The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 7, Issue 2, Article 25, pp. 1-8, 2011

ON AN ELLIPTIC OVER-DETERMINED PROBLEM IN DIMENSION TWO

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Received 9 January, 2010; accepted 28 June, 2010; published 9 May, 2011.

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ABSTRACT. We extend the method of Weinberger for a non-linear over-determined elliptic problem in R^2. We prove that the domain in consideration is a ball. The tool of this investigation are maximum principles and P-functions.

Key words and phrases: Maximum principle, Elliptic problems, Weinbeger's technique.

2000 *Mathematics Subject Classification.* 28D10, 35B05, 35B50, 35J25, 35J60, 35J65.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Maximum principles of E. Hopf [15, 16], first and second, for solutions of certain classes of second order elliptic problems have been known for a while. These tools in partial differential equations are investigated by Serrin [23] and Weinbeger [27]. In 1971, Serrin [23] assumed that u is a classical solution of $\Delta u = -1$ in Ω, $u = 0$ on the boundary $\partial\Omega$ where Ω is a bounded, regular domain of $\boldsymbol{R}^N$ simply connected. He furthermore supposed that the normal derivative of u is constant on the boundary $\partial\Omega$ and showed that the only configuration of the domain Ω is an N-ball. This popular technique which is called moving plane method works for the non-linear case as

$$a(u, |\boldsymbol{\nabla} u|)\Delta u + h(u, |\boldsymbol{\nabla} u|)u_{,i}u_{,j}u_{,ij} = f(u, |\boldsymbol{\nabla} u|),$$

where a, h and f are continuously differentiable functions in each variable.

For the particular case " the Saint-Venant problem" Weinberger [27] in a short note used an elementary argument based on maximum principles of E. Hopf [15, 16] and on a construction of some combination of u (called P-function) solution of the over-determined problem under consideration and its gradient $|\boldsymbol{\nabla} u|$. The advantage here, we have a weaker condition of regularity which is u of class C^1 only on $\bar{\Omega}$. Employing moreover Green's theorem to establish an auxiliary identity. Weinberger was able to prove that the combination constructed in terms of u and its gradient $|\boldsymbol{\nabla} u|$ is constant. To build an analogous function as in [27] require more attention for different reasons, in particular to get a similar integral identity seems with limited success.

The contribution made in this paper is to extend the argument used in [27] to more general setting, the non-linear case. Also we note that the extension that we look for is to consider non-linearity in dimension 2 as made in [27] for the linear case. Our technique require maximum principles of E. Hopf [15, 16] for non-linear second order elliptic equations and several applications of Green's theorem. In the same direction, symmetry results are also derived by A. Greco in [11, 12, 13, 14] using different technique. The device of his proof is based on a comparison with the radial case which is achieved by means of maximum principles. He investigates the question: which over-determined condition can force the domain to be a ball centered at a prescribed point? Results are provided for some quasi-linear elliptic equations involving the p Laplace as well as the minimal surface operator. In particular with a boundary condition depending on a radius r, he proved the following theorem where u is assumed to be a classical solution of the Saint-Venant problem, formulated by

$$\Delta u = -1 \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Theorem 1.1. *Let Ω be a bounded domain of class C^1 in $\boldsymbol{R}^N$, $N \geq 2$, containing the origin. Consider problem (1.1) over-determined by the condition*

$$-\frac{\partial u}{\partial n} = c|x| \quad \text{on } \partial\Omega, \tag{1.2}$$

where c is a positive constant. If problem $(1.1) - (1.2)$ has a solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ then Ω is a ball centered at 0.

For related work as existence, uniqueness and geometrical form of domains for a class of elliptic problems we refer the reader to [7, 10, 14, 17, 18, 19, 21, 22, 26]. To this end, we mention that two new ways to prove symmetry results in a class of elliptic problems involving constant and radial boundary conditions are the use of domain derivative which is classical tool of shape optimization, namely the differentiation with respect to the domain, due to M.Choulli and A.Henrot [6] and continuous Steiner symmetrization investigated in a brilliant series of F.

Brock [1, 2, 3, 4, 5]. As a variant of Serrin's technique, F. Brock defined a new kind of symmetry " Local symmetry" for more details see [1].

2. MAIN RESULT

In this section, we consider the non-linear case for the Saint-Venant problem. We assume that u is a classical solution of

$$\Delta u + f(u) = 0 \quad \text{in } \Omega, \qquad u = 0 \text{ on } \quad \partial\Omega, \tag{2.1}$$

where f is a positive function, non-increasing. The domain Ω is supposed to be bounded, regular and simply connected in $\boldsymbol{R}^2$. The statement of the result that we will prove is formulated in the following theorem

Theorem 2.1. *Let u be a classical solution of (2.1), at least of class C^2 on Ω and of class C^1 on $\bar{\Omega}$. Moreover if we assume that u satisfies the extra-condition*

$$-\frac{\partial u}{\partial n} = c \text{ on } \partial\Omega, \tag{2.2}$$

then Ω is a ball and u is radially symmetric provided that:

$$f \,>\, 0, \quad f' \,\leq\, 0,$$

and

$$f(u) \,\geq\, \frac{c^2\,+\,1}{u}.$$

As noted before, keeping constant sign for the non-linear function f was a necessary condition in proving Serrin's theorem [23] for a class of elliptic over-determined problems, while in Weinberger's proof it was not. It seems to us as we did in this note, that to do not assume that f is increasing or decreasing leads the general result fails to be true. This is due to some difficulties arising in performing some convenient elliptic differential inequality. Without monotonicity of the function f, difficulties are usually met even if we do not use Serrin's argument. In other words, we need some sufficient information on the given non-linear function in order to lead the maximum principles of E. Hopf applied. We split the proof of Theorem 2.1 in some lemmas. The first result stated in Lemma 2.1 below express an auxiliary equality that we need to use jointly with maximum principles of E. Hopf [15, 16] and the second result provide an elliptic differential inequality which is a crucial tool of the maximum principles. The identity arising in Lemma 2.1 is somewhat a kind of Rellich's type identity.

Lemma 2.1. *Let u be a classical solution of (2.3) then the following integral equality is satisfied*

$$2\int_\Omega F\, d\mathbf{x} = c^2 V, \tag{2.3}$$

where V denotes the volume $|\Omega|$ of the domain Ω and F satisfies

$$F(u) \,:=\, \int_0^u f(s)\, d\mathbf{s}. \tag{2.4}$$

For the proof of Lemma 2.1, recall that u has constant sign if the function f is non-increasing (resp. non-decreasing). Whereas this fact was necessary in the Serrin'proof [23], in the Weinberger's proof this condition is dropped. Moreover as we will see here this investigation leads the method used in [27] works as well as that of [23] for the general case. To start, let us make use of the following observation

$$\Delta\,(r\frac{\partial u}{\partial r}) = r\frac{\partial \Delta\,u}{\partial r} + 2\Delta\,u = -r\frac{\partial f}{\partial r} - 2f, \tag{2.5}$$

upon the governing equation (2.1) is inserted and r denotes the distance from a fixed origin. We remark that the inserted function $r\frac{\partial u}{\partial r}$ in (2.7) is super-harmonic when u is a solution of the Saint-Venant problem (1.1), (2.2). Based on this observation, it was proved in [18] that the domain Ω is an N-ball without using the Technique of moving plane. The starting proof of that of Payne and Schaefer [18] is to transform the given over-determined problem to the new one written in an equivalent integral form where Green's and Rellich's identities are the essence of this method.
Now multiplying (2.5) by $-u$ and using Green's theorem, we get

$$\begin{aligned}\int_\Omega (-u\Delta\,(r\frac{\partial u}{\partial r}) + r\frac{\partial u}{\partial r}\Delta\,u)\,\mathrm{d}\mathbf{x} &= \int_\Omega (ru\frac{\partial f}{\partial r} + 2uf - rf\frac{\partial u}{\partial r}))\,\mathrm{d}\mathbf{x} \\ &= \int_\Omega (r\frac{\partial uf}{\partial r} + 2uf - 2rf\frac{\partial u}{\partial r})\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{2.6}$$

It is easy to see that equality (2.6) can be rewritten explicitly as

$$\int_\Omega (-u\Delta\,(r\frac{\partial u}{\partial r}) + r\frac{\partial u}{\partial r}\Delta\,u)\,\mathrm{d}\mathbf{x} = \int_\Omega (r\frac{\partial uf}{\partial r} + 2uf)\mathrm{d}\mathbf{x} - 2\int_\Omega \boldsymbol{\nabla}\,\frac{r^2}{2}\boldsymbol{\nabla}\,F\,\mathrm{d}\mathbf{x}. \tag{2.7}$$

Next we express the last term in (2.5) from the right in light of classical formula of Green, we obtain

$$\begin{aligned}\int_\Omega r\frac{\partial uf}{\partial r}\,\mathrm{d}\mathbf{x} &= \int_\Omega \boldsymbol{\nabla}\,(\frac{r^2}{2})\boldsymbol{\nabla}\,(uf)\,\mathrm{d}\mathbf{x} \\ &= -2\int_\Omega uf\,\mathrm{d}\mathbf{x},\end{aligned} \tag{2.8}$$

while for the same argument in view of (2.1) and (2.2) the integrals equalities in (2.6) take the form

$$\begin{aligned}\int_\Omega (-u\Delta\,(r\frac{\partial u}{\partial r}) + r\frac{\partial\,u}{\partial r}\Delta\,u)\,\mathrm{d}\mathbf{x} &= \int_{\partial\Omega} (-u\frac{\partial}{\partial n}(r\frac{\partial u}{\partial r}) + r\frac{\partial u}{\partial r}\frac{\partial u}{\partial n})\,\mathrm{d}\mathbf{s} \\ &= \int_{\partial\Omega} r\frac{\partial r}{\partial n}(\frac{\partial u}{\partial n})^2\,\mathrm{d}\mathbf{s} \\ &= 2c^2V.\end{aligned} \tag{2.9}$$

Thus we conclude that

$$2\int_\Omega F\,\mathrm{d}\mathbf{x} = c^2V. \tag{2.10}$$

The second step that we need in proving Theorem 2.1 relies heavily on the use of maximum principles of E. Hopf [15, 16]. For this, we construct a new function Φ which is combination of

u and its gradient $\boldsymbol{\nabla}\, u$ by setting

$$\Phi \;:= |\boldsymbol{\nabla}\, u|^2 + F(u). \tag{2.11}$$

Hereafter in order to prove that the combination Φ in (2.11) is constant, we need an auxiliary result formulated in the following lemma.

Lemma 2.2. *Let Φ be defined in (2.11), then Φ satisfies the following elliptic differential inequality*

$$\Phi_{,kk} \;\geq\; 0, \tag{2.12}$$

provided that the functions f and its derivatives are subject to

$$f \;>\; 0, \quad f' \;\leq\; 0. \tag{2.13}$$

For the proof of Lemma 2.2, let us compute

$$\Phi_{,k} = 2u_{,ki}u_{,i} + u_{,k}f, \tag{2.14}$$

$$\begin{aligned} \Phi_{,kk} &= 2u_{,ki}u_{,ki} + 2u_{,i}\Delta\, u_{,i} \\ &+ u_{,kk}f + u_{,k}u_{,k}f', \end{aligned} \tag{2.15}$$

(2.16)

or equivalently, we have

$$\Phi_{,kk} = 2u_{,ki}u_{,ki} - f'u_{,k}u_{,k} - f^2. \tag{2.17}$$

Hence in view of (2.15) and since $2u_{,ki}u_{,ki} - (\Delta\, u)^2 \;\geq\; 0$, (see Sperb's book [24]), we deduce that

$$\Phi_{,kk} \;>\; 0 \text{ in } \Omega. \tag{2.18}$$

Consequently, applying the maximum principle for a class of elliptic equations [15, 16, 24] we conclude that the combination Φ defined in (2.11) attains its maximum value on the boundary $\partial\Omega$ of Ω unless Φ is constant. This result may be formulated as follows

$$\Phi \;<\; c^2 \text{ in } \Omega, \tag{2.19}$$

or

$$\Phi \;=\; c^2 \text{ in } \Omega. \tag{2.20}$$

If the first alternative (2.19) occurred, we get

$$\int_\Omega \Phi \,\mathrm{d}\mathbf{x} \;<\; c^2 V, \tag{2.21}$$

where V denotes the volume of the domain Ω. Again classical formula of Green and (2.3) yield

$$\int_\Omega \Phi \,\mathrm{d}\mathbf{x} \;=\; \int_\Omega u f(u)\,\mathrm{d}\mathbf{x} + \int_\Omega F(u)\,\mathrm{d}\mathbf{x}. \tag{2.22}$$

From (2.12) we handle the first term in (2.22) from the right as follows

$$\int_\Omega F(u)\,\mathrm{d}\mathbf{x} = \frac{1}{2}c^2 V. \tag{2.23}$$

Combining $(2.21) - (2.23)$ together, we are conducted to

$$\begin{aligned} \int_\Omega \Phi \,\mathrm{d}\mathbf{x} \;&=\; \int_\Omega u\, f(u)\,\mathrm{d}\mathbf{x} + \frac{1}{2}c^2 V \\ &<\; c^2 V. \end{aligned} \tag{2.24}$$

The argue by contradiction will be completed if we assume that

$$f(u) \geq \frac{c^2 + 1}{u}, \text{ for } u \text{ positive solutions of } (2.3) - (2.4). \tag{2.25}$$

In fact we use (2.23), (2.24) and (2.25) in order to get

$$\begin{aligned} 2\int_\Omega u\, f(u)\,\mathbf{dx} + 2\int_\Omega F(u)\,\mathbf{dx} &= 2\int_\Omega u\, f(u)\,\mathbf{dx} + c^2 V \\ &= 2\int_\Omega \Phi\,\mathbf{dx} \\ &< 2c^2\, V. \end{aligned} \tag{2.26}$$

By our assumptions (2.25) on f and its derivatives (2.13) we observe easily that we are conducted to a contradiction. This is due to the positivity of u, f, F and the fact that $F \geq u\, f$ since f is non-increasing. Indeed the differential inequality (2.26) is reduced to

$$4(c^2 + 1)\, V < 4\int_\Omega F(u)\ \mathrm{d\,x} < 2\, c^2\, V.$$

Thus, the combination Φ is necessarily constant in the domain Ω. Similarly to the final part of Weinberger [27], to determine the geometric nature of the domain Ω we employ a result of Spivack [25]. Namely, we show that the mean curvature of the boundary $\partial\Omega$ is constant. To do this, we make successive partial differentiation of the combination Φ. In fact, we have

$$\Phi := |\boldsymbol{\nabla}\, u|^2 + F(u) = \text{const. in } \Omega. \tag{2.27}$$

From (2.29), we express the normal derivative of the combination Φ on the boundary $\partial\Omega$ as follows

$$2(u_n u_{nn} + \frac{1}{2} u_n f) = 0, \tag{2.28}$$

where u_n and u_{nn} denote respectively the normal and second derivatives of u. Now since the boundary $\partial\Omega$ is sufficiently smooth, the differential equation (2.1) takes the form

$$u_{nn} + K u_n + f(0) = 0. \tag{2.29}$$

Combining (2.28) and (2.29), we obtain

$$K = \frac{f(0)}{2c} \tag{2.30}$$

which in view of [25] implies that Ω is a ball and u is radially symmetric. To this end we give an example in order to illustrate Theorem 2.1.

Example 2.1. Let f be a function defined by: $f(u) := 18\,\frac{\ln(u + c^2 + 2)}{(u + c^2 + 2)}$, positive for $u > e - c^2 + 2$, where
$e = 2.718....$ and let u be a classical solution of the following over-determined elliptic problem

$$\Delta\, u + 18\,\frac{\ln(u + c^2 + 2)}{(u + c^2 + 2)} = 0 \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{2.31}$$

where Ω is a bounded, regular, simply connected domain of $\boldsymbol{R}^2$, and

$$-\frac{\partial u}{\partial n} = c \text{ where, } c := \frac{\int_\Omega f(u)dx}{|\partial\Omega|}. \tag{2.32}$$

Then f is a non-increasing function satisfying $u\,f(u) \geq 3$. Indeed, we consider $g(u) := u\,f(u) = 18\,u\,\frac{\ln(u+c^2+2)}{(u+c^2+2)}$, and therefore $g'(u) = 18\,\frac{u+\ln(u+c^2+2)}{(u+c^2+2)^2}$ which is positive for $u > e - c^2 + 2 > 0$. Then we deduce that $g(u) > g(e-1) = \frac{9}{e} > 2$. Now applying Theorem 2.1, we conclude that Ω is a ball and u is radially symmetric.

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