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SEVERAL $q-$INTEGRAL INEQUALITIES

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ABSTRACT. In the present paper several $q-$integral inequalities are presented, some of them are new and others are generalizations of known results.

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1. INTRODUCTION

The $q-$analog $(0 < q < 1)$ of the derivative, denoted by D_q is defined (see [7]) by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0. \tag{1.1}$$

If $f'(0)$ exists, then $D_q f(0) = f'(0)$. As $q \to 1$, the $q-$derivative reduces to the usual derivative.

The $q-$analog of integration may be given (see [8]) by

$$\int_0^1 f(x) d_q x = (1-q) \sum_{i=0}^{\infty} f(q^i) q^i, \tag{1.2}$$

which reduces to $\int_0^1 f(x)dx$ as $q \to 1$.

The $q-$Jackson integral from 0 to $a \in \Re$ can be defined (see [2, 3]) by

$$\int_0^a f(x) d_q x = a(1-q) \sum_{i=0}^{\infty} f(aq^i) q^i \tag{1.3}$$

provided the sum converges absolutely. The $q-$Jackson integral on a general interval $[a, b]$ may be defined (see [2, 3]) by

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x \tag{1.4}$$

The $q-$Jackson integral and $q-$derivative are related by the "fundamental theorem of quantum calculus" which can be stated (see [3, p. 73]) as follows: If F is an anti $q-$derivative of the function f, namely $D_q F = f$, continuous at $x = a$, then

$$\int_a^b f(x) d_q x = F(b) - F(a). \tag{1.5}$$

For any function f one has

$$D_q \left(\int_a^x f(t) d_q t \right) = f(x). \tag{1.6}$$

The $q-$analog of Leibniz's rule is also valid

$$D_q(f(x)g(x)) = f(x) D_q g(x) + g(qx) D_q f(x). \tag{1.7}$$

For $b > 0$ and $a = bq^n$ with $n \in N$, denote

$$[a, b]_q = \{bq^k : 0 \leq k \leq n\} \quad and \quad (a, b]_q = [aq^{-1}, b]_q. \tag{1.8}$$

In [6], the following results were proved:

Theorem 1.1. *If $f(x)$ is a non-negative and increasing function on $[a, b]_q$ and satisfies*

$$(\alpha - 1) f^{\alpha - 2}(qx) D_q f(x) \geq \beta(\beta - 1) f^{\beta - 1}(x)(x - a)^{\beta - 2} \tag{1.9}$$

for $\alpha \geq 1$ and $\beta \geq 1$, then

$$\int_a^b f^{\alpha}(x) d_q x \geq \left(\int_a^b f(x) d_q x \right)^{\beta}. \tag{1.10}$$

Theorem 1.2. *If $f(x)$ is a non-negative and increasing function on $[bq^{n+m}, b]_q$ for $m, n \in N$ and satisfies*

$$(\alpha-1)D_q f(x) \geq \beta(\beta-1)f^{\beta-\alpha+1}(q^m x)(x-a)^{\beta-2} \tag{1.11}$$

for $\alpha \geq 1$ and $\beta \geq 1$, then

$$\int_a^b f^{\alpha}(x)d_q x \geq \left(\int_a^b f(q^m x)d_q x\right)^{\beta}. \tag{1.12}$$

Theorem 1.3. *If $f(x)$ is a non-negative function on $[0, b]_q$ and satisfies*

$$\int_0^b f^{\beta}(t)d_q t \geq \int_0^b t^{\beta}d_q t \tag{1.13}$$

for $x \in [0, b]_q$ and $\beta > 0$, then the inequality

$$\int_0^b f^{\alpha+\beta}(x)d_q x \geq \int_0^b x^{\alpha}f^{\beta}(x)d_q x. \tag{1.14}$$

holds for all α and β.

2. RESULTS.

We are assuming that $\alpha > 0$ is fixed, and start by giving an alternative proof for the following lemma.

Lemma 2.1. [1] *Let $p \geq 1$ and $g(x)$ be a non-negative, non-decreasing function on $[a, b]_q$. Then*

$$pg^{p-1}(qx)D_q g(x) \leq D_q(g^p(x)) \leq pg^{p-1}(x)D_q g(x), \quad x \in (a, b]_q. \tag{2.1}$$

Proof. Since $p \geq 1$, then it is sufficient to prove the inequality for p integer, as any non integer lies between two integers and has the same property. We have

$$\begin{aligned} D_q g^p(x) &= \frac{g^p(x) - g^p(qx)}{x(1-q)} = \frac{g^p(x) - g^p(qx)}{g(x) - g(qx)} \times \frac{g(x) - g(qx)}{x(1-q)} \\ &= D_q g(x)\sum_{j=1}^{p} g^{j-1}(x)g^{p-j}(qx) \leq pg^{p-1}(x)D_q g(x), \end{aligned}$$

as g is non-decreasing. Also, we have

$$D_q g^p(x) \geq pg^{p-1}(qx)D_q g(x).$$

∎

The result of Lemma 2.1 can be obtained also by using the following proposition.

Proposition 2.2. *If $a > b > 0, p > 1$, then*

$$pb^{p-1} < \frac{a^p - b^p}{a-b} < pa^{p-1}.$$

Proof. Let

$$f(a) = p(a-b)a^{p-1} - a^p + b^p.$$

By keeping b fixed and letting a vary, we have

$$f'(a) = p^2a^{p-1} - p(p-1)ba^{p-2} - pa^{p-1} = 0,$$

whenever $a = b$.

$$f''(a) = p^2(p-1)a^{p-2} - bp(p-1)(p-2)a^{p-3} - p(p-1)a^{p-2}.$$
$$f''(a) = p(p-1)a^{p-2} > 0,$$

whenever $a = b$.
Therefore $f(a)$ attains its minimum when $a = b$ which is 0. That is $f(a) \geq 0$. The left inequality follows the same steps by keeping a fixed and b variable, and therefore is omitted. The proof of the first part of the Lemma follows by applying the proposition with $a = g(x), b = g(qx)$ in the following step:

$$D_q g^p x = \frac{g^p(x) - g^p(qx)}{g(x) - g(qx)} D_q g(x).$$

∎

It may be mentioned that in Theorem 1.1, β should be greater or equal 2, otherwise the step in line 4 page 119 is not true in general.

The following is a good generalization of Theorem 1.1:

Theorem 2.3. *If $f(x)$ is a non-negative increasing function on $[a,b]_q$ and satisfies*

(2.2) $$(\gamma - \alpha) f^{\gamma-\alpha+1}(qx) D_q f(x) - \beta(\beta-1) f^{\alpha(\beta-1)}(x)(x-a)^{\beta-2} \geq 0$$

for $1 \leq \alpha < \gamma, \beta \geq 2$, then

(2.3) $$\int_a^b f^\gamma(x) d_q x \geq \left(\int_a^b f^\alpha(x) d_q x \right)^\beta.$$

Proof. Let

$$F(x) = \int_a^x f^\gamma(t) d_q t - \left(\int_a^x f^\alpha(t) d_q t \right)^\beta, \quad x \in [a,b]_q,$$

$$g(x) = \int_a^x f^\alpha(t) d_q t.$$

By virtue of Lemma 2.1, we have

$$\begin{aligned} D_q F(x) &= f^\gamma(x) - D_q g^\beta(x) \\ &\geq f^\gamma(x) - \beta g^{\beta-1}(x) f^\alpha(x) \\ &= f^\alpha(x)(f^{\gamma-\alpha}(x) - \beta g^{\beta-1}(x)) = f^\alpha(x) h(x). \end{aligned}$$

$$D_q h(x) \geq (\gamma - \alpha) f^{\gamma-\alpha-1}(qx) D_q f(x) - \beta(\beta-1) g^{\beta-2}(x) f^\alpha(x).$$

As $g^{\beta-2}(x) = \left(\int_a^x f^\alpha(t) dt \right)^{\beta-2} \leq f^{\alpha(\beta-2)}(x)(x-a)^{\beta-2}$, then

$$D_q h(x) \geq (\gamma - \alpha) f^{\gamma-\alpha-1}(qx) D_q f(x) - \beta(\beta-1) f^{\alpha(\beta-1)}(x)(x-a)^{\beta-2} \geq 0.$$

This shows that $h(x)$ is non-decreasing, and hence $h(x) \geq h(a) \geq 0$. Therefore $F(x)$ is non-decreasing and so $F(x) \geq F(a) = 0$. This completes the proof. ∎

The following Lemmas are needed for the coming results.

Lemma 2.4. *Let $f, g \geq 0, g$ is non-decreasing with $g(a) = 0$. Then either of the two conditions*

$$\int_x^b f^\alpha(t) d_q t \geq \int_x^b g^\alpha(t) d_q t, \quad \forall x \in [a, b]_q, \tag{2.4}$$

$$\int_a^x f^\alpha(t) d_q t \leq \int_a^x g^\alpha(t) d_q t, \quad and \quad \int_a^b f^\alpha(t) d_q t = \int_a^b g^\alpha(t) d_q t, \quad \forall x \in [a, b]_q, \tag{2.5}$$

implies

$$\int_a^b f^\alpha(t) g^\beta(t) d_q t \geq \int_a^b g^{\alpha+\beta}(t) d_q t, \quad \forall \beta > 0. \tag{2.6}$$

Proof. We define

$$\frac{f(g(x)) - f(g(qx))}{g(x) - g(qx)} = D_q(f, g).$$

Since

$$\begin{aligned} D_q f \circ g(x) &= \frac{f(g(x)) - f(g(qx))}{x - qx} \\ &= \frac{f(g(x)) - f(g(qx))}{g(x) - g(qx)} \times \frac{g(x) - g(qx)}{x - qx} = D_q(f, g) D_q g, \end{aligned}$$

then

$$f \circ g(x) = \int D_q(f, g) D_q g d_q x. \tag{2.7}$$

Suppose (2.4) is satisfied and define $h(x) = x^\beta$, then $g^\beta(x) = h(g(x))$, and we have

$$\begin{aligned} \int_a^b f^\alpha(x) g^\beta(x) d_q x &= \int_a^b f^\alpha(x) h(g(x)) d_q x = \int_a^b f^\alpha(x) \int_a^x D_q(h, g) D_q g(u) d_q u d_q x \\ &= \int_a^b D_q(h, g) D_q g(u) \int_u^b f^\alpha(x) d_q x d_q u \\ &\geq \int_a^b D_q(h, g) D_q g(u) \int_u^b g^\alpha(x) d_q x d_q u \\ &= \int_a^b g^\alpha(x) \int_a^x D_q(h, g) D_q g(u) d_q u d_q x \\ &= \int_a^b g^{\alpha+\beta}(x) d_q x. \end{aligned}$$

Now let (2.5) be satisfied, then we have

$$\begin{aligned}
\int_a^b f^\alpha(x)g^\beta(x)d_qx &= \int_a^b f^\alpha(x)\int_a^x D_q(h,g)D_qg(u)d_qud_qx \\
&= \int_a^b f^\alpha(x)\left(\int_a^b D_q(h,g)D_qg(u)d_qu - \int_x^b D_q(h,g)D_qg(u)d_qu\right)d_qx \\
&= g^\beta(b)\int_a^b f^\alpha(x)d_qx - \int_a^b D_q(h,g)D_qg(u)\int_a^u f^\alpha(x)d_qxd_qu \\
&\geq g^\beta(b)\int_a^b f^\alpha(x)d_qx - \int_a^b D_q(h,g)D_qg(u)\int_a^u g^\alpha(x)d_qxd_qu \\
&= g^\beta(b)\int_a^b f^\alpha(x)d_qx - \int_a^b D_q(h,g)D_qg(u) \\
&\times \left(\int_a^b g^\alpha(x)d_qx - \int_u^b g^\alpha(x)d_qx\right)d_qu \\
&= \int_a^b D_q(h,g)D_qg(u)\int_u^b g^\alpha(x)d_qxd_qu \\
&= \int_a^b g^\alpha(x)\int_a^x D_q(h,g)D_qg(u)d_qud_xx \\
&= \int_a^b g^{\alpha+\beta}(x)d_qx.
\end{aligned}$$

∎

Lemma 2.5. *Let $f,g\geq 0, g$ is non-decreasing with $g(a)=0$. Then either of the two conditions*

(2.8) $$\int_x^b f^\alpha(t)d_qt \leq \int_x^b g^\alpha(t)d_qt, \quad \forall x\in[a,b]_q,$$

(2.9) $$\int_a^x f^\alpha(t)d_qt \geq \int_a^x g^\alpha(t)d_qt \quad and \int_a^b f^\alpha(t)d_qt = \int_a^b g^\alpha(t)d_qt, \quad \forall x\in[a,b]_q,$$

implies

(2.10) $$\int_a^b f^\alpha(t)g^\beta(t)d_q(t) \leq \int_a^b g^{\alpha+\beta}(t)d_qt, \quad \forall\beta>0.$$

Proof. The proof is similar to that given in Lemma 2.4.

∎

The following are the main results:

Theorem 2.6. *Suppose $f,g\geq 0, g$ is non-decreasing, $g(a)=0$. If either of the two conditions* (2.4) *or (2.5) is satisfied, then*

(2.11) $$\int_a^b f^{\alpha+\beta}(x)d_qx \geq \int_a^b f^\alpha(x)g^\beta(x)d_qx, \quad \forall\beta>0.$$

(2.12) $$\int_a^b f^{\alpha+\beta}(x)d_qx \geq \int_a^b g^{\alpha+\beta}(x)d_qx, \quad \forall\beta>0.$$

In (2.11) if we are replacing α by γ, provided $\gamma+\beta\geq\alpha, \forall\gamma,\beta>0,$ then (2.11) remains true. If g is non-increasing and (2.4), (2.5) reverses, then (2.11), (2.12) reverses.

Proof. By the AG inequality,

$$\frac{\alpha}{\alpha+\beta}f^{\alpha+\beta}(x)+\frac{\beta}{\alpha+\beta}g^{\alpha+\beta}(x)\geq f^{\alpha}(x)g^{\beta}(x),$$

or

$$f^{\alpha+\beta}(x)\geq(1+\beta/\alpha)f^{\alpha}(x)g^{\beta}(x)-(\beta/\alpha)g^{\alpha+\beta}(x).$$

Integrating the above inequality and hence making use of Lemma 2.4 gives

$$\begin{aligned}\int_a^b f^{\alpha+\beta}(x)d_qx &\geq (1+\beta/\alpha)\int_a^b f^{\alpha}(x)g^{\beta}(x)d_qx-(\beta/\alpha)\int_a^b g^{\alpha+\beta}(x)d_qx\\ &\geq (1+\beta/\alpha)\int_a^b f^{\alpha}(x)g^{\beta}(x)d_qx-(\beta/\alpha)\int_a^b f^{\alpha}(x)g^{\beta}(x)d_qx\\ &= \int_a^b f^{\alpha}(x)g^{\beta}(x)d_qx. \qquad (2.13)\end{aligned}$$

By replacing β by $\beta+\gamma-\alpha>0$ in (2.12), we obtain

$$\int_a^b f^{\gamma+\beta}(x)d_qx\geq\int_a^b g^{\gamma+\beta}(x)d_qx.$$

By the AG inequality,

$$f^{\gamma+\beta}(x)\geq(1+\beta/\gamma)f^{\gamma}(x)g^{\beta}(x)-(\beta/\gamma)g^{\gamma+\beta}(x).$$

Integrating, we get

$$\begin{aligned}\int_a^b f^{\gamma+\beta}(x)d_qx &\geq (1+\beta/\gamma)\int_a^b f^{\gamma}(x)g^{\beta}(x)d_qx-(\beta/\gamma)\int_a^b g^{\gamma+\beta}(x)d_qx\\ &\geq (1+\beta/\gamma)\int_a^b f^{\gamma}(x)g^{\beta}(x)d_qx-(\beta/\gamma)\int_a^b f^{\gamma+\beta}(x)d_qx\end{aligned}$$

which implies

$$\int_a^b f^{\gamma+\beta}(x)d_qx\geq\int_a^b f^{\gamma}(x)g^{\beta}(x)d_qx.$$

Similarly, (2.12) follows from (2.13) and the proof is complete.
∎

The reverse follows from the coming result.

Theorem 2.7. *Suppose $f,g\geq0, g$ is non-decreasing, $g(a)=0$. If either (2.4) or (2.5), with α replaced by $-\alpha$ is satisfied, then*

$$\int_a^b f^{\beta-\alpha}(x)d_qx \leq \int_a^b f^{-\alpha}(x)g^{\beta}(x)d_qx, \quad \forall\beta>\alpha>0. \qquad (2.14)$$

$$\int_a^b f^{\beta-\alpha}(x)d_qx \geq \int_a^b g^{\beta-\alpha}(x)d_qx, \quad \forall\beta>\alpha>0. \qquad (2.15)$$

Proof. From Lemma 2.4, we have

$$\int_a^b f^{-\alpha}(t)g^{\beta}(t)d_qt \quad \geq \quad \int_a^b g^{\beta-\alpha}(t)d_qt, \quad \forall\beta > 0. \tag{2.16}$$

Now, by making use of the AG inequality, with $\beta > \alpha > 0$, we have

$$\frac{\beta}{\beta-\alpha}f^{\beta-\alpha}(x) - \frac{\alpha}{\beta-\alpha}g^{\beta-\alpha}(x) \leq f^{-\alpha}(x)g^{\beta}(x),$$

that is

$$f^{\beta-\alpha}(x) \leq (1-\alpha/\beta)f^{-\alpha}(x)g^{\beta}(x) + (\alpha/\beta)g^{\beta-\alpha}(x).$$

By integrating the above inequality, and then making use of (2.16), we obtain

$$\int_a^b f^{\beta-\alpha}(x)d_qx \leq \int_a^b f^{-\alpha}(x)g^{\beta}(x)d_qx, \quad 0 < \alpha < \beta.$$

∎

Theorem 2.8. *Suppose $f, g \geq 0, g$ is non-decreasing, $g(a) = 0$. If either (2.8) or (2.9), with α replaced by $-\alpha$ is satisfied, then*

$$\int_a^b f^{\beta-\alpha}(x)d_qx \leq \int_a^b g^{\beta-\alpha}(x)d_qx. \tag{2.17}$$

Proof. From Lemma 2.5, we have

$$\int_a^b f^{-\alpha}(x)g^{\beta}(x)d_qx \leq \int_a^b g^{\beta-\alpha}(x)d_qx, \quad 0 < \alpha < \beta. \tag{2.18}$$

By using the AG inequality with $0 < \alpha < \beta$, we have

$$f^{\beta-\alpha}(x) \leq (1-\alpha/\beta)f^{-\alpha}(x)g^{\beta}(x) + (\alpha/\beta)g^{\beta-\alpha}(x).$$

Integrating the above inequality and then making use of (2.7), we obtain

$$\begin{aligned}\int_a^b f^{\beta-\alpha}(x)d_qx &\leq (1-\alpha/\beta)\int_a^b f^{-\alpha}(x)g^{\beta}(x)d_qx + (\alpha/\beta)\int_a^b g^{\beta-\alpha}(x)d_qx \\ &\leq (1-\alpha/\beta)\int_a^b g^{\beta-\alpha}(x)d_qx + (\alpha/\beta)\int_a^b g^{\beta-\alpha}(x)d_qx \\ &= \int_a^b g^{\beta-\alpha}(x)d_qx.\end{aligned}$$

∎

The coming result gives an analogous result to Theorem 2.6.

Theorem 2.9. *Suppose $f, g \geq 0, g$ is non-decreasing. If*

$$\int_x^b f(t)d_qt \geq \int_x^b g(t)d_qt, \quad \forall x \in [a,b], \tag{2.19}$$

then

$$\int_a^b f^{\alpha+\beta}(x)d_qx \geq \int_a^b f^{\alpha}(x)g^{\beta}(x)d_qx, \quad \forall\alpha,\beta \geq 0, \alpha+\beta \geq 1. \tag{2.20}$$

Proof. On putting $\alpha = 1$ in (2.4), we obtain via Lemma 2.4

$$\int_x^b f(t)d_qt \geq \int_x^b g(t)d_qt \implies \int_a^b f(t)g^\beta(t)d_qt \geq \int_a^b g^{1+\beta}(t)d_qt, \quad \forall\beta > 0$$
$$\implies \int_a^b f(t)g^{\alpha-1}(t)dt \geq \int_a^b g^\alpha(t)dt, \quad \forall\beta > 0, \alpha \geq 1.$$

Therefore, by the AG inequality, for $\alpha \geq 1$, we have

$$\begin{aligned}\int_a^b f^\alpha(x)d_qx &\geq \alpha\int_a^b f(x)g^{\alpha-1}(x)d_qx - (\alpha-1)\int_a^b g^\alpha(x)d_qx \\ &\geq \int_a^b g^\alpha(x)d_qx - (\alpha-1)\int_a^b g^\alpha(x)d_qx = \int_a^b g^\alpha(x)d_qx.\end{aligned}$$

Again by the AG inequality for $\alpha + \beta \geq 1$,

$$\begin{aligned}\int_a^b f^{\alpha+\beta}(x)d_qx &= \frac{\alpha}{\alpha+\beta}\int_a^b f^{\alpha+\beta}(x)d_qx + \frac{\beta}{\alpha+\beta}\int_a^b f^{\alpha+\beta}(x)d_qx \\ &\geq \frac{\alpha}{\alpha+\beta}\int_a^b f^{\alpha+\beta}(x)d_qx + \frac{\beta}{\alpha+\beta}\int_a^b g^{\alpha+\beta}(x)d_qx \\ &\geq \int_a^b f^\alpha(x)g^\beta(x)d_qx.\end{aligned}$$

∎

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