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THE SUPERSTABILITY OF THE PEXIDER TYPE TRIGONOMETRIC FUNCTIONAL EQUATION

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ABSTRACT. The aim of this paper is to investigate the stability problem for the Pexider type (hyperbolic) trigonometric functional equation $f(x+y)+f(x+\sigma y)=\lambda g(x)h(y)$ under the conditions : $|f(x+y)+f(x+\sigma y)-\lambda g(x)h(y)| \le \varphi(x),\ \varphi(y)$, and $\min\{\varphi(x),\varphi(y)\}$.

As a consequence, we have generalized the results of stability for the cosine(d'Alembert), sine, and the Wilson functional equations by J. Baker, P. Găvruta, R. Badora and R. Ger, Pl. Kannappan, and G. H. Kim.

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1. INTRODUCTION

J. Baker and et al. in [4] introduced the following: if f satisfies the inequality $|E_1(f) - E_2(f)| \leq \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. This is frequently referred to as superstability.

The superstability of the cosine functional equation (also called the d'Alembert equation)

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{C}$$

is studied by J. Baker [3], R. Badora and R. Ger ([1], [2]), and P. Găvruta [6], and the superstability of the sine functional equation

$$f(x)f(y) = f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 \tag{S}$$

is by P.W. Cholewa [5], Badora and Ger [2], and Kim ([11], [12]), respectively.

The cosine functional equation (C) is generalized to the following functional equations

$$f(x+y) + f(x-y) = 2f(x)g(y), \tag{C_{fg}}$$
$$f(x+y) + f(x-y) = 2g(x)f(y), \tag{C_{gf}}$$

and

$$f(x+y) + f(x-y) = 2g(x)g(y) \tag{C_{gg}}$$

and their stabilities are explored by Kannappan and Kim ([9], [10], [13]). The equation (C_{fg}), introduced by Wilson, is sometimes referred to as the Wilson equation. This work is the improved result of author's paper [10].

The hyperbolic trigonometric functions and some exponential functions satisfy the above mentioned equations, and, thus, they can be called the hyperbolic cosine, trigonometric and exponential functional equations, respectively.

For example,

$$\cosh(x+y) + \cosh(x-y) = 2\cosh(x)\cosh(y)$$
$$\sinh(x+y) + \sinh(x-y) = 2\sinh(x)\cosh(y)$$
$$\sinh^2\left(\tfrac{x+y}{2}\right) - \sinh^2\left(\tfrac{x-y}{2}\right) = \sinh(x)\sinh(y)$$
$$ca^{x+y} + ca^{x-y} = 2\tfrac{ca^x}{2}(a^y + a^{-y}) = 2ce^x\tfrac{a^y+a^{-y}}{2}$$
$$e^{x+y} + e^{x-y} = 2\tfrac{e^x}{2}(e^y + e^{-y}) = 2e^x\cosh(y)$$
$$(n(x+y)+c) + (n(x-y)+c) = 2(nx+c) : \text{Jensen equation, for } f(x) = nx + c,$$

where a and c are constants.

We will consider the Pexider type (hyperbolic) trigonometric functional equation as follows:

$$f(x+y) + f(x+\sigma y) = \lambda g(x)h(y). \tag{$\widetilde{C_{gh}}$}$$

In this paper, let $(G,+)$ be an Abelian group, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers, and let σ be an endomorphism of G with $\sigma(\sigma(x)) = x$ for all $x \in G$ with notation $\sigma(x) = \sigma x$. The properties $g(x) = g(\sigma x)$ and $g(x) = -g(\sigma x)$ with respect to σ will be represented as the even and odd function properties for convenience, respectively. In all equations, if "$-x$" is replaced by "σx", for example, the equation (C) becomes $(\widetilde{C})$:

$$(\widetilde{C}) \qquad f(x+y)+f(x+\sigma y)=\lambda f(x)f(y)$$

$$(\widetilde{S}) \qquad f\Big(\frac{x+y}{2}\Big)^2-f\Big(\frac{x+\sigma y}{2}\Big)^2=f(x)f(y).$$

Whenever we deal with (S), we need to assume additionally that $(G,+)$ is a uniquely 2-divisible group. We will write then "under 2-divisibility" for short. We may assume that $f, g,$ and h are non-zero functions, that ε is a nonnegative real constant, and that $\varphi : G \to \mathbb{R}$ is a mapping.

Given mappings $f, g,$ and $h : G \to \mathbb{C}$, we will denote their difference by an operator $D\widetilde{C_{gh}} : G\times G \to \mathbb{C}$ as

$$D\widetilde{C_{gh}}(x,y) := f(x+y)+f(x+\sigma y)-\lambda g(x)h(y).$$

Our goal is to investigate the stability problem for the Pexider type (hyperbolic) trigonometric functional equation $(\widetilde{C_{gh}})$ under conditions : $|D\widetilde{C_{gh}}(x,y)| \le \varphi(x), \varphi(y)$, and $\min\{\varphi(x),\varphi(y)\}$. Then we extend the obtained results to the Banach algebra. As a consequence, we obtain the superstability of the cosine, sine, and the Wilson functional equations that is examined by Baker, Badora, Ger, Găvruta, Kannappan, Kim, etc ([1], [2], [3], [6], [7], [9], [10], [13]).

2. STABILITY ON THE EQUATION $(\widetilde{C_{gh}})$

In this section, we talk about the stability of the equation $(\widetilde{C_{gh}})$ related to the d'Alembert type equation $(\widetilde{C})$, the Wilson type equations $(\widetilde{C_{fg}})$ and $(\widetilde{C_{gf}})$, and the sine type functional equation $(\widetilde{S})$.

Theorem 2.1. *Suppose that $f, g,$ and $h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y)+f(x+\sigma y)-\lambda g(x)h(y)| \le \varphi(x) \quad \forall\, x,y\in G. \tag{2.1}$$

If h fails to be bounded, then

(i) g satisfies $(\widetilde{S})$ under 2-divisibility and one of the cases $g(0)=0$ or $f(\sigma x)=-f(x)$,

(ii) In addition, if h satisfies $(\widetilde{C})$, then g and h are solutions of the Wilson equation $g(x+y)+g(x+\sigma y)=\lambda g(x)h(y)$.

Proof. Let h be unbounded solution of the inequality (2.1). Then, there exists a sequence $\{y_n\}$ in G such that $0\neq |h(y_n)| \to \infty$ as $n\to\infty$.

Taking $y=y_n$ in the inequality (2.1), dividing both sides by $|\lambda h(y_n)|$, and passing to the limit as $n\to\infty$ we obtain that

$$g(x)=\lim_{n\to\infty}\frac{f(x+y_n)+f(x+\sigma y_n)}{\lambda h(y_n)}, \quad x\in G. \tag{2.2}$$

Using (2.1), we have

$$\begin{aligned}&\big|f\left(x+(y+y_n)\right)+f\left(x-(y+y_n)\right)-\lambda g(x)h(y+y_n)\\ &\quad f\left(x+(y+\sigma y_n)\right)+f\left(x-(y+\sigma y_n)\right)-\lambda g(x)h(y+\sigma y_n)\big| \le 2\varphi(x)\end{aligned}$$

so that

$$\begin{aligned}&\left|\frac{f\left((x+y)+y_n\right)+f\left((x+y)+\sigma y_n\right)}{\lambda h(y_n)}\right.\\&\left.+\frac{f\left((x+\sigma y)+y_n\right)+f\left((x+\sigma y)+\sigma y_n\right)}{\lambda h(y_n)}-\lambda g(x)\cdot\frac{h(y+y_n)+h(y+\sigma y_n)}{\lambda h(y_n)}\right|\\&\leq\frac{2\varphi(x)}{|\lambda||h(y_n)|}\end{aligned}\tag{2.3}$$

for all $x, y \in G$.

We conclude that, for every $y \in G$, there exists a limit function

$$k_1(y) := \lim_{n\to\infty}\frac{h(y+y_n)+h(y+\sigma y_n)}{\lambda h(y_n)},$$

where the function $k_1 : G \to \mathbb{C}$ satisfies the equation

$$g(x+y)+g(x+\sigma y)=\lambda g(x)k_1(y)\quad\forall x,y\in G.\tag{2.4}$$

Applying the case $g(0) = 0$ in (2.4), it implies that g is an odd w.r.t. for σ. Keeping this in mind, by means of (2.4), we infer the equality

$$\begin{aligned}g(x+y)^2-g(x+\sigma y)^2&=\lambda g(x)k_1(y)[g(x+y)-g(x+\sigma y)]\\&=g(x)\big[g(x+2y)-g(x-2y)\big]\\&=g(x)\big[g(2y+x)+g(2y-x)\big]\\&=\lambda g(x)g(2y)k_1(x).\end{aligned}\tag{2.5}$$

Putting $y = x$ in (2.4) we get the equation

$$g(2x)=\lambda g(x)k_1(x),\qquad x\in G.$$

This, in return, leads to the equation

$$g(x+y)^2-g(x+\sigma y)^2=g(2x)g(2y)\tag{2.6}$$

valid for all $x, y \in G$ which g, in the light of the unique 2-divisibility of G, states nothing else but $(\widetilde{S})$.

Next, in particular case $f(\sigma x) = -f(x)$, it is enough to show that $g(0) = 0$. Suppose that this is not the case.

Putting $x = 0$ in (2.1), due to $g(0) \neq 0$ and $f(\sigma x) = -f(x)$, we obtain the inequality

$$|h(y)|\leq\frac{\varphi(0)}{|\lambda||g(0)|},\qquad y\in G.$$

This inequality means that h is globally bounded – a contradiction. Thus the claimed $g(0) = 0$ holds.

(ii) In the case h satisfies $(\widetilde{C})$, the limit k_1 states nothing else but h, so (2.4) validates $g(x + y) + g(x + \sigma y) = \lambda g(x)h(y)$. ∎

Replacing h by f, g by f, and h by g in Theorem 2.1, we obtain the following corollaries.

Corollary 2.1. *Suppose that f and $g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y)+f(x+\sigma y)-\lambda g(x)f(y)|\leq\varphi(x)\quad\forall\, x,y\in G.$$

Then either f is bounded or g satisfies $(\widetilde{C})$.

Proof. Replacing h by f in Theorem 2.1, the limit equation (2.2) transfer to

$$g(x) = \lim_{n\to\infty} \frac{f(x+y_n) + f(x+\sigma y_n)}{\lambda f(y_n)}. \tag{2.7}$$

An obvious slight change in the steps of the proof applied after (2.2) of Theorem 2.1, with (2.4) and an application of k_1, gives us the required result. ∎

Corollary 2.2. *Suppose that f and $h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x+\sigma y) - \lambda f(x)h(y)| \le \varphi(x) \quad \forall\, x, y \in G.$$

If h fails to be bounded, then

(i) f satisfies $(\widetilde{S})$ under 2-divisibility and one of the cases $f(0) = 0$ or $f(\sigma x) = -f(x)$,

(ii) In addition, if h satisfies $(\widetilde{C})$, then f and h are solutions of $f(x+y) + f(x+\sigma y) = \lambda f(x)h(y)$.

Corollary 2.3. *Suppose that f and $g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x+\sigma y) - \lambda g(x)g(y)| \le \varphi(x) \quad \forall\, x, y \in G.$$

Then either g is bounded or g satisfies $(\widetilde{S})$ under 2-divisibility and one of the cases $g(0) = 0$ or $f(\sigma x) = -f(x)$.

Corollary 2.4. *Suppose that $f : G \to \mathbb{C}$ satisfies the inequality*

$$|f(x+y) + f(x+\sigma y) - \lambda f(x)f(y)| \le \varphi(x) \qquad \forall\, x, y \in G.$$

Then either f is bounded or f satisfies $(\widetilde{C})$.

Remark 2.1. *In Theorem 2.1 and Corollaries 2.1 $\sim$ 2.4,*

(i) Applying $\sigma(y) = -y$, or $\lambda = 2$, respectively, we can obtain same number of corollaries.

(ii) Applying $\sigma(y) = -y$ and $\lambda = 2$, we can obtain same number of corollaries. Some of which can be found in the papers ([6], [9], [10]*).*

(iii) In all of the Theorem 2.1, Corollaries 2.1 $\sim$ 2.4, in all of the results obtained from (i) and (ii), applying $\varphi(x) = \varepsilon$, we obtain twenty corollaries, respectively. Some of which can be found in the papers ([1], [2], [3], [6], [7], [9], [10]*).*

Theorem 2.2. *Suppose that f, g and $h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x+\sigma y) - \lambda g(x)h(y)| \le \varphi(y) \quad \forall\, x, y \in G. \tag{2.8}$$

If g fails to be bounded, then

(i) h satisfies $(\widetilde{S})$ under 2-divisibility and one of the cases $h(0) = 0$ or $f(\sigma x) = -f(x)$,

(ii) In addition, if g satisfies $(\widetilde{C})$, then h and g are solutions of the Wilson equation $h(x+y) + h(x+\sigma y) = \lambda h(x)g(y)$.

Proof. Let g be unbounded solution of the inequality (2.8). Then, there exists a sequence $\{x_n\}$ in G such that $0 \ne |g(x_n)| \to \infty$ as $n \to \infty$.

Taking $x = x_n$ in the inequality (2.8), dividing both sides by $|\lambda g(x_n)|$, and passing to the limit as $n \to \infty$ we obtain that

$$h(y) = \lim_{n\to\infty} \frac{f(x_n+y) + f(x_n+\sigma y)}{\lambda g(x_n)}, \qquad x \in G. \tag{2.9}$$

In (2.8), replacing x by $x_n + y$, replacing y by x, and also replacing x by $x_n + \sigma y$, replacing y by x, respectively.

Due to similar reasoning as in the proof applied in Theorem 2.1, we conclude that, for every $y \in G$, there exists a limit function

$$k_2(x) := \lim_{n\to\infty} \frac{[g(x_n + y) + g(x_n + \sigma y)]}{\lambda g(x_n)},$$

where the function $k_2 : G \to \mathbb{C}$ satisfies the equation

$$h(x + y) + h(x + \sigma y) = \lambda h(x)k_2(y) \quad \forall x, y \in G. \tag{2.10}$$

The next of the proof runs along a similar step as Theorem 2.1, h satisfies (S). ∎

Corollary 2.5. *Suppose that f and $g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x + y) + f(x + \sigma y) - \lambda g(x)f(y)| \le \varphi(y) \quad \forall\, x, y \in G.$$

If g fails to be bounded, then

(i) f satisfies ($\widetilde{S}$) under 2-divisibility and one of the cases $f(0) = 0$ or $f(\sigma x) = -f(x)$,

(ii) In addition, if g satisfies ($\widetilde{C}$), then f satisfies ($\widetilde{C}$).

Corollary 2.6. *Suppose that f and $h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x + y) + f(x + \sigma y) - \lambda f(x)h(y)| \le \varphi(y) \quad \forall\, x, y \in G.$$

Then either f is bounded or h satisfies ($\widetilde{C}$).

Corollary 2.7. *Suppose that f and $g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x + y) + f(x + \sigma y) - \lambda g(x)g(y)| \le \varphi(y) \quad \forall\, x, y \in G.$$

Then either g is bounded or g satisfies ($\widetilde{S}$) under 2-divisibility and one of the cases $g(0) = 0$ or $f(\sigma x) = -f(x)$.

Corollary 2.8. ([1], [7]) *Suppose that $f : G \to \mathbb{C}$ satisfies the inequality*

$$|f(x + y) + f(x + \sigma y) - \lambda f(x)f(y)| \le \varphi(y) \qquad \forall\, x, y \in G.$$

Then either f is bounded or f satisfies ($\widetilde{C}$).

Remark 2.2. *By same application as Remark 2.1 for Theorem 2.2 and Corollaries 2.5 $\sim$ 2.8, we obtain same number of corollaries.*

Theorem 2.3. *Suppose that f, g and $h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x + y) + f(x + \sigma y) - \lambda g(x)h(y)| \le \min\{\varphi(x), \varphi(y)\} \qquad \forall\, x, y \in G. \tag{2.11}$$

Then,

(a) If g fails to be bounded, then

(i) h satisfies ($\widetilde{S}$) under 2-divisibility and one of the cases $h(0) = 0$ or $f(\sigma x) = -f(x)$,

(ii) In addition, if h satisfies ($\widetilde{C}$), then h and g are solutions of the Wilson equation $h(x + y) + h(x + \sigma y) = \lambda h(x)g(y)$.

(b) If h fails to be bounded, then

(i) g satisfies ($\widetilde{S}$) under 2-divisibility and one of the cases $g(0) = 0$ or $f(\sigma x) = -f(x)$,

(ii) In addition, if h satisfies ($\widetilde{C}$), then g and h are solutions of the Wilson equation $g(x + y) + g(x + \sigma y) = \lambda g(x)h(y)$.

Corollary 2.9. *Suppose that f and $g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x + y) + f(x + \sigma y) - \lambda g(x)f(y)| \le \min\{\varphi(x), \varphi(y)\} \quad \forall\, x, y \in G.$$

Then,

(a) If g fails to be bounded, then

(i) f satisfies ($\widetilde{S}$) under 2-divisibility and one of the cases $f(0) = 0$ or $f(\sigma x) = -f(x)$,

(ii) In addition, if f satisfies ($\widetilde{C}$), then f and g are solutions of the Wilson equation $f(x + y) + f(x + \sigma y) = \lambda f(x)g(y)$.

(b) If f fails to be bounded, then

(i) g satisfies ($\widetilde{S}$) under 2-divisibility and one of the cases $g(0) = 0$ or $f(\sigma x) = -f(x)$,

(ii) In addition, if f satisfies ($\widetilde{C}$), then g and f are solutions of the Wilson equation $g(x + y) + g(x + \sigma y) = \lambda g(x)f(y)$.

Corollary 2.10. *Suppose that f and $h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x + y) + f(x + \sigma y) - \lambda f(x)h(y)| \le \min\{\varphi(x), \varphi(y)\} \qquad \forall\, x, y \in G.$$

Then,

(a) If f fails to be bounded, then

(i) h satisfies ($\widetilde{S}$) under 2-divisibility and one of the cases $h(0) = 0$ or $f(\sigma x) = -f(x)$,

(ii) In addition, if h satisfies ($\widetilde{C}$), then h and f are solutions of the Wilson equation $h(x + y) + h(x + \sigma y) = \lambda h(x)f(y)$.

(b) If h fails to be bounded, then

(i) f satisfies ($\widetilde{S}$) under 2-divisibility and one of the cases $f(0) = 0$ or $f(\sigma x) = -f(x)$,

(ii) In addition, if h satisfies ($\widetilde{C}$), then f and h are solutions of the Wilson equation $f(x + y) + f(x + \sigma y) = \lambda f(x)h(y)$.

Corollary 2.11. *Suppose that f and $g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x + y) + f(x + \sigma y) - \lambda g(x)g(y)| \le \min\{\varphi(x), \varphi(y)\} \qquad \forall\, x, y \in G.$$

Then either g is bounded or g satisfies ($\widetilde{C}$).

Corollary 2.12. *Suppose that $f : G \to \mathbb{C}$ satisfies the inequality*

$$|f(x + y) + f(x + \sigma y) - \lambda f(x)f(y)| \le \varepsilon \quad \forall\, x, y \in G. \tag{2.12}$$

Then either f is bounded or f satisfies ($\widetilde{C}$).

Remark 2.3. *By same application as Remark 2.2 for Theorem 2.3 and Corollaries 2.9 $\sim$ 2.12, we obtain same number of corollaries.*

3. EXTENSION TO THE BANACH ALGEBRA

All of the results obtained in sections 2 can be extended to the Banach algebra. For simplicity, we will present only some of them, and the application to other corollaries will be omitted.

Theorem 3.1. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that f, g and $h : G \to E$ satisfy the inequality*

$$\|f(x + y) + f(x + \sigma y) - \lambda g(x)h(y)\| \le \varphi(x) \quad \forall\, x, y \in G. \tag{3.1}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,
if the superposition $x^* \circ h$ fails to be bounded, then

(i) g satisfies (S) under 2-divisibility and one of the cases $g(0) = 0$ or $f(-x) = -f(x)$,

(ii) if, additionally, $x^* \circ h$ satisfies (C), then g and h are solutions of the Wilson equation $g(x + y) + g(x + \sigma y) = \lambda g(x)h(y)$.

Proof. (i) Fix arbitrarily a linear multiplicative functional $x^* \in E^*$, we have $\|x^*\| = 1$ as known. In (3.1), we have

$$\begin{aligned}\varphi(x) &\geq \|f(x+y) + f(x+\sigma y) - \lambda g(x)h(y)\| \\ &= \sup_{\|y^*\|=1} \left|y^*\big(f(x+y) + f(x+\sigma y) - \lambda g(x)h(y)\big)\right| \\ &\geq \left|x^*\big(f(x+y)\big) - x^*\big(f(x+\sigma y)\big) - 2x^*\big(g(x)\big)x^*\big(h(y)\big)\right|.\end{aligned}$$

In the above inequality, we know that the superpositions $x^* \circ f$, $x^* \circ g$, and $x^* \circ h$ yield solutions of inequality (2.1) in Theorem 2.1.

Assume that the superposition $x^* \circ h$ is unbounded, then Theorem 2.1 forces the superposition $x^* \circ g$ solves (S). These statements mean, keeping the linear multiplicativity of x^* in mind, that the difference $DS_g(x,y) := g(x)g(y) - g\big(\frac{x+y}{2}\big)^2 + g\big(\frac{x+\sigma y}{2}\big)^2$ for all $x, y \in G$ falls into the kernel of x^*.

Since x^* is arbitrary, we deduce that

$$DS_g(x,y) \in \bigcap\{\ker x^* : x^* \in E^*\}$$

for all $x, y \in G$.

Since the Banach algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$DS_g(x,y) = 0 \quad \forall\, x, y \in G,$$

as claimed.

(ii) Under the assumption that the superposition $x^* \circ h$ satisfies (C), we know from Theorem 2.1 that the superpositions $x^* \circ g$ and $x^* \circ h$ are solutions of the equation

$$x^*\big(g(x+y)\big) - x^*\big(g(x-y)\big) = 2x^*\big(g(x)\big)x^*\big(h(y)\big).$$

Namely,

$$g(x+y) - g(x+\sigma y) - \lambda g(x)h(y) \in \bigcap\{\ker x^* : x^* \in E^*\}.$$

The other argument is similar. ∎

Corollary 3.1. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that f and $g : G \to E$ satisfy the inequality*

$$\|f(x+y) + f(x+\sigma y) - \lambda g(x)f(y)\| \leq \varphi(x) \quad \forall\, x, y \in G.$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,
if the superposition $x^* \circ f$ fails to be bounded, then

(i) g satisfies (S) under 2-divisibility and one of the cases $g(0) = 0$ or $f(-x) = -f(x)$,

(ii) if, additionally, $x^* \circ h$ satisfies (C), then g and f are solutions of the Wilson equation $g(x+y) + g(x+\sigma y) = \lambda g(x)f(y)$.

Corollary 3.2. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that f and $h : G \to E$ satisfy the inequality*

$$\|f(x+y) + f(x+\sigma y) - \lambda f(x)h(y)\| \leq \varphi(x) \quad \forall\, x, y \in G.$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,
if the superposition $x^* \circ h$ fails to be bounded, then

(i) f satisfies (S) under 2-divisibility and one of the cases $f(0) = 0$ or $f(-x) = -f(x)$,

(ii) if, additionally, $x^* \circ h$ satisfies (C), then f and h are solutions of the Wilson equation $f(x+y) + f(x+\sigma y) = \lambda f(x)h(y)$.

Corollary 3.3. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that f and $g : G \to E$ satisfy the inequality*

$$\|f(x+y) + f(x+\sigma y) - \lambda g(x)g(y)\| \leq \varphi(x) \quad \forall\, x, y \in G. \tag{3.2}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,
if the superposition $x^* \circ h$ fails to be bounded, then
(i) g satisfies (S) under 2-divisibility and one of the cases $g(0) = 0$ or $f(-x) = -f(x)$,
(ii) if, additionally, $x^* \circ g$ satisfies (C), then g satisfies the cosine type equation $g(x+y) + g(x+\sigma y) = \lambda g(x)g(y)$.

Corollary 3.4. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f : G \to E$ satisfies the inequality*

$$\|f(x+y) + f(x+\sigma y) - \lambda f(x)f(y)\| \leq \varphi(x) \quad \forall\, x, y \in G. \tag{3.3}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,
if the superposition $x^* \circ f$ fails to be bounded, then f satisfies the cosine type equation $f(x+y) + f(x+\sigma y) = \lambda f(x)f(y)$.

Remark 3.1. *(i) As Theorems 2.2, 2.3 and Corollaries 2.5 $\sim$ 2.12, if we replace $\varphi(y)$, or $\min\{\varphi(x), \varphi(y)\}$ for $\varphi(x)$ in the stability inequality* (3.1)*, then we can obtain the same number of results.*

(ii) The same applications ($\sigma(y) = -y, \lambda = 2, \sigma(y) = -y$ and $\lambda = 2$, and $\varphi(x) = \varepsilon$) *as above remarks applied in theorems and corollaries give us the same number of corollaries. As a consequence, this paper inclosed about one hundred eighty results.*

(iii) If we consider the Kannappan's condition $f(x+y+z) = f(x+z+y)$ (see [8]*), then our results on the group will be held on the semigroup.*

REFERENCES

[1] R. BADORA, On the stability of cosine functional equation, *Rocznik Naukowo-Dydak., Prace Mat.*, **15**, (1998), pp. 1–14.

[2] R. BADORA and R. GER, On some trigonometric functional inequalities, *Functional Equations - Results and Advances*, (2002), pp. 3–15.

[3] J. A. BAKER, The stability of the cosine equation, *Proc. Amer. Math. Soc.*, **80**, (1980), pp. 411–416.

[4] J. A. BAKER, J. LAWRENCE and F. ZORZITTO, The stability of the equation $f(x+y) = f(x)f(y)$, *Proc. Amer. Math. Soc.*, **74**, (1979), pp. 242–246.

[5] P. W. CHOLEWA, The stability of the sine equation, *Proc. Amer. Math. Soc.*, **88**, (1983). pp. 631–634.

[6] P. GĂVRUTA, On the stability of some functional equations, in : Th. M. Rassias and J. Tabor, *Stability of mappings of Hyers-Ulam type*, Hadronic Press, 1994, pp. 93–98.

[7] P. GĂVRUTA, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, **184**, (1994), pp. 431–436.

[8] PL. KANNAPPAN, The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups, *Proc. Amer. Math. Soc.*, **19**, (1968), pp. 69–74.

[9] PL. KANNAPPAN and G. H. KIM, On the stability of the generalized cosine functional equations, *Annales AcadedmiæPaedagogicae Cracoviensis - Studia mathematica*, **1**, (2001), pp. 49–58.

[10] G. H. KIM, The Stability of the d'Alembert and Jensen type functional equations, *Jour. Math. Anal & Appl.*, **325**, (2007), pp. 237–248.

[11] G. H. KIM, A Stability of the generalized sine functional equations, *Jour. Math. Anal & Appl.*, **331**, (2007), pp. 886–894.

[12] G. H. KIM, On the Stability of the generalized sine functional equations, *Acta Math. Sin.*, **25**, (2009), pp. 965–972

[13] G. H. KIM and S. S. DRAGOMIR, On the Stability of generalized d'Alembert and Jensen functional equation, *Intern. Jour. Math. & Math. Sci.*, Article ID 43185, DOI 10.1155 (2006), pp. 1–12.