The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 8, Issue 1, Article 12, pp. 1-3, 2011

ON THE INEQUALITY WITH POWER-EXPONENTIAL FUNCTION

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Received 2 June, 2010; accepted 30 September, 2010; published 28 November, 2011.

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ABSTRACT. In this paper, we prove the inequality $a^{b^{a^{\cdot^{\cdot^{\cdot^{a}}}}}} < b^{a^{b^{\cdot^{\cdot^{\cdot^{b}}}}}}$ for $0 < a < b$. Other related conjectures are also presented.

Key words and phrases: Power-exponential function.

2000 *Mathematics Subject Classification.* 26D15.

ISSN (electronic): 1449-5910

1. INTRODUCTION

In the article [1], the following problem was submitted.

Is the inequality

$$2^{3^{2^{3^{2^{3}}}}} > 3^{2^{3^{2^{3^{2}}}}}$$

true or false?

We generalize this problem.

Definition 1.1. Let a, b $(a < b)$ be positive real numbers. Define the sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ in the following way,

$$a_1 = a, \; b_1 = b \quad \text{and} \quad a_{n+1} = a^{b_n}, \; b_{n+1} = b^{a_n} \quad n = 1, 2, \dots.$$

We consider the problem of comparisons between a_n and b_n.

2. RESULTS AND PROOFS

Theorem 2.1. *If* $1 < a$, *then*

$$\frac{b_{2m}}{a_{2m}} < \frac{b_{2m-1}}{a_{2m-1}} < \frac{b_{2m+1}}{a_{2m+1}} \quad m = 1, 2, \dots. \tag{2.1}$$

Proof. The proof is by induction. First, we show that the inequality (2.1) is true for $m = 1$, that is, we prove that

$$\frac{b_2}{a_2} < \frac{b_1}{a_1} < \frac{b_3}{a_3}. \tag{2.2}$$

The left side of the inequality (2.2) is equivalent to $f(a_1) > f(b_1)$, where

$$f(x) = \frac{\ln x}{x - 1} \quad \text{for} \quad x > 1.$$

From $f'(x) < 0$, it follows that $f(x)$ is strictly decreasing, and hence $f(a_1) > f(b_1)$.

Now we prove the right side of the inequality (2.2).

Case $1 \geq \frac{b_2}{a_2}$. The right side of the inequality (2.2) is equivalent to

$$(b_2 - 1) \ln a_1 < (a_2 - 1) \ln b_1. \tag{2.3}$$

From $b_2 - 1 \leq a_2 - 1$, we have the inequality (2.3).

Case $1 < \frac{b_2}{a_2}$. The right side of the inequality (2.2) is equivalent to

$$b_2 a_1 g(a_2) < a_2 b_1 g(b_2), \tag{2.4}$$

where

$$g(x) = \frac{x \ln x}{x - 1} \quad \text{for} \quad x > 1.$$

From $g'(x) > 0$, it follows that $g(x)$ is strictly increasing, and hence $g(a_2) < g(b_2)$. Since $\frac{b_2}{a_2} < \frac{b_1}{a_1}$, the inequality (2.4) holds.

Assuming the inequality (2.1) is true for $m = k$, we show that the inequality (2.1) is also true for $m = k + 1$, that is, we prove that

$$\frac{b_{2k+2}}{a_{2k+2}} < \frac{b_{2k+1}}{a_{2k+1}} < \frac{b_{2k+3}}{a_{2k+3}}. \tag{2.5}$$

The left side of (2.5) is equivalent to

$$b_{2k}(a_{2k} - a_{2k-1}) f\left(\frac{a_{2k+1}}{a_{2k}}\right) > a_{2k}(b_{2k} - b_{2k-1}) f\left(\frac{b_{2k+1}}{b_{2k}}\right). \tag{2.6}$$

Since $f(x)$ is strictly decreasing and $\frac{b_{2k}}{a_{2k}} < \frac{b_{2k+1}}{a_{2k+1}}$, $f\left(\frac{a_{2k+1}}{a_{2k}}\right) > f\left(\frac{b_{2k+1}}{b_{2k}}\right)$. From $\frac{b_{2k}}{a_{2k}} < \frac{b_{2k-1}}{a_{2k-1}}$, we have

$$b_{2k}(a_{2k} - a_{2k-1}) > a_{2k}(b_{2k} - b_{2k-1}).$$

Thus (2.6) holds.

We prove the right side of (2.5).

Case $\frac{b_{2k}}{a_{2k}} \geq \frac{b_{2k+2}}{a_{2k+2}}$. The right side of (2.5) is equivalent to

$$a_{2k}(b_{2k+2} - b_{2k}) \ln a_{2k+1} < b_{2k}(a_{2k+2} - a_{2k}) \ln b_{2k+1}. \tag{2.7}$$

Since $\frac{b_{2k}}{a_{2k}} \geq \frac{b_{2k+2}}{a_{2k+2}}$, we have

$$a_{2k}(b_{2k+2} - b_{2k}) \leq b_{2k}(a_{2k+2} - a_{2k}).$$

From $1 < \frac{b_1}{a_1} < \frac{b_{2k+1}}{a_{2k+1}}$, we have $\ln a_{2k+1} < \ln b_{2k+1}$. Thus (2.7) holds.

Case $\frac{b_{2k}}{a_{2k}} < \frac{b_{2k+2}}{a_{2k+2}}$. The right side of the inequality (2.5) is equivalent to

$$b_{2k+2}(a_{2k+1} - a_{2k-1}) g\left(\frac{a_{2k+2}}{a_{2k}}\right) < a_{2k+2}(b_{2k+1} - b_{2k-1}) g\left(\frac{b_{2k+2}}{b_{2k}}\right). \tag{2.8}$$

Since $g(x)$ is strictly increasing and $\frac{b_{2k}}{a_{2k}} < \frac{b_{2k+2}}{a_{2k+2}}$, $g(\frac{a_{2k+2}}{a_{2k}}) < g(\frac{b_{2k+2}}{b_{2k}})$. From $\frac{b_{2k-1}}{a_{2k-1}} < \frac{b_{2k+1}}{a_{2k+1}}$ and $\frac{b_{2k+2}}{a_{2k+2}} < \frac{b_{2k+1}}{a_{2k+1}}$, we have $b_{2k+2}(a_{2k+1} - a_{2k-1}) < a_{2k+2}(b_{2k+1} - b_{2k-1})$. Thus the inequality (2.8) holds. ∎

Proposition 2.2. *If $a \leq 1$, then $a_n < b_n$.*

Proof. *Case* $b > 1$. $a_n \leq 1 < b_n$.

Case $b = 1$. $a_n < 1 = b_n$.

Case $b < 1$. We can show this by induction. ∎

Corollary 2.3. *If n is an odd number, then $a_n < b_n$.*

Proof. *Case* $a \leq 1$. From Proposition 2.2, it is obvious.

Case $1 < a$. Using Theorem 2.1, $1 < \frac{b_1}{a_1} \leq \frac{b_{2m-1}}{a_{2m-1}}$. ∎

Corollary 2.4. *If n is an even number and $a \leq 1$, then $a_n < b_n$.*

Proof. *Case* $a \leq 1$. From Proposition 2.2, it is obvious. ∎

3. CONJECTURES

Conjecture 3.1. *If n is an even number and $1 < a$, there is only one real $\epsilon_{a,n}$ such that $b \lesseqgtr a + \epsilon_{a,n} \iff a_n \lesseqgtr b_n$.*

Using Theorem 2.1,

$$a_{2m+2} < b_{2m+2} \iff \frac{b_{2m+1}}{a_{2m+1}} < \frac{\ln b}{\ln a} \implies \frac{b_{2m-1}}{a_{2m-1}} < \frac{\ln b}{\ln a} \iff a_{2m} < b_{2m}.$$

If Conjecture 3.1 is true, $\epsilon_{a,2m} \geq \epsilon_{a,2m+2}$. So there exists $\alpha = \lim_{m\to\infty} \epsilon_{a,2m}$.

Conjecture 3.2. *If $1 < a$, then $\lim_{m\to\infty} \epsilon_{a,2m} = 0$.*

REFERENCES

[1] *Crux Mathematicorum*, **5** (24) (1998), 259.