{6.5}$$

In order to prove the convergence of the sequence $\{2^{-n} f(2^{-n}x)\}$, we have if $n > p > 0$, then

$$\left\| 2^{-n} f(2^{-n}x) - 2^{-p} f(2^{-p}x) \right\| = 2^{-p} \left\| 2^{-n+p} f(2^{-n}x) - f(2^{-p}x) \right\|$$

holds for all $x \in X$ and $n, p \in \mathbb{N}$. Setting $2^{-p}x = y$ in this relation and using (6.5), we obtain

$$\begin{aligned} \left\| 2^{-n} f(2^{-n}x) - 2^{-p} f(2^{-p}x) \right\| &= 2^{-p} \left\| 2^{-n+p} f(2^{-n+p}y) - f(y) \right\| \\ &\leq 2^{-p(2\alpha+1)} c \|x\|^{2\alpha}. \end{aligned} \tag{6.6}$$

As $\alpha > -\frac{1}{2}$, the right-hand side of (6.6) tends to 0 as $p \to \infty$. This shows that the sequence $\{2^{-n} f(2^{-n}x)\}$ is a Cauchy sequence. To show that r satisfies (1.2), setting $r(x) =_{n\to\infty}^{lim} 2^{-n} f(2^{-n}x)$, replacing (x, y) by $(2^{-n}x, 2^{-n}y)$ in (6.1) and dividing by 2^n, we obtain

$$\begin{aligned} &2^{-n} \left\| f\left(2^{-n}\left(\frac{x+y}{2}\right)\right) - f(2^{-n}(x+y)) - \frac{f(2^{-n}x) f(2^{-n}y)}{f(2^{-n}x) + f(2^{-n}y)} \right\| \\ &\qquad \leq 2^{-n(2\alpha+1)} k \left(\|x\|^{\alpha} \|y\|^{\alpha} + \left(\|x\|^{2\alpha} + \|y\|^{2\alpha} \right) \right). \end{aligned} \tag{6.7}$$

Allowing $n \to \infty$ in (6.7), we see that r satisfies (1.2) for all $(x, y) \in X^2$. To prove r is a unique reciprocal function satisfying (1.2) subject to (6.2). Let $s : X \to Y$ be another reciprocal function which satisfies (1.2) and the inequality (6.2). Clearly $s(2^{-n}x) = 2^n s(x)$, $r(2^{-n}x) = 2^n r(x)$ and using (6.2), we arrive

$$\begin{aligned} \|s(x) - r(x)\| &= 2^{-n} \left\| s(2^{-n}x) - r(2^{-n}x) \right\| \\ &\leq 2^{-n} \left(\left\| s(2^{-n}x) - f(2^{-n}x) \right\| + \left\| f(2^{-n}x) - r(2^{-n}x) \right\| \right) \\ &\leq 2^{-n(2\alpha+1)+1} c \|x\|^{2\alpha} \end{aligned} \tag{6.8}$$

for all $x \in X$. Allowing $n \to \infty$ in (6.8), we find that r is unique. This completes the proof of the Theorem 6.1. ∎

Theorem 6.2. *Let $f : X \to Y$ be a mapping on the spaces of non-zero real numbers. If there exist k and α with $k > 0$ and $\alpha < -\frac{1}{2}$ such that*

$$\left\| f\left(\frac{x+y}{2}\right) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)} \right\| \leq k \left(\|x\|^{\alpha} \|y\|^{\alpha} + \left(\|x\|^{2\alpha} + \|y\|^{2\alpha} \right) \right) \tag{6.9}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \to Y$ such that

$$\|f(x) - r(x)\| \leq c \|x\|^{2\alpha} \tag{6.10}$$

and r satisfies (1.2) for all $x, y \in X$ where $c = \frac{6k}{1-2^{2\alpha+1}}$.

Proof. Replacing (x, y) by (x, x) in (6.9) and multiplying by 2 , we obtain

$$\|f(x) - 2f(2x)\| \leq 6k \|x\|^{2\alpha}. \tag{6.11}$$

Now replacing x by $2x$ in (6.11), multiplying by 2 and summing the resulting inequality with (6.11),we get $\|f(x) - 2^2 f(2^2 x)\| \leq 6k \sum_{i=0}^{1} 2^{i(2\alpha+1)} \|x\|^{2\alpha}$. Using induction on a positive integer n, we get

$$\begin{aligned} \|f(x) - 2^n f(2^n x)\| &\leq 6k \sum_{i=0}^{n-1} 2^{i(2\alpha+1)} \|x\|^{2\alpha} \\ &\leq 6k \sum_{i=0}^{\infty} 2^{i(2\alpha+1)} \|x\|^{2\alpha} \\ &\leq \frac{6k}{1 - 2^{2\alpha+1}} \|x\|^{2\alpha}. \end{aligned} \tag{6.12}$$

Setting $c = \frac{6k}{1-2^{2\alpha+1}}$ then the equation (6.12) reduces to

$$\|f(x) - 2^n f(2^n x)\| \leq c \|x\|^{2\alpha}. \tag{6.13}$$

In order to prove the convergence of the sequence $\{2^n f(2^n x)\}$, we have if $n > p > 0$, then

$$\|2^n f(2^n x) - 2^p f(2^p x)\| = 2^p \left\|2^{n-p} f(2^n x) - f(2^p x)\right\|$$

holds for all $x \in X$ and $n, p \in \mathbb{N}$. Setting $2^p x = y$ in this relation and using (6.13), we obtain

$$\begin{aligned} \|2^n f(2^n x) - 2^p f(2^p x)\| &= 2^p \left\|2^{n-p} f(2^{n-p} y) - f(y)\right\| \\ &\leq 2^{p(2\alpha+1)} c \|x\|^{2\alpha}. \end{aligned} \tag{6.14}$$

As $\alpha < -\frac{1}{2}$, the right-hand side of (6.14) tends to 0 as $p \to \infty$. This shows that the sequence $\{2^n f(2^n x)\}$ is a Cauchy sequence. To prove r satisfies (1.2) and it is unique, the proof is similar to that of Theorem 6.1. This completes the proof of the Theorem 6.2. ∎

7. GENERALIZED ULAM STABILITY OF RAF EQUATION (1.3)

Theorem 7.1. *Let $f : X \to Y$ be a mapping on the spaces of non-zero real numbers. If there exist $a, b : \rho = a + b > -1$ and $c_1 \geq 0$ such that*

$$\left\| f\left(\frac{x+y}{2}\right) + f(x+y) - \frac{3f(x)f(y)}{f(x) + f(y)} \right\| \leq c_1 \|x\|^a \|y\|^b \tag{7.1}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \to Y$ such that

$$\|f(x) - r(x)\| \leq c \|x\|^\rho \tag{7.2}$$

holds and r satisfies (1.3), for all $x, y \in X$ where $c = \frac{2c_1}{2^{\rho+1}-1}$.

Proof. Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (7.1), we obtain

$$\left\| f(x) - \frac{1}{2} f\left(\frac{x}{2}\right) \right\| \leq \frac{c_1}{2^\rho} \|x\|^\rho. \tag{7.3}$$

Now replacing x by $\frac{x}{2}$ in (7.3), dividing by 2 and summing the resulting inequality with (7.3),we get $\left\|f(x) - \frac{1}{2^2} f\left(\frac{x}{2^2}\right)\right\| \leq \frac{c_1}{2^\rho} \sum_{i=0}^{1} \frac{1}{2^{i(\rho+1)}} \|x\|^\rho$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned} \left\|f(x) - \frac{1}{2^n} f\left(\frac{x}{2^n}\right)\right\| &\leq \frac{c_1}{2^\rho} \sum_{i=0}^{n-1} \frac{1}{2^{i(\rho+1)}} \|x\|^\rho \\ &\leq \frac{c_1}{2^\rho} \sum_{i=0}^{\infty} \frac{1}{2^{i(\rho+1)}} \|x\|^\rho \\ &\leq \frac{2c_1}{2^{\rho+1}-1} \|x\|^\rho . \end{aligned} \tag{7.4}$$

Setting $c = \frac{2c_1}{2^{\rho+1}-1}$ then the equation (7.4) reduces to

$$\left\|f(x) - \frac{1}{2^n} f\left(\frac{x}{2^n}\right)\right\| \leq c \|x\|^\rho . \tag{7.5}$$

The proof of the rest of Theorem 7.1 goes through the same way as in Theorem 5.1. This completes the proof of the Theorem 7.1. ∎

Theorem 7.2. *Let $f:X\to Y$ be a mapping on the spaces of non-zero real numbers. If there exist $a, b : \rho = a + b < -1$ and $c_1 \geq 0$ such that*

$$\left\|f\left(\frac{x+y}{2}\right) + f(x+y) - \frac{3f(x)f(y)}{f(x)+f(y)}\right\| \leq c_1 \|x\|^a \|y\|^b \tag{7.6}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r:X\to Y$ such that

$$\|r(x) - f(x)\| \leq c \|x\|^\rho \tag{7.7}$$

and r satisfies (1.3), for all $x, y \in X$ where $c = \frac{2c_1}{1-2^{\rho+1}}$.

Proof. Replacing (x, y) by (x, x) in (7.6) and multiplying by 2, we obtain

$$\|2f(2x) - f(x)\| \leq 2c_1 \|x\|^\rho . \tag{7.8}$$

Now replacing x by $2x$ in (7.8), multiplying by 2 and summing the resulting inequality with (7.8), we get $\|2^2 f(2^2 x) - f(x)\| \leq 2c_1 \sum_{i=0}^{1} 2^{i(\rho+1)} \|x\|^\rho$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned} \|2^n f(2^n x) - f(x)\| &\leq 2c_1 \sum_{i=0}^{n-1} 2^{i(\rho+1)} \|x\|^\rho \\ &\leq 2c_1 \sum_{i=0}^{\infty} 2^{i(\rho+1)} \|x\|^\rho \\ &\leq \frac{2c_1}{1-2^{\rho+1}} \|x\|^\rho . \end{aligned} \tag{7.9}$$

Setting $c = \frac{2c_1}{1-2^{\rho+1}}$ then the equation (7.9) reduces to

$$\|2^n f(2^n x) - f(x)\| \leq c \|x\|^\rho . \tag{7.10}$$

The proof of the rest of Theorem 7.2 can be done by similar arguments as in Theorem 5.2. This completes the proof of the Theorem 7.2. ∎

Theorem 7.3. *Let $f:X\to Y$ be a mapping for which there exists a constant $\theta>0$ and f satisfies*

$$\left\| f\left(\frac{x+y}{2}\right)+f(x+y)-\frac{3f(x)f(y)}{f(x)+f(y)}\right\| \leq \theta H(x,y) \tag{7.11}$$

where $H:X^2\to Y$ be a function such that

$$\phi(x)=\sum_{i=0}^{\infty}\frac{1}{2^i}H\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\right) \tag{7.12}$$

with the condition

$$\lim_{n\to\infty}\frac{1}{2^n}H\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)=0 \tag{7.13}$$

holds. Then there exists a unique reciprocal mapping $A:X\to Y$ which satisfies (1.3) and the inequality

$$\|f(x)-A(x)\|\leq\theta\phi(x) \tag{7.14}$$

for all $x\in X$.

Proof. Replacing (x,y) by $(\frac{x}{2},\frac{x}{2})$ in (7.11), we obtain

$$\left\| f(x)-\frac{1}{2}f\left(\frac{x}{2}\right)\right\| \leq \theta H\left(\frac{x}{2},\frac{x}{2}\right). \tag{7.15}$$

Again replacing x by $\frac{x}{2}$ in (7.15), dividing by 2 and summing the resulting inequality with (7.15), we get $\left\|f(x)-\frac{1}{2^2}f\left(\frac{x}{2^2}\right)\right\| \leq \theta\sum_{i=0}^{1}\frac{1}{2^i}H\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\right)$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned}\left\| f(x)-\frac{1}{2^n}f\left(\frac{x}{2^n}\right)\right\| &\leq \theta\sum_{i=0}^{n-1}\frac{1}{2^i}H\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\right)\\ &\leq \theta\sum_{i=0}^{\infty}\frac{1}{2^i}H\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\right) \end{aligned}\tag{7.16}$$

for all $x\in X$. The proof of the rest of Theorem 7.3 goes through the same way as in Theorem 5.3. This completes the proof of the Theorem 7.3. ∎

Theorem 7.4. *Let $f:X\to Y$ be a mapping for which there exists a constant $\theta>0$ and f satisfies*

$$\left\| f\left(\frac{x+y}{2}\right)+f(x+y)-\frac{3f(x)f(y)}{f(x)+f(y)}\right\| \leq \theta H(x,y) \tag{7.17}$$

where $H:X^2\to Y$ be a function such that

$$\phi(x)=\sum_{i=o}^{\infty}2^iH(2^{i+1}x,2^{i+1}x) \tag{7.18}$$

with the condition

$$\lim_{n\to\infty}2^nH(2^{n+1}x,2^{n+1}x)=0 \tag{7.19}$$

holds. Then there exists a unique reciprocal mapping $A:X\to Y$ which satisfies (1.2) and the inequality

$$\|A(x)-f(x)\|\leq\theta\phi(x) \tag{7.20}$$

for all $x\in X$.

Proof. Replacing (x, y) by (x, x) in (7.17) and multiplying by 2, we obtain

$$\|2f(2x) - f(x)\| \leq \theta H(x, x). \tag{7.21}$$

Again replacing x by $2x$ in (7.21), multiplying by 2 and summing the resulting inequality with (7.21), we get $\|2^2 f(2^2 x) - f(x)\| \leq \theta \sum_{i=0}^{1} 2^i H(2^{i+1}x, 2^{i+1}x)$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned} \|2^n f(2^n x) - f(x)\| &\leq \theta \sum_{i=0}^{n-1} 2^i H(2^{i+1}x, 2^{i+1}x) \\ &\leq \theta \sum_{i=0}^{\infty} 2^i H(2^{i+1}x, 2^{i+1}x) \end{aligned} \tag{7.22}$$

for all $x \in X$. The proof of the rest of Theorem 7.4 is similar to that of Theorem 5.4. This completes the proof of the Theorem 7.4. ∎

8. EXTENDED ULAM STABILITY OF RAF EQUATION (1.3)

Theorem 8.1. *Let $f: X \to Y$ be a mapping on the spaces of non-zero real numbers. If there exist k and α with $k > 0$ and $\alpha > -\frac{1}{2}$ such that*

$$\left\| f\left(\frac{x+y}{2}\right) + f(x+y) - \frac{3f(x)f(y)}{f(x)+f(y)} \right\| \leq k\left(\|x\|^{\alpha}\|y\|^{\alpha} + \left(\|x\|^{2\alpha} + \|y\|^{2\alpha}\right)\right) \tag{8.1}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r: X \to Y$ such that

$$\|f(x) - r(x)\| \leq c\|x\|^{2\alpha} \tag{8.2}$$

and r satisfies (1.3), for all $x, y \in X$ where $c = \frac{6k}{2^{2\alpha+1}-1}$.

Proof. Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (8.1), we obtain

$$\left\| f(x) - \frac{1}{2} f\left(\frac{x}{2}\right) \right\| \leq \frac{3k}{2^{2\alpha}} \|x\|^{2\alpha}. \tag{8.3}$$

Now replacing x by $\frac{x}{2}$ in (8.3), dividing by 2 and summing the resulting inequality with (8.3), we get $\left\| f(x) - \frac{1}{2^2} f\left(\frac{x}{2^2}\right) \right\| \leq \frac{3k}{2^{2\alpha}} \sum_{i=0}^{1} \frac{1}{2^{i(2\alpha+1)}} \|x\|^{2\alpha}$. Using induction on a positive integer n, we get

$$\begin{aligned} \left\| f(x) - \frac{1}{2^n} f\left(\frac{x}{2^n}\right) \right\| &\leq \frac{3k}{2^{2\alpha}} \sum_{i=0}^{n-1} \frac{1}{2^{i(2\alpha+1)}} \|x\|^{2\alpha} \\ &\leq \frac{3k}{2^{2\alpha}} \sum_{i=0}^{\infty} \frac{1}{2^{i(2\alpha+1)}} \|x\|^{2\alpha} \\ &\leq \frac{6k}{2^{2\alpha+1}-1} \|x\|^{2\alpha}. \end{aligned} \tag{8.4}$$

Setting $c = \frac{6k}{2^{2\alpha+1}-1}$ then the equation (8.4) reduces to

$$\left\| f(x) - \frac{1}{2^n} f\left(\frac{x}{2^n}\right) \right\| \leq c\|x\|^{2\alpha}. \tag{8.5}$$

The proof of the rest of Theorem 8.1 is similar to that of Theorem 6.1. This completes the proof of the Theorem 8.1. ∎

Theorem 8.2. *Let $f:X\to Y$ be a mapping on the spaces of non-zero real numbers. If there exist k and α with $k>0$ and $\alpha<-\frac{1}{2}$ such that*

$$\left\|f\left(\frac{x+y}{2}\right)+f(x+y)-\frac{3f(x)f(y)}{f(x)+f(y)}\right\|\leq k\left(\|x\|^{\alpha}\|y\|^{\alpha}+\left(\|x\|^{2\alpha}+\|y\|^{2\alpha}\right)\right) \tag{8.6}$$

for all $x,y\in X$, then there exists a unique reciprocal mapping $r:X\to Y$ such that

$$\|r(x)-f(x)\|\leq c\|x\|^{2\alpha} \tag{8.7}$$

and r satisfies (1.3), for all $x,y\in X$ where $c=\frac{6k}{1-2^{2\alpha+1}}$.

Proof. Replacing (x,y) by (x,x) in (8.6) and multiplying by 2 , we obtain

$$\|2f(2x)-f(x)\|\leq 6k\|x\|^{2\alpha}. \tag{8.8}$$

Now replacing x by $2x$ in (8.8), multiplying by 2 and summing the resulting inequality with (8.8),we get $\|2^2f(2^2x)-f(x)\|\leq 6k\sum_{i=0}^{1}2^{i(2\alpha+1)}\|x\|^{2\alpha}$. Using induction on a positive integer n, we get

$$\begin{aligned}\|2^nf(2^nx)-f(x)\| &\leq 6k\sum_{i=0}^{n-1}2^{i(2\alpha+1)}\|x\|^{2\alpha}\\ &\leq 6k\sum_{i=0}^{\infty}2^{i(2\alpha+1)}\|x\|^{2\alpha}\\ &\leq \frac{6k}{1-2^{2\alpha+1}}\|x\|^{2\alpha}. \end{aligned} \tag{8.9}$$

Setting $c=\frac{6k}{1-2^{2\alpha+1}}$ then the equation (8.9) reduces to

$$\|2^nf(2^nx)-f(x)\|\leq c\|x\|^{2\alpha}. \tag{8.10}$$

The proof of the rest of Theorem 8.2 is similar to that of Theorem 6.2. This completes the proof of the Theorem 8.2. ∎

REFERENCES

[1] J. ACZEL and J. DHOMBRES, *Functional Equations in Several Variables,* Cambridge Univ.Press, 1989.

[2] J. H. BAE and K. W. JUN, On the generalized Hyers-Ulam-Rassias stability of quadratic functional equation, *Bull. Koeran. Math. Soc.,* **38** (2001), pp. 325-336.

[3] B. BOUIKHALENE and E. ELQORACHI, Ulam-Gavruta-Rassias stability of the Pexider functional equation, *Internat. J. Appl. Math. Stat.*, **7** (2007), pp. 27-39.

[4] I. S. CHANG and H. M. KIM, On the Hyers-Ulam stability of quadratic functional equations, *J. Ineq. Appl. Math.,* **3** (2002), pp. 1-12.

[5] I. S. CHANG and Y. S. JUNG, Stability of functional equations deriving from cubic and quadratic functions, *J. Math. Anal. Appl.,* **283** (2003), pp. 491-500.

[6] P. W. CHOLEWA, Remarks on the stability of functional equations, *Aequationes Math.,* **27** (1984), pp. 76-86.

[7] CHOONKIL PARK, JONG SU AN and JIANLIAN CUI, Isomorphisms and Derivations in Lie C*-Algebras, *Abstract and Applied Analysis,* Article ID 85737, 14 pages(2007).

[8] CH. PARK, Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras, *Bull. Sci. Math.*, **132** (2008), pp. 87-96.

[9] S. CZERWIK, On the stability of the quadratic mappings in normed spaces, *Abh. Math. Sem. Univ. Hamburg.,* **62** (1992), pp. 59-64.

[10] P. GAVRUTA, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.,* **184** (1994), pp. 431-436.

[11] P. GAVRUTA, An answer to a question of John. M. Rassias concerning the stability of Cauchy equation, *Hadronic Math. Ser.* (1999), pp. 67-71.

[12] P. M. GRUBER, Stability of isometries, *Trans. Amer. Math. Soc.,* **245** (1978), pp. 263-277.

[13] D. H. HYERS, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci., U.S.A..* **27** (1941), pp. 222-224.

[14] D. H. HYERS, G. ISAC and TH. M. RASSIAS, *Stability of Functional Equations in Several Variables,* Birkhauser, Basel, 1998.

[15] D. H. HYERS and TH. M. RASSIAS, Approximate homomorphisms, *Aequationes. Math.,* **44** (1992), pp. 125-153.

[16] K. W. JUN and Y. H. LEE, On the Hyers-Ulam-Rassias stability of a pexiderized quadratic inequality, *Math. Ineq. Appl.,* **4(1)** (2001), pp. 93-118.

[17] S. M. JUNG, On the Hyers-Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. Appl.,* **222** (1998), pp. 126-137.

[18] S. M. JUNG, On the Hyers-Ulam -Rassias stability of a quadratic functional equation, *J. Math. Anal. Appl.,* **232** (1999), pp. 384-393.

[19] S. M. JUNG and B. KIM, On the stability of the quadratic functional equation on bounded domains, *Abh. Math. Sem. Univ. Hamburg.,* **69** (1999), pp. 293-308.

[20] S. M. JUNG and P. K. SAHOO, Hyers-Ulam-Rassias stability of the quadratic functional equation of pexider type, *J. Korean. Math. Soc.,* **38** (2001), pp. 645-656.

[21] PL. KANNAPPAN, Quadratic functional equation and inner product spaces, *Results Math.,* **27** (1995), pp. 368-372.

[22] Y.-S. JUNG, On the generalized Hyers-Ulam stability of moudule derivations, *J. Math. Anal. Appl.,* **339** (2008), pp. 108-114.

[23] Y.-S. LEE and S.-Y. CHUNG, Stability of an Euler-Lagrange-Rassias Equation in the spaces of generalized functions, *Appl. Math. Lett.*, **21** (2008), pp. 694-700.

[24] P. NAKMAHACHALASINT, Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities of an Additive Functional Equation in Several variables, *Internat. J. Math. and Math. Sc. (IJMMS),* Article ID 13437, 6 pages (2007).

[25] P. NAKMAHACHALASINT, On the generalized Ulam-Gavruta-Rassias stability of mixed-type linear and Euler-Lagrange-Rassias functional equations, *Internat. J. Math. and Math. Sc. (IJMMS),* Article ID 63239, 10 pages (2007).

[26] A. NAJATI and C. PARK, Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation, *J. Math. Anal. Appl.*, **35** (2007), pp. 763-778.

[27] C. G. PARK, Stability of an Euler-Lagrange-Rassias type additive mapping, *Internat. J. Appl. Math. Stat.*, **7** (2007), pp. 101-111.

[28] M. J. RASSIAS and J. M. RASSIAS, On the Ulam stability for Euler-Lagrange type quadratic functional equations, *The Australian Journal of Mathematical Analysis and Applications*, **2,** no. 1, article 11, 10 pages (2005).

[29] J. M. RASSIAS, On approximation of approximately linear mappings by linear mappings, *J. Funct. Anal.*, **46** (1982), pp. 126-130.

[30] J. M. RASSIAS, On approximation of approximately linear mappings by linear mappings, *Bull. Sci. Math.*, **108** (1984), pp. 445-446.

[31] J. M. RASSIAS, Solution of a problem of Ulam, *J. Approx. Theory,* **57** (1989), pp. 268-273.

[32] J. M. RASSIAS, Hyers-Ulam stability of the quadratic functional equation in several variables, *J. Ind. Math. Soc.,* **68** (1992), pp. 65-73.

[33] J. M. RASSIAS, On the stability of the Euler-Lagrange functional equation, *Chinese J. Math.,* **20** (1992), pp. 185-190.

[34] J. M. RASSIAS, Solution of a stability problem of Ulam, *Discussiones Mathematicae,* **12** (1992), pp. 95-103.

[35] J. M. RASSIAS, Complete solution of the multi-dimensional problem of Ulam, *Discussiones Mathematicae,* **14** (1994), pp. 101-107.

[36] J. M. RASSIAS, On the stability of the non-linear Euler-Lagrange functional equation in real normed linear spaces, *J. Math. Phys. Sci.*, **28** (1994), pp. 231-235.

[37] J. M. RASSIAS, On the stability of the general Euler-Lagrange functional equation, *Demonstratio Math.,* **29** (1996), pp. 755-766.

[38] J. M. RASSIAS, Solution of the Ulam stability problem for Euler-Lagrange quadratic mappings, *J. Math. Anal. Appl.,* **220** (1998), pp. 613-639.

[39] J. M. RASSIAS, On the stability of the multi-dimensional Euler-Lagrange Functional equation, *J. Indian Math. Soc.,* **66** (1999), pp. 1-9.

[40] J. M. RASSIAS, On the Euler stability problem, *J. Indian Math. Soc.*, **67** (2000), pp. 1-15.

[41] J. M. RASSIAS, On some approximately quadratic mappings being exactly quadratic, *J. Indian Math. Soc.,* **69** (2002), pp. 155-160.

[42] J. M. RASSIAS and M. J. RASSIAS, Asymptotic behaviour of alternative Jensen and Jensen type functional equations, *Bulletin des Sciences Mathematiques,* **129**, no. 7 (2005), pp. 545-558.

[43] J. M. RASSIAS, Solution of the Hyers-Ulam stability problem for quadratic type functional equations in several variables, *The Australian Journal of Mathematical Analysis and Applications,* **2**, no. 2, article 11 (2005), pp. 1-9.

[44] J. M. RASSIAS and M. J. RASSIAS, The Ulam problem for 3-dimensional quadratic mappings, *Tatra. Mt. Math. Publ.* **34** (2006), pp. 333-337.

[45] K. RAVI and M. ARUNKUMAR, On the Ulam-Gavruta-Rassias stability of the orthogonally Euler-Lagrange type functional equation, *Internat. J. Appl. Math. Stat.*, **7** (2007), pp. 143-156.

[46] K. RAVI and B. V. SENTHIL KUMAR, Ulam-Gavruta-Rassias stability of Rassias Reciprocal functional equation, *Glob. J. App. Math. and Math. Sci.*, **3** (2010), pp. 57-79.

[47] M. A. SIBAHA, B. BOUIKHALENE and E. ELQORACHI, Ulam-Gavruta-Rassias stability for a linear functional equation, *Internat. J. Appl. Math. Stat.*, **7** (2007), pp. 157-168.

[48] TH. M. RASSIAS, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.,* **72** (1978), pp. 297-300.

[49] TH. M. RASSIAS, On the stability of the functional equations in Banach spaces, *J. Math. Anal. Appl.,* **251** (2000), pp. 264-284.

[50] F. SKOF, Proprieta locali e approssimazione di operatori, *Rend. Sem. Mat. Fis. Milano,* **53** (1983), pp. 113-129.

[51] S. M. ULAM, Problems in Modern Mathematics, Chapter VI, *Wiley-Interscience,* New York, 1964.