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GENERALIZING POLYHEDRA TO INFINITE DIMENSION

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ABSTRACT. This paper generalizes polyhedra to infinite dimensional separable Hilbert spaces as countable intersections of closed semispaces. We show that a polyhedron is the sum of convex proper subset, which is compact in the product topology, plus a closed pointed cone plus a closed subspace. In the final part the dual range space technique is extended to solve infinite dimensional LP problems.

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1. INTRODUCTION

The purpose of this paper is to lay down a significant part of the theory of polyhedra in infinite dimensions in separable Hilbert space environment. Naturally there are profound differences with the finite dimensional case, but also, the theory is in some way more natural. It allows for countable intersections and includes more general classes of closed convex sets. Other important differences derive from the fact that the positive cone of the order has a void interior.

To obtain void interior compact polyhedra, a sort of infinite dimensional polytopes, we have to weaken the underlying topology beyond the weak topology and use the (relativized) product topology. This requires some preliminary technical work.

In the central part of the paper we use the range space approach to develop a theory of decomposition of polyhedra. Incidentally, to pull back topologically the results to the domain space it is needed that the operator defining the polyhedron has closed range. This excludes convex bodies like the unit ball (which is a polyhedron in our setting), because the positive cone of the order has void interior.

We prove that in the range space a non-void polyhedron (indeed the slack polyhedron) always has extreme points, and is the sum of a convex closed proper subset, which is compact in the product topology and contains all the extreme points of the polyhedron, plus (in the cases we will specify) a pointed cone, which is always the intersection of the range of the operator defining the inequality system intersected with the positive cone of the range space. The pull-back in the domain spaces adds to the inverse image of those components a closed linear subspace which is always the null space of the operator defining the inequality system

In the final section we give some basic results on LP, based on the dual range space conditions (in our setting dual means: involving polar cones). Naturally, even in the cases where there are suprema, we cannot take for granted, as in finite dimensions, that they are attained. However, some conditions for this to happen are provided.

Finite dimensional concepts along similar lines, are developed in [3], see also [5], [6] and [4]. However, the infinite dimensional theory, presented here, is selfcontained.

1.1. Notations.

$\mathfrak{N}$ is the set of positive integers

For $y \in l_2$

$$ip(y) = \{i : y_i > 0\}$$

$$iz(y) = \{i : y_i = 0\}$$

$$in(y) = \{i : y_i < 0\}$$

e^i is the i-th versor in l_2.

Cone means convex cone.

H is a real Hilbert space.

B^r is the closed sphere of radius r.

$\mathcal{C}(A)$ and $\mathcal{C}^-(A)$ are, respectively, the convex and closed convex hull of A.

$\mathcal{C}o(A)$ and $\mathcal{C}o^-(A)$ are, respectively, the convex and closed conical hull of A.

$\mathcal{C}o(\{x\}) = r(x)$ is the ray generated by x. The set $z + r(x)$ is also called ray.

$\mathcal{B}(A)$ is the boundary of A.

Operator means continuous linear transformation with respect to the specified topologies

iff=if and only if.

$ex(C)$ = the set of extreme points of C.

$x_F = P_F x$ = orthogonal projection of x on the closed convex set F

Primal in the present context means no polarization of cones is involved. Dual means that a condition is obtained involving some polar cones.

2. PRELIMINARIES

The lineality space $lin(C)$ of a cone C in H is the linear subspace $C \cap -C$ (the maximal linear subspace contained in C). A cone C is said to be pointed if $lin(C) = \{0\}$. If a cone is closed, then, obviously, its lineality space is closed as well.

A first technical lemma is:

Lemma 2.1. *A closed cone in a Hilbert space H is proper if and only if it is contained in a closed half-space.*

Proof. Let C be a closed proper cone. Then there is a singleton $\{y\}$ disjoint from C. Singletons are convex and compact and therefore the Strong Separation Corollary 14.4 in [11] applies. The rest is immediate. ∎

The well-known decomposition of non-pointed cones [14] extends easily in Hilbert spaces. We restate it in our context:

Proposition 2.2. *Let C be a non-pointed closed cone and F its lineality space then:*

$$C = F + P_{F^\perp}C = F + (C \cap F^\perp)$$

where the cone $C \cap F^\perp$ is pointed and closed. If C is not closed but its lineality space is closed the same expression holds and the cone $C \cap F^\perp$ is pointed.

Next we recall the definition of polar cone of a set:

Definition 2.1. Given an arbitrary subset S of H, the *polar cone* of the set, denoted by S^p is given by:

$$S^p = \{n : (n, y) \leq \ 0, \forall y \in S\}$$

Note that S^p *is a closed* cone. This is a consequence of continuity of the inner product.

Elementary computations on polarization are similar to their finite dimensional counterpart (see e.g.[14]). Here are a few , which we mostly state without proof.

Proposition 2.3. *The following formulas hold for polarization of cones in Hilbert spaces. All sets in the formulas are cones.*

$$\textit{if } C_1 \subset C_2 \textit{ then } C_1^p \supset C_2^p$$

$$C^{-p} = C^p$$

$$C^{pp} = C^-$$

$$(C_1 + C_2)^p = C_1^p \cap C_2^p$$

Let T be an operator $H \to H$ and C a cone in H. Then

$$(TC)^p = T^{*-1}C^p$$

Let $\{g_\alpha : \alpha \in \mathcal{A}\}$ be an arbitrary family of vectors in H. Then

$$\mathcal{C}o(\{g_\alpha\})^p = \cap\{x : (g_\alpha, x) \leq 0\}$$

and

$$(\cap\{x : (g_\alpha, x) \leq 0\})^p = \mathcal{C}o^-(\{g_\alpha\})$$

Proof. Polar of TC:

$$(TC)^p = \{y : (y, Tx) \leq 0, x \in C\} =$$
$$\{y : (T^* y, x) \leq 0, x \in C\} = T^{*-1}C^p$$

Last statement: Take $\beta \in \mathcal{A}$, $\mathcal{C}o(\{g_\beta\}) \subset \mathcal{C}o(\{g_\alpha\})$ implies

$$\{x : (g_\beta, x) \leq 0\} = (\mathcal{C}o(\{g_\beta\}))^p \supset (\mathcal{C}o(\{g_\alpha\}))^p$$

and so:

$$(\mathcal{C}o(\{g_\alpha\}))^p \subset \cap\{x : (g_\alpha, x) \leq 0\}$$

because the reverse inclusion is obvious, the first formula is proved. The second follows taking polars on both sides and applying the elementary computations. ∎

Theorem 2.4. *Let C be a closed cone. Then $C+ C^p = H$. Moreover $\forall x$, $\exists! x_C \in C$ and $\exists! x_{C^p} \in C^p$ with $x_C \perp x_{C^p}$, such that $x = x_C + x_{C^p}$. Moreover, $x_C = P_C(x)$ and $x_{C^p} = P_{C^P}(x)$.*

Proof. If $x = 0$, it can only be decomposed as $x = 0 + 0$, by the orthogonality constraint. Thus in what follows $x \neq 0$. If $P_C(x) = 0$, then $x \in C^p$ and can be decomposed as $x = 0 + x$. If we attempt another decomposition $x = w + (x - w)$, with $w \in C$, then, by orthogonality:

$$(w, x - w) = (w, x) - \|w\|^2 = 0$$

and because $(w, x) \leq 0$ it follows $w = 0$. A similar argument applies if $x \in C$. At this point we can assume that $x \notin C \cup C^p$. Thus , $P_C(x) \neq 0$. If we write:

$$x = P_C(x) + (x - P_C(x))$$

by the preceding Proposition, $((x - P_C(x)), P_C(x)) = 0$, thus $x - P_C(x) \in C^p$ and the decomposition is orthogonal. Next we claim that the projection of x onto C^p is $x - P_C(x)$. In fact for $z \in C^p$,

$$(x - (x - P_C(x)), z - (x - P_C(x))) = (P_C(x), z) - (P_C(x), (x - P_C(x)))$$

Now the first term is ≤ 0 by the definition of polar cone and the second is zero by the Lemma. Thus in view of the Projection Theorem our claim is proved. It remains to show that there is no other orthogonal decomposition. Suppose that there is another decomposition in addition to this one

$$x = [P_C(x)] + [x - P_C(x)]$$

. Clearly we can write the new decomposition in the form

$$x = [P_C(x) + w] + [(x - P_C(x)) + z]$$

with $[P_C(x) + w] \in C$ and $[(x - P_C(x)) + z] \in C^p$. Subtracting these two relations, $w + z = 0$. We are assuming that the new decomposition is orthogonal so that, taking the inner product of the two components and bearing in mind that the former decomposition was orthogonal

$$(w, (x - P_C(x))) + (w, z) + (P_C(x), z) = 0$$

On the other hand, by polarity, $((w+ P_C(x)), (x - P_C(x))) = (w, (x - P_C(x))) \leq 0$. Similarly $([(x - P_C(x)) + z], P_C(x)) = (P_C(x), z) \leq 0$ and $w + z = 0$ implies $(w, z) \leq 0$, because $(w, z) = -1\|w\|^2 = -1\|z\|^2$. Hence all terms in the above sum must be zero. In particular, $0 = (w, z) = -1\|w\|^2 = -1\|z\|^2$. It follows $w = z = 0$ and the proof is finished. ∎

In Hilbert spaces the Induced Map Theorem can be put in a more useful form. We start with the following:

Proposition 2.5. *Assume that F be a closed subspace of the Hilbert space H.Then H/F is linearly and topologically isomorphic to $F^\perp$(with the relative topology).*

Proof. Incidentally, recall that the quotient topology is Hausdorff because F is closed. Define the map:

$$S : H/F \to F^{\perp} \text{ by: } S(x+F) = x_{F^{\perp}}$$

Because $\forall x$, $x + F = x_{F^{\perp}} + F$, and because for two vectors z and w in $F^{\perp}$:

$$z + F = w + F \Rightarrow z = w$$

it follows that S is linear one to one and onto. Moreover,

$$S \circ Q = P_{_{F^{\perp}}}$$

and thus, by the Induced Map Theorem, S is continuous. On the other hand:

$$S^{-1} = Q \circ P_{_{F^{\perp}}}$$

and so $S^{-1}$is continuous too. Thus S is a linear topological isomorphism and we are done. ∎

At this point we use the Open mapping Theorem:

Theorem 2.6. *(Open Mapping Theorem) . Consider a continuos map T: $H_1 \to H_2$. Then $\mathcal{R}(T)$ is closed if and only T is an open mapping.*

Proof. Assume $\mathcal{R}(T)$ closed. Call I_q the topological isomorphism $H_1/\mathcal{N}(T) \to \mathcal{N}(T)^{\perp}$ of Proposition 2.5. Then clearly

$$T = T|_{\mathcal{N}(T)^{\perp}} I_q Q$$

Notice that $T|_{\mathcal{N}(T)^{\perp}} I_q$ is, by the preceding proposition, continuous and one to one and onto the Hilbert space $\mathcal{R}(T)$. By the closed graph Theorem its graph is closed and therefore the graph of its inverse is closed. Hence the inverse is a continuous operator on the Hilbert space $\mathcal{R}(T)$. It follows that $T|_{F^{\perp}} I_q$ is an open map. But, as we know, the quotient map is always open and hence T is open too. Conversely if T is open, write according to the Induced Map Theorem $T = \widetilde{T} \circ Q$. By the same Theorem $\widetilde{T}$ is a topological isomorphism and hence $\mathcal{R}(T)$ is closed. ∎

Note that, obviously $I_q Q = P|_{\mathcal{N}(T)^{\perp}}$ and thus we can state the corollary:

Corollary 2.7. *Consider a continuos map T: $H_1 \to H_2$. Then we can write:*

$$T = T|_{\mathcal{N}(T)^{\perp}} P|_{\mathcal{N}(T)^{\perp}}$$

and the map $T|_{\mathcal{N}(T)^{\perp}}$is a linear topological isomorphism if and only if $\mathcal{R}(T)$ is closed

3. THE PRODUCT TOPOLOGY

In our analysis of polyhedra we will assume H separable, and so we will work directly in l_2, which is isometric to H. Every operator in l_2 is represented by an infinite matrix. When we will need to establish the converse, a simple sufficient condition will be enough. We note in passing that l_2 is not a closed subset of R^{∞}.

In l_2 we need to consider three topologies: the native (strong) topology $\mathcal{S}$, the weak topology $\mathcal{W}$, and the relativized product topology of R^{∞}, $\mathcal{X}$. Note that:

$$\mathcal{X} \subset \mathcal{W} \subset \mathcal{S}$$

If we consider a linear transformation $l_2 \to l_2$, it is well known that $\mathcal{S} - \mathcal{S}$ continuity $\Leftrightarrow$ $\mathcal{W} - \mathcal{W}$ continuity (e.g. [8]). Thus, assuming, of $\mathcal{S} - \mathcal{S}$ continuity all the following types of continuity are implied:

$$\mathcal{S} - \mathcal{W}; \mathcal{S} - \mathcal{X}; \mathcal{W} - \mathcal{W}; \mathcal{W} - \mathcal{X}$$

We will use the same symbol for a given topology and the same topology relativized to a subspace. The space $(R^{\infty}, \mathcal{X})$ is locally convex, but not normable.

The main motivation for weakening (beyond the weak) the original topology is to facilitate compactness. In this respect the weak topology is not enough, as we shall see.

If we consider a (matrix) operator $G\colon (l_2, \mathcal{S}) \to (l_2, \mathcal{S})$, since the adjoint exists, it is obviously necessary that both its rows and columns are in l_2. If we assume that the rows only are in l_2 then G is an operator (is continuous) $(l_2, \mathcal{S}) \to (R^\infty, \mathcal{X})$ (because the composition with each projection on a factor is continuous).

The dual space of R^∞ is the direct sum of duals of the factors. This means that a vector g is in this dual if and only if it has finitely many non-zero components (and so also $g \in l_2$).

We next provide a few technical Lemmas involving the $\mathcal{X}$ topology for l_2.

In connection with the subsequent discussion bear in mind that, as recalled in the preliminaries, in a Hilbert space a closed range operator is necessarily an open map.

Lemma 3.1. *Suppose that G is a $(l_2, \mathcal{S}) \to (l_2, \mathcal{S})$ operator. Then G is open relative to the $\mathcal{X}$ topology in its domain and any vector topology in the space it maps into.*

Proof. The open sets in domain are cylinder set with finite dimensional open base. Thus we may represent and open set as the sum $B + L$, where B is an open set in a finite dimensional coordinate space F and and $L = F^\perp$ (also a coordinate space).Thus:

$$G(B + L) = G|_F(B) + G(L) =$$

$$\cup\{y + G|_F(B) : y \in G(L)\}$$

and this latter set is an union of open sets, because $G|_F$ is a finite dimensional range transformation and hence open. ∎

Corollary 3.2. *A topological isomorphism T $(l_2, \mathcal{S}) \to (l_2, \mathcal{S})$ is a topological isomorphism $(l_2, \mathcal{X}) \to (l_2, \mathcal{X})$. If F and Γ are closed subspaces of l_2, a topological isomorphism $(F, \mathcal{S}) \to (\Gamma, \mathcal{S})$ is a topological isomorphism $(F, \mathcal{X}) \to (\Gamma, \mathcal{X})$.*

Proof. We observe that as a map T is invertible and both T and its inverse map are continuous in the strong topology and hence open in the $\mathcal{X}$ topology. As to the second part it suffices to remind that closed subspaces are isometric to l_2 (or R^n according to the cases) via a the choice of an orthonormal base. ∎

A closed subspace in the $\mathcal{X}$ topology for l_2 is also closed in the $\mathcal{S}$ topology. The converse is true: all $\mathcal{S}$ - closed subspaces are $\mathcal{X}$ - closed.

Lemma 3.3. *A closed subspace of $(l_2, \mathcal{S})$ is also closed in the $\mathcal{X}$ topology. Thus $(l_2, \mathcal{S})^* = (l_2, \mathcal{X})^*$. Consequently any closed semispace in $(l_2, \mathcal{S})$ is also closed in the $\mathcal{X}$ topology.*

Proof. Note that any closed subspace can be represented as the null space of a projector which a self adjoint operator. By a change of base and invoking the Toepliz Theorem [8] the matrix of the operator is transformed (by an unitary equivalence) in a self-adjoint matrix with both finite rows and columns. Thus in the new l_2 space the subspace is the intersection of subspaces closed in the product topology. But we have just proved that a topological isomorphism $(l_2, \mathcal{S}) \to (l_2, \mathcal{S})$ is a topological isomorphism $(l_2, \mathcal{X}) \to (l_2, \mathcal{X})$. Thus the original subspace is also closed in the $\mathcal{X}$ topology. ∎

Theorem 3.4. *Suppose that G is a $(l_2, \mathcal{S}) \to (l_2, \mathcal{S})$ operator with closed range. Then we can write:*

$$G = G|_{\mathcal{N}(G)^\perp} P_{\mathcal{N}(G)^\perp}$$

and $G|_{\mathcal{N}(G)^\perp}$, is not only a linear isomorphism $(\mathcal{N}(G)^\perp, \mathcal{S}) \to (\mathcal{R}(G), \mathcal{S})$, but also a linear isomorphism $(\mathcal{N}(G)^\perp, \mathcal{X}) \to (\mathcal{R}(G), \mathcal{X})$.

Proof. We know that $G|_{\mathcal{N}(G)^{\perp}}$ is a linear isomorphism with respect to the native topology and, by the preceding Corollary, it is also a linear isomorphism $(\mathcal{N}(G)^{\perp},\mathcal{X}) \to (\mathcal{R}(G),\mathcal{X})$. ∎

4. POLYHEDRA AND POLYHEDRAL CONES AND THEIR GENERALITY

In R^n a polyhedron is a finite intersection of closed semispaces, that is, a set of the form:

$$\cap\{x : (g^i, x) \le v_i, i = 1, .., n\} = \{x : Gx \le v\}$$

where G is a real matrix formed by the rows g^i, $v \in R^n$ has components v_i and the order $\le$ is the product order (and the corresponding positive cone is the non-negative orthant).

We generalize this to a real separable Hilbert spaces in the natural way, substituting countable intersections to finite intersections. Not only this is natural but it is also indispensable if, for example, want to make the positive cone itself a polyhedron. For simplicity we may work directly in l_2. In the l_2 setting we can consider the natural versors $\{e^i\}$ base and the consequent matrix representation of $(l_2,\mathcal{S}) \to (l_2,\mathcal{S})$ operators.

Definition 4.1. A polyhedron $\mathcal{G}$ (in l_2) is a countable intersection of closed semispaces:

$$\mathcal{G} = \cap\,\{x : (g^i, x) \le v_i, i = 1, 2, ...\}$$

where $g^i \in l_2$ and $g^i \ne 0$, $\forall i$. If $v_i = 0$, $\forall i$ then $\mathcal{G}$ is a cone and is called polyhedral cone.

Without restriction of generality we can divide each inequality by $\|g^i\|$, and hence we can assume, whenever convenient, that $\|g^i\| = 1$.

We may rewrite the set as:

$$\mathcal{G} = \{x : Gx \le v\}$$

where G : is an infinite matrix whose rows are the g^i and $v \in R^\infty$ has components v_i. Note that G is continuous $(l_2,\mathcal{S}) \to (R^\infty,\mathcal{X})$ (it is not in general an operator $(l_2,\mathcal{S}) \to (l_2,\mathcal{S})$). The order $\le$ denotes the product vector ordering in R^∞, and v is called the bound vector.

Remark 4.1. Note that in view of the structure of the dual space of R^∞, polyhedra defined relative to the $\mathcal{X}$ topology for l_2 are also polyhedra in l_2.

A noteworthy fact about polyhedra is that any closed convex subsets in a separable Hilbert space is a polyhedron and any closed convex cone is a polyhedral cone.

Theorem 4.1. *Consider a non-void strongly closed convex subset $C \ne H$ in a separable Hilbert space H and let D be a countable dense subset of H. Then $\forall \zeta \in \mathcal{B}(C)$ there exists a sequence of support points $\{z_i\}$ such that $\{z_i\} \to \zeta$ strongly where $\{z_i\}$ is in the same countable set $P_C(D\backslash C)$. Thus there exists a countable set of support points dense in $\mathcal{B}(C)$. Moreover, the countable intersection of supporting semispaces defined by the points of the countable set $P_C(D\backslash C)$ and the corresponding normals (i.e. if $y \in D\backslash C$ the normal is $y - P_C y$) coincides with C. Thus any non-void closed convex set C is a polyhedron and any non-void closed cone is a polyhedral cone. In particular, any strongly closed convex set in l_2 is also $\mathcal{X}$-closed and hence, also, the closed convex extension in all three topologies ($\mathcal{X}$, $\mathcal{W}$, and $\mathcal{S}$) coincide.*

Proof. Observe that, because $\overline{C}$ is open and consider an open sphere $S^{\zeta}_{1/i}$ (of radius $1/i$) around ζ. This sphere will contain the open set $\overline{C} \cap S^{\zeta}_{1/i}$ and so take a point z in this intersection. If this is not already in D there is another sphere S^z_r entirely contained in $\overline{C} \cap S^{\zeta}_{1/i}$, which will intersect D. Take y_i in this intersection. In this way we define a sequence $\{y_i\}$ which converges to ζ, and, by continuity of projection, the sequence of support points $\{z_i\} = \{P_C(y_i)\}$ converges to ζ. This proves the first part. As to the second statement assume that there is a vector z in the given intersection such that $z \notin C$. Because $z \in \overline{C}$, arguing in a similar way, we can take $\{z_i\} \to z$, with $\{z_i\}$ in $\overline{C} \cap D$. By continuity of P_C, $\{P_C(z_i)\} \to P_C(z)$. Define $n_i = (z_i - P_C(z_i))$, and notice that:

$$(n_i, z - P_C(z_i)) = (n_i, z - z_i) + (n_i, z_i - P_C(z_i))$$

Next the second term on the rhs goes to $\|z - P_C(z)\|^2 = \delta > 0$ while the first term on the rhs goes to zero. Thus, for sufficiently high i,

$$(n_i, z) > (n_i, P_C(z_i))$$

It follows that z is outside the supporting semispace corresponding to z_i. This contradiction completes the proof for polyhedra. The remaining part of the proof is straightforward and is omitted. ∎

This Theorem has an important consequence, under the restriction of considering void interior sets. This is less unusual than it may appear at first sight, considering the well known fact that the positive cone in l_2 has void interior even in the strong topology (as will be recalled in the next Section).

Theorem 4.2. *Suppose C is a a strongly closed convex subset of H, which has a void $\mathcal{X}$-interior. Then C is the $\mathcal{X}$-closure (or equivalently $\mathcal{S}$-closure) of the countable subset of its points $P_C(D \backslash C)$.*

Proof. By the preceding Theorem, C convex and strongly closed implies that C is $\mathcal{X}$-closed. Thus, with obvious meaning of symbols, $\mathcal{B}_{\mathcal{X}}(C) = C$. By the preceding Theorem the countable subset of C given by $P_C(D \backslash C)$ is dense in C in the strong topology and hence also in the $\mathcal{X}$ topology. From this the thesis immediately follows. ∎

Proposition 4.3. *Finite and countable intersections of polyhedra are polyhedra. Closed subspaces are polyhedra*

Proof. For finite intersections the proof is based on block matrix operators. In the case of countable intersection we have a countable set of block each of which has a countable number of rows. This results in a matrix with a countable set of rows. Next, expressing a closed subspace F as the kernel of the projection on the orthogonal complement we obtain:

$$F = \{x : P_{F^\perp} x = 0\} =$$

$$\{x : \begin{pmatrix} P_{F^\perp} \\ -P_{F^\perp} \end{pmatrix} x \leq 0\}$$

∎

At this point we introduce a useful technical Lemma:

Lemma 4.4. *The translate of a polyhedron is a polyhedron. In particular if we translate a polyhedron $\mathcal{G}$ by the opposite one of its points $-t$ (so that $0 \in \mathcal{G} - t$) then, in the representation of $\mathcal{G} - t$, the bound vector is non-negative.*

Proof. Just note that

$$-t + \mathcal{G}(G, v) = \mathcal{G}(G, v - Gt)$$

∎

The following Theorem shows that a large subclass of polyhedra can be represented working completely within l_2, in the sense that we can take $v \in l_2$ and G a (matrix) operator $(l_2, \mathcal{S}) \to (l_2, \mathcal{S})$.

By what we proved earlier the closed sphere B^r of radius r around the origin in l_2 is a polyhedron. However, in the next proof it will be convenient to use the obvious alternative representation:

$$B^r = \{x : (g, x) \leq r, \|g\| \leq 1\}$$

Theorem 4.5. *The class of polyhedra defined by:*

$$\mathcal{G} = \{x : Gx \leq v\}$$

where $v \in l_2$, G a (matrix) operator $l_2 \to l_2$, and $\leq$ is the product vector ordering restricted to l_2, includes polyhedral cones, closed subspaces and bounded, and hence weakly compact, polyhedra.

Proof. First we prove the statement for polyhedral cones. Thus consider a polyhedron of the form: $\cap\{y : (g^i, y) \leq 0\}$ and an arbitrary vector $z \in l_2$ with $z_i > 0$. Then:

$$\cap\{y : (g^i, y) \leq 0\} = \cap\{y : (z_i g^i, y) \leq 0\}$$

For simplicity use the same symbol g^i to denote the new vectors $z_i g^i$, and form the infinite matrix G whose rows are the new vectors g^i. Because $\sum_{ij} g_{ij}^2 = \|z\|^2 < \infty$, and this is a sufficient condition for G to represent a continuous linear operator (see [8]), we are done. Next consider a closed subspace F. Then $P_{F^\perp}$ is a continuous operator and hence can be represented by a matrix operator. But:

$$F = \{x : \begin{pmatrix} P_{F^\perp} \\ -P_{F^\perp} \end{pmatrix} x \leq 0\}$$

and since the block matrix represent a continuous linear operator, the second statement is proved too. Finally, let $\mathcal{G}$ be a weakly compact closed convex set and, without restriction of generality suppose that $0 \in \mathcal{G}$. Then, for some $r > 0$:

$$\mathcal{G} = \cap \{x : (g^i, x) \leq v_i\} \subset B^r$$

Observe that the closed sphere B^r of radius r around the origin in l_2 can be represented as:

$$B^r = \{x : (g, x) \leq r, \|g\| \leq 1\}$$

Now all $v_i \geq 0$. If some i, $v_i > r$, we can substitute v_i with r, without altering $\mathcal{G}$, thanks to the above inclusion relation. But this implies that $v \in l_\infty$. At this point, arguing as before, take $z \in l_2$ with $z_i > 0$. Then we can write:

$$\mathcal{G} = \cap \{x : (z_i g^i, x) \leq z_i v_i\}$$

which is equivalent to say that, if G is the matrix whose rows are the vectors $z_i g^i$ and γ is the vector defined by $\gamma_i = z_i v_i$, then:

$$\mathcal{G} = \{x : Gx \leq \gamma\}$$

with $\gamma \in l_2$, and G is a $(l_2, \mathcal{S}) \to (l_2, \mathcal{S})$ matrix operator since

$$\sum_{ij} g_{ij}^2 = \|z\|^2 < \infty$$

The proof is now finished. ∎

It is convenient to extract from the above proof and state formally the following sufficient condition to represent a polyhedron by means of an operator G: $(l_2, \mathcal{S}) \to (l_2, \mathcal{S})$ and a bound $v \in l_2$.

Proposition 4.6. *Consider a polyhedron $\mathcal{G}$ and assume that the bound vector $v \in l_\infty$, then the polyhedron is representable as*

$$\mathcal{G} = \{x : \Gamma x \leq \gamma\}$$

where $\gamma \in l_2$ and Γ is an operator $(l_2, \mathcal{S}) \to (l_2, \mathcal{S})$. In particular All polyhedral cones (and all closed subspaces) can be (and will be) represented as

$$\mathcal{G} = \{x : \Gamma x \leq 0\}$$

where Γ is an operator $(l_2, \mathcal{S}) \to (l_2, \mathcal{S})$.

We single out the class of polyhedra that are representable within l_2 by the following:

Definition 4.2. A polyhedron $\mathcal{G}$ is called range-l_2 (briefly rl2) if it admits a representation as

$$\mathcal{G} = \{x : Gx \leq v\}$$

where G is an operator $(l_2, \mathcal{S}) \to (l_2, \mathcal{S})$ and $v \in l_2$.

In what follows we study this special class of polyhedra. Thus all polyhedra will be rl2, unless otherwise specified. We have just shown that polyhedral cones, closed subspaces and bounded, and hence weakly compact, polyhedra are rl2.

The fact that the polyhedral cones are rl2 immediately provides the fundamental passage from the external to the internal description (i.e. the pertinent type of Weyl Theorem, as it is called in finite dimensions [14]). Of course the following result also settles internal representation of closed subspaces, which are a special class of closed cones.

Corollary 4.7. *Any closed cone is countably generated.*

Proof. Just apply the computation $C = C^{pp}$ and then express C^p as a countable intersection of semispaces. Now the thesis follows from Theorem 2.3. ∎

5. THE POSITIVE CONE OF THE PRODUCT ORDER

The vector ordering in R^∞ we are using is the product ordering; in l_2 it is the restriction of the same order to l_2 Such orders are uniquely specified by their positive cones:

$$P_{R^\infty} = \{s : s \in R^\infty, s_i \geq 0\}$$

$$P_{l_2} = \{s : s \in l_2, s_i \geq 0\}$$

We work mostly on l_2 and, for simplicity, indicate all positive cones by P, leaving to the context to specify the space it is referred to.

Some properties of the finite dimensional case go through, like the fact the P is pointed and closed, but there are fundamental differences as well. First and foremost, as is well known, the fact that P has a void interior in l_2 (and also in R^∞).

To remedy the absence of interior, we introduce two subsets of P, which we consider its quasi-interior and quasi-boundary (this concept has already been used, not only for cones, see e.g. [16], but also for other kind of sets, see [17]). They will be referred to as intern and extern of P.

Definition 5.1. The set of all positive vectors $\{y : y_i > 0, \forall i\}$ is denoted by $P^\vee$ and called the intern of P. The set $P \backslash P^\vee$ is denoted by $P^\wedge$ and called the extern of P.

We will also need the cone $\mathcal{P}$:

$$\mathcal{P} = \mathcal{C}o(\{e_i : i = 1, ..\})$$

which is is properly contained in P. Note that $\mathcal{L}^-(P) = l_2$.

The vector ordering has all the desirable properties and, in particular the Banach lattice structure is in place in $(l_2, \mathcal{S})$. Note that pointedness of P implies:

$$x \leq y \text{ and } y \leq x \Rightarrow x = y$$

We collect the properties of P and its vector ordering in the following Proposition (well-known, see e.g.[11], [13] and [15]). We only prove that the strong interior is void (and hence such is the interior in any weaker topology), because is more directly related to subsequent work.

Proposition 5.1. *The positive cone P of l_2 is pointed. In all three topologies it is closed, has void interior and:*

$$P = \mathcal{P}^- = \mathcal{C}o^-(\{e_i : i = 1, ..\})$$

Moreover:

$$P^p = -P$$

and

$$(-P)^p = P$$

Finally, the vector ordering defined by P is reproducing, Archimedean and defining a Banach lattice.

Proof. Because P contains an orthonormal base it spans the whole space. If $x \in P \backslash P^\vee$ it is obvious that any neighborhood of x intersect $\overline{P}$. If $x \in P^\vee$, let T_N be the operator $I - P_N$ where P_N is the orthogonal projection on the space spanned by $\{e_1,..,e_n\}$ and notice that the sequence $\{x - 2T_N x\}$ is in $\overline{P}$ and converges to x. ∎

6. FACES OF P

Faces of P are subcones of P. Among them there are the extreme rays $r(e_i)$ and $\{0\}$, which is the only extreme point. These are obviously closed faces.

Although $P^\vee \cup \{0\}$ is a subcone of P, it is not a face of P and neither subcones of $P^\vee \cup \{0\}$ can be faces of P. In fact, if $f \in P^\vee$ and P_i is the orthogonal projection on $\mathcal{L}(\{e_i\})$, then:

$$f = \frac{1}{2} 2P_i f + \frac{1}{2} 2(I - P_i) f$$

We formalize this and more in the following:

Theorem 6.1. *Neither $P^\vee \cup \{0\}$ nor any of its subcones are faces of P. A closed proper face of P cannot intersect $P^\vee$. Consequently, all closed proper faces of P are contained in the extern $P^\wedge$ of P.*

Proof. Consider a closed face $\mathcal{F}$ and suppose $f \in \mathcal{F} \cap P^\vee \neq \phi$. Then consider any $g \in \mathcal{P}$. Because it is immediate that $\exists z \neq g$ such that:

$$[g : f] \subset [g : z] \subset P$$

we can conclude that g, $z \in \mathcal{F}$ (since $\mathcal{F}$ is a face). Therefore that $\mathcal{P} \subset \mathcal{F}$. But we are talking of closed proper faces and so, because $\mathcal{P}^- = P$, we have a contradiction, and so the proof is done. ∎

We will show that the sublattice of the closed faces is complete and has a simple representation, reminding the finite dimensional case, although this sublattice is not even countable and is properly contained in the lattice of faces.

Consider the family of subsets of $\mathfrak{N}$. Associate to $\mathfrak{N}$ the positive cone P itself (this is the upper bound of the lattice of faces) and to the void set ϕ associate $\{0\}$ (the lower bound of the lattice). To any other subset $\Omega \subset \mathfrak{N}$ associate the cone $\mathcal{F}_\Omega$:

$$\mathcal{F}_\Omega = \{f : f \in P \text{ and } ip(f) \subset \Omega\}$$

There is a one to one correspondence between the subsets of $\mathfrak{N}$ and this family of cones. We can define lattice operations by union and intersections of subsets of $\mathfrak{N}$. Formally and with self-evident symbols

$$\vee\{\mathcal{F}_\Omega : \Omega \in \Psi\} = \mathcal{F}_{\cap\Psi}$$

where Ψ is any family of subsets of $\mathfrak{N}$ and $\vee$ indicates the lub. Dually:

$$\wedge\{\mathcal{F}_\Omega : \Omega \in \Psi\} = \mathcal{F}_{\cup\Psi}$$

where $\wedge$ indicates the glb. Moreover, we can state:

Theorem 6.2. *Except P itself, each above defined cone $\mathcal{F}_\Omega$ with $\Omega \subset \mathfrak{N}$ is a closed proper and exposed face of P. Moreover:*

$$\mathcal{F}_\Omega = \mathcal{C}o^-(\{e_i : i \in \Omega\})$$

The correspondence between a closed proper face $\mathcal{F}$ and the set:

$$\Omega(\mathcal{F}) = \cup\{ip(f) : f \in \mathcal{F}\}$$

is bi-univocal. There are no other closed faces of P. Finally, the union of proper closed faces of P is equal to its extern $P^\wedge$ of P.

Proof. That each $\mathcal{F}_\Omega$ is closed is readily seen. Consider a net $\{f_\alpha\}$ in $\mathcal{F}_\Omega$ and suppose that $\{f_\alpha\} \to f$ where, necessarily $f \in P$. Because $\{f_\alpha\} \to f$ also weakly, $\{(f_\alpha, e_i)\} = \{f_{\alpha i}\} \to (f, e_i) = f_i$. Thus if $i \in \overline{\Omega}$, then $f_i = 0$ and, therefore, $f \in \mathcal{F}_\Omega$. Because if $i \in \Omega$, $e_i \in \mathcal{F}_\Omega$, it follows $\mathcal{F}_\Omega \supset \mathcal{C}o^-(\{e_i : i \notin \Omega\}$. The converse is also true because if $f \in \mathcal{F}_\Omega$ then $f_i = 0$, $\forall i \notin \Omega$, and so, clearly, $f \in \mathcal{C}o^-(\{e_i : i \notin \Omega\}$. To show that these faces are exposed, consider an arbitrary $z \in P^\vee$ and denote by P_Ω the orthogonal projection operator which zeroes all the components with index in Ω (or $P_\Omega f = f\chi_{\overline{\Omega}}$). Let $n = -P_\Omega z \in -P = P^p$. Then $(n, .)$ is a continuous linear functional separating $\mathcal{F}_\Omega$ and P, because $(n, \mathcal{F}_\Omega) = \{0\}$ and $(n, y) < 0$ for any $y \in P \backslash \mathcal{F}_\Omega$. Hence in particular:

$$\mathcal{F}_\Omega = \{f : f \in P \text{ and } (n, f) = 0\}$$

It remains to show that there are no other closed faces. Consider a closed face of P, and let it be $\mathcal{F}$. Define:

$$\Omega = \cup\{ip(f) : f \in \mathcal{F}\}$$

Take $i \in \Omega$ and $f \in \mathcal{F}$ with $f_i > 0$. If by chance $f = r(e_i)$ then $e_i \in \mathcal{F}$. Otherwise, if f has also other nonzero components, reasoning as we did earlier, we see that we can find $z \neq f$ such that:

$$[e_i : f] \subset [e_i : z] \subset P$$

so that we are forced to conclude that $e_i \in \mathcal{F}$. At this point it is readily seen that:

$$\mathcal{F} = \mathcal{C}o^-(\{e_i : i \in \Omega\}$$

The remaining statements are self-evident and so the proof is finished. ∎

Remark 6.1. Notice that this Theorem entails that the set of closed faces of P is not countable.

Definition 6.1. Consider a closed proper face $\mathcal{F}$ of P so that for some $\Omega \subset \mathfrak{N}$, $\mathcal{F} = \mathcal{F}_\Omega$. A vector y such that $ip(y) = \Omega$ is said to belong to the relative intern of $\mathcal{F}_\Omega$.

Notice that

$$\Omega_1 \cap \Omega_1 = \phi \Rightarrow \digamma_{\Omega_1} \perp \digamma_{\Omega_2}$$

In view of this latter it make sense to define orthogonal complementation within the lattice of closed faces.

Definition 6.2. For any closed face $\digamma_\Omega$ the orthogonal complement face in P is

$$\digamma_{\overline{\Omega}} = \digamma_\Omega^\perp \cap P$$

In the sequel if a face is denoted by M we briefly denote by $M^\perp$ its orthogonal complement face in P.

Note also that, obviously:

$$\digamma_\Omega = \mathcal{L}^-(\digamma_\Omega) \cap P$$

Are there non closed faces of P? The answer is yes. In fact:

Proposition 6.3. *For any non finite subset Ψ of $\mathfrak{N}$ the cone:*

$$C_\Psi = \mathcal{C}o(\{e_i : i \in \Psi\}$$

is a non-closed face of P.

Proof. The function $ip(.)$ is monotone increasing under convex combinations. If $f \in C_\Psi$ and is the convex combination:

$$f = \alpha f_1 + \beta f_2$$

with $f_1 \in P$ and $f_2 \in P$ and with $\alpha, \beta > 0$, then $ip(f) = ip(f_1) \cup ip(f_2)$ so that $ip(f_1) \subset ip(f)$ and $ip(f_2) \subset ip(f)$ and this implies that $f_1 \in C_\Psi$ and $f_2 \in C_\Psi$. And we are done. ∎

7. FEASIBILITY AND FIRST DECOMPOSITION OF POLYHEDRA IN l_2

Primal range-space feasibility conditions are formally the same as in finite dimension:

$$\mathcal{G}(G, v) \neq \phi \Leftrightarrow v \in \mathcal{R}(G) + P \Leftrightarrow$$

$$(v + \mathcal{R}(G)) \cap P \neq \phi \Leftrightarrow (v - P) \cap \mathcal{R}(G) \neq \phi$$

In view of the first condition, the cone $\mathcal{R}(G) + P$ is called the cone of feasible bound vectors (that is, the cone of bound vectors for which the polyhedron is non-void). In contrast to the finite dimensional case this cone is not necessarily closed. However if we assume that $\mathcal{R}(G)$ is closed then it is true that $\mathcal{R}(G) + P$ is closed. We state now this important result, postponing the proof to the Section dedicated to the cone capping theory.

Theorem 7.1. *If a linear subspace F is closed so is $F + P$. Consequently, if F is not closed:*

$$(F + P)^- = F^- + P$$

The set $S = (v + \mathcal{R}(G)) \cap P$, which is called the *slack set* (essentially the slack set is the polyhedron as viewed from the Range Space side), is of course closed if $\mathcal{R}(G)$ is closed.

Another important set, which determines a polyhedron by a suitable inverse image is

$$\Sigma = (v - P) \cap \mathcal{R}(G) = v - S$$

In fact we can write:

$$\mathcal{G}(G, v) = G^{-1}(\Sigma)$$

Next bear in mind Corollary 2.7 and write

$$G = G|_{\mathcal{N}(G)^\perp} P_{\mathcal{N}(G)^\perp}$$

Where both map are continuous, $G|_{\mathcal{N}(G)^\perp}$ is one to one from $\mathcal{N}(G)^\perp$ to $\mathcal{R}(G)$ and also a topological isomorphism if and only if $\mathcal{R}(G)$ is closed.

Even if $\mathcal{R}(G)$ is not closed, we can still decompose $G = G|_{\mathcal{N}(G)^\perp} P_{\mathcal{N}(G)^\perp}$ in the same way, $G|_{\mathcal{N}(G)^\perp}$ will be continuous *but not a linear topological isomorphism*. Substituting

$$\mathcal{G} = (G|_{\mathcal{N}(G)^\perp})^{-1}(\Sigma) + \mathcal{N}(G)$$

Bearing in mind:

$$S^- = (v + \mathcal{R}(G)^-) \cap P$$
$$\Sigma^- = (v - P) \cap \mathcal{R}(G)^-$$

as long as we take the above inverse image, it is legal to substitute to $\mathcal{R}(G)$ its closure.

Thus a first level of decomposition of polyhedra, which in the case of cones, reinterprets differently Proposition 2.2 is given by the following:

Proposition 7.2. *Any (non-void) polyhedron is the sum of a closed linear subspace, always given by $\mathcal{N}(G)$, plus a closed convex subset of $\mathcal{N}(G)^\perp$ given by $(G|_{\mathcal{N}(G)^\perp})^{-1}(\Sigma)$, which contains no linear subspaces. A closed cone is the sum of $\mathcal{N}(G)$ plus the closed pointed cone contained in $\mathcal{N}(G)^\perp$ given by $(G|_{\mathcal{N}(G)^\perp})^{-1}(\mathcal{R}(G)^- \cap P)$.*

This proposition indicates whether or not there is a linear component in a polyhedron and who this linear component is. A polyhedron where the linear subspace is present is called a *stripe*, the closed convex set containing no subspaces of the decomposition is called the *base of the stripe*. In the range space, the slack set and hence Σ cannot contain linear subspaces: *it is never a stripe*. As trivial finite dimensional cases indicates the addition of a subspace to form a stripe may well destroy the fine structure of the base. Not only extreme points and rays disappear but we may end up with a polyhedron that has no extreme sets at all, as for example a sandwich (non void intersection of closed semispaces defined by opposite normals).

If we want a topologically isomorphic pull back we must assume that $\mathcal{R}(G)$ be closed. In this case both the slack set and Σ are closed and such is the inverse image of Σ. Moreover such inverse images will necessarily have void interiors.

8. RELATIVE POSITION OF A SUBSPACE AND P

Next we investigate the structure of the base of the stripe. Such finer structure depends on the relative position of $\mathcal{R}(G)$ (which for simplicity we will mostly denote by F) and P. In this paper we develop the range space theory of polyhedra that assumes F (which is usually identified with $\mathcal{R}(G)$) is closed. The results on the on the $\mathcal{X}$ topology for l_2 are instrumental and almost all of the finite dimensional results go through, albeit with harder proofs. However, the optimization results of last Section are general and do not require this hypothesis. The requirement of closure of range spaces only determines whether or not suprema are attained.

Remark 8.1. In the simplex model the role of $\mathcal{R}(G)$ is taken by the kernel of an linear transformation. Thus all our results apply to this model with much more generality, since we only need the the transformation in question be continuous.

The present Section is concluded with a few technical Lemmas.

Definition 8.1. We say that a closed linear subspace F is strictly tangent to P, if:

$$F \cap P = \{0\}$$

Dually we say that a closed linear subspace F is intern to P if

$$F \cap P^\vee \neq \phi$$

We say that a closed subspace F is weakly tangent or extern to P if it is neither strictly tangent nor intern.

Theorem 8.1. *Suppose that F is intern to P. Then:*

$$F + P = H \text{ and this is equivalent to}$$

$$(F + P)^p = F^\perp \cap (-P) = \{0\}$$

Thus F is intern to P if and only if $F^\perp$ is strictly tangent to P. Moreover, if a closed subspace is strictly tangent to P, then it is contained in a closed hyperplane strictly tangent to P.

Proof. In this proof bear in mind that, by Theorem 7.1, $F + P$ is closed. Note that $e_i \in F + P$, $\forall i$. By hypothesis $\exists w$, $w \neq 0$ and $w \in F \cap P^\vee$ and also $-w \in F + P$. Apply now Lemma 2.1 to $F + P$. Suppose $\exists z \neq 0$ such that:

$$F + P \subset \{y : (z, y) \leq 0\}$$

This implies $z_i \leq 0$ and that it must exist a j such that $z_j < 0$. But then $(z, w) > 0$. This contradiction shows that $F + P = H$. This implies $(F + P)^p = F^\perp \cap (-P) = \{0\}$. Thus $F^\perp$ is strictly tangent to P. Conversely, by taking polars:

$$F^\perp \cap P = \{0\} \Rightarrow H = \{0\}^p = (F + P)^- = F + P$$

Finally, if a closed subspace F is strictly tangent to P, so that $F^\perp$ is intern to P, consider a vector $n \neq 0$ with $n \in F^\perp \cap P^\vee$. Then:

$$\{y : (n, y) = 0\} \supset F$$

and because $\mathcal{L}(\{n\})$ is intern to P such an hyperplane is strictly tangent to P. ∎

Theorem 8.2. *If a closed subspace F is strictly tangent to P then $F + P$ is a proper cone and:*

$$lin(F + P) = F$$

If F is an hyperplane $F + P$ is a closed semispace.

Proof. Notice that in general:

$$lin(F + P) \supset F$$

First assume that F is an hyperplane, which we denote by L. Then

$$L + P = (L + P)^{pp} = (L^\perp \cap -P)^p = (r(n))^p$$

where n is a unit vector in $L^\perp \cap -P$. Thus $L + P$ is closed semispace and its lineality space is L. For the general case note that, by the preceding Theorem, $F \subset L$ where L is a closed hyperplane strictly tangent. Thus $F + P$ is a proper cone. Next take $n \in P^\vee \cap L^\perp$, and so it will be true that

$$F + P \subset L + P \subset \{y : (y, n) \geq 0\}$$

Next note that for $y \in F + P \subset L + P$ it is true that $y = w + z$ with $w \in F$ and $z \in P$ and then with obvious meaning of symbols we can write $y = w + z_L + \alpha n$. Because if $\alpha > 0$ then $(n, -y) < 0$, $y \in F + P$ and $-y \in F + P$ implies $\alpha = 0$. But then $y \in L$, $y - w = z_L \in P$ implies $z_L = 0$ and so $y = w$. Thus $lin(F + P) = F$. ∎

Theorem 8.3. *Suppose that F is extern to P. Then in the lattice of closed faces of P, there exist a unique maximal face F_Ω whose relative intern is met by F. Moreover:*

$$F \cap P \subset F_\Omega$$

Proof. Define the order $\succ$ on $F \cap P$ by $y \succ w$ if $ip(y) \supset ip(w)$. We now apply the Maximal Principle (e.g. [10]). Consider a tower in this set and a vector $z \in P^{\vee}$. Also we can normalize all elements of the tower without any harm to the argument and also assume that the tower is countable. In fact if it is not simply take the subtower obtained choosing only one element for each value of the function ip, and our argument will go through anyway. Denote the tower by $\{w^j\}$. Define the series:

$$\sum z_j w^j$$

This series is Cauchy in norm and therefore it converges to some vector ξ in $F \cap P$, which follows in the order any element in the tower. Because for each tower there is vector in $F \cap P$ which follows all vectors in the tower, it follow from the Maximal Principle that there is a maximal element μ in $F \cap P$. Such vector identifies the claimed maximal face by means of the index set $ip(\mu)$. If there were more than one call σ and $\tau$the vectors for which $ip(\sigma)$ and $ip(\tau)$ are maximal. Then $ip((\sigma + \tau)/2)$ properly contains both $ip(\sigma)$ and $ip(\tau)$. This contradiction concludes the proof. ∎

We now need a technical Lemma, because it is often useful to consider relaxations of the system $Gx \leq v$, which are systems obtained from the original one by deleting some of the inequalities and/or adding to v a vector in P. Dealing with the first case, does the fact that $\mathcal{R}(G)$ is closed imply that, if G_2 is the matrix obtained from G deleting some of its rows (which obviously also represent an operator), then $\mathcal{R}(G_2)$ is closed? The answer is affirmative.

Lemma 8.4. *If $\mathcal{R}(G)$ is closed then, denoting by $G_2 x \leq v_2$ any relaxation of $Gx \leq v$, $\mathcal{R}(G_2)$ is closed.*

Proof. It is clearly equivalent to argue applying to both sides of the system the projector P_2 into the coordinate space individuated by the relaxation. Then write:

$$G_2 = P_2 G|_{\mathcal{N}(G)^{\perp}} P|_{\mathcal{N}(G)^{\perp}}$$

and notice that all maps on the right hand side are open. Thus G_2 is open and thus, by virtue of the Open Mapping Theorem has closed range. ∎

Suppose that $F = \mathcal{R}(G)$ is extern to P, and let $M \leftrightarrow \Psi$ be the maximal face of P, whose relative intern is met by F.

Theorem 8.5. *Suppose that $F = \mathcal{R}(G)$ is extern to P. If in the system $Gx \leq v$ we delete the inequalities corresponding to Ψ the ensuing relaxation $G_2 x \leq v_2$ is strictly tangent. Moreover if $M \leftrightarrow \Psi$ is the maximal face whose relative intern is met by F then $M^{\perp} \leftrightarrow \overline{\Psi}$ is the maximal face of P, whose relative intern is met by $F^{\perp}$.*

Proof. By the first duality principle (F strictly tangent $\Leftrightarrow$ $F^{\perp}$ intern or F intern $\Longleftrightarrow$ $F^{\perp}$strictly tangent), we deduce that $F^{\perp}$ is extern, and in addition, the maximal face whose relative intern is met by $F^{\perp}$has an index set $\Gamma \subset \overline{\Psi}$, for otherwise we would have two vectors in F and $F^{\perp}$ with positive inner product. Suppose $\Gamma \subset \overline{\Psi}$ properly. Now consider the direct sum (which is stated by polarity theory):

$$(F^{\perp} \cap P) \oplus (F - P)$$

and consider a vector in the relative intern of $M^{\perp}$. The first set cannot contribute with a vector with all the positive components and the second set can contribute at most with a vector in the relative interior of M. Thus the direct sum is contradicted and so it must be $\Gamma = \overline{\Psi}$ or the maximum face whose relative intern is met by $F^{\perp}$ is $M^{\perp}$. Next , let Q be the matrix, whose rows generate the cone $F \cap P$. All rows have all zeros for indexes in $\overline{\Psi}$. Since $F^{\perp} + P = \{x : Qx \leq 0\}$, then we can write:

$$F^{\perp} + P = F^{\perp} + \mathcal{L}^{-}(M^{\perp}) + P$$

and by polarization:

$$F \cap P = [F \cap \mathcal{L}^{-}(M)] \cap P$$

thus to the effect of intersecting F with P we can use the subspace $F \cap \mathcal{L}^{-}(M)$ in place of F. But this implies that if we project $\mathcal{R}(G)$ on $\mathcal{L}^{-}(M^{\perp})$, as we do when we consider the relaxation individuated by the subscript 2, we find a strictly tangent system. Thus the proof is finished. ∎

Corollary 8.6. *Suppose that $F = \mathcal{R}(G)$ is extern to P, then the cone of feasible bound vectors $\mathcal{R}(G) + P$ is proper and hence contained in a closed semispace.*

Proof. Obvious from the fact that the system has a strictly tangent relaxation. ∎

Lemma 8.7. *Given two closed cones C_1 and C_2 and two vectors y_1 and y_2:*

$$y_1 + C_1 \subset y_2 + C_2 \Leftrightarrow y_1 - y_2 \in C_2 \textit{ and } C_1 \subset C_2$$

$$y_1 + C_1 = y_2 + C_2 \Leftrightarrow y_1 - y_2 \in lin(C_2) \textit{ and } C_1 = C_2$$

Proof. The second statement is an immediate consequence of the first one. As to this latter sufficiency is obvious. Next suppose that although

$$y_1 - y_2 + C_1 = y + C_1 \subset C_2$$

there is a vector z in C_1 that does not belong to C_2. Thus kz is in C_1 but not in C_2. By hypothesis $y + kz \in C_2$, for any positive integer k. Therefore $\{(y/k) + z\}$ is in C_2. But this sequence converges to z and so, being C_2 closed $z \in C_2$. This contradiction concludes the proof. ∎

The following Lemma is the essential tool to determine who is the conical component of a slack polyhedron.

Lemma 8.8. *Suppose that F is either intern or extern to P and that $(v + F) \cap P$ is non-void. Then*

$$\forall w \in (v + F) \cap P, w + (F \cap P) \subset (v + F) \cap P$$

and for any closed cone C

$$\forall w \in (v + F) \cap P, w + C \subset (v + F) \cap P \Leftrightarrow C \subset (F \cap P)$$

Proof. First statement. Suppose a vector z belongs to the lhs. Then $z = v + y + \gamma$ with $y \in F$, $v + y \in P$ and $\gamma \in F \cap P$, so that $z \in P$. But, also, $v + y \in v + F$ and $\gamma \in F$ imply $z \in v + F$. Second statement. It must be $C \subset P$. For otherwise there would be a vector z in C with at least a negative component. Then for sufficiently large $\alpha > 0$, $w + \alpha z$ cannot be in P, which is a contradiction. On the other hand

$$w + C \subset v + F \Rightarrow C \subset F$$

in view of the Lemma on inclusion of translated closed cones. Thus $C \subset F \cap P$ as stated. ∎

9. POLYHEDRA IN RANGE SPACE, GENERALITIES

A major goal is to prove that the slack polyhedron $(v+F)\cap P$ is either a generalized polytope (a convex set that is compact in the $\mathcal{X}$ topology) or a cup, that is, the sum of a generalized polytope plus a closed pointed cone (contained in P).

There are also major differences though, especially in the approach. In finite dimension polytopes have a finite set of extreme points and the convex hull of a finite set is always compact. This fails in infinite dimension, where the set of extreme points can even be uncountable.

Our approach is first to settle the case where $F = \mathcal{R}(G)$ is a closed hyperplane, which is an easier case, and then use the result as a Lemma to settle in turn the general case. In this way we will be able to show that the finite dimensional classification goes through: in fact in the strictly tangent case we can only have polytopes; all other cases generate cups. The cone of the cups is always $F \cap P$. Their compact components always contain the set $ex((v+F)\cap P)$.

However, as we know, both the cone $F \cap P$ and the compact base are countably generated, whereas extreme points and rays might be uncountable. Internal descriptions by extreme rays and extreme points remain a fundamental theoretical issue, but might loose some appeal especially from the numerical point of view. There might be special cases, where countability is preserved, but this goes beyond the present purposes.

10. COMPACT POLYHEDRA: THE STRICTLY TANGENT CASE

In this Section we investigate the case in which $F = \mathcal{R}(G)$ is strictly tangent to P. We show that the slack polyhedron is compact in the $\mathcal{X}$ topology. This is the infinite dimensional counterpart of the concept of polytope. For brevity, we make only the statements relative to the range space, from which the statements on domain spaces follow immediately via Theorem 3.4.

Theorem 10.1. *Suppose that $\mathcal{R}(G)$ is strictly tangent to P. Then:*

$$S = (v+F)\cap P$$

is compact in the $\mathcal{X}$ topology. Assume, without restriction of generality, $v \perp F$. If F is an hyperplane, then the set of extreme points of S is:

$$\Lambda = \{\frac{\|v\|^2}{v_i} e^i : i = 1, 2, ..\}$$

and so

$$S = \mathcal{C}^-(\Lambda)$$

If F is any closed subspace, let $\digamma$ be a closed strictly tangent hyperplane, which contains F. In the definition of Λ, substitute v by $P_{\digamma^\perp} v$. Then $(v+F)\cap P$ is a convex closed subset of $\mathcal{C}^-(\Lambda)$, and hence it is $\mathcal{X}$-compact. It follows that S has extreme points and :

$$S = \mathcal{C}^-(ex(S))$$

Proof. If F is an hyperplane then

$$v + F = \{y : (v, y) = \|v\|^2\}$$

The ith coordinate axis (extreme ray) of P intersect $v + F$ *only once* at the points

$$\lambda_i = \frac{\|v\|^2}{v_i} e^i$$

and it easy to verify that these points are extreme for $(v + F) \cap P$ and that there are no other extreme points. At this point notice that:

$$(v+F)\cap P \subset X\{[0, \frac{\|v\|^2}{v_i}]\}$$

which, by the Tychonoff Theorem is a compact set in the product topology. Thus the statement dealing with hyperplanes is now proved. If F is not an hyperplane, by Theorem 8.1, it is contained in a closed hyperplane $\mathcal{F}$ strictly tangent to P. Thus

$$v + F \subset v + \mathcal{F} = P_{\mathcal{F}^{\perp}} v + \mathcal{F}$$

From this the remaining part of the thesis follows at once. ∎

Theorem 10.2. *The orthogonal projection of P on a a closed linear subspace F intern to P is closed.*

Proof. We know that $F + P$ is closed and that $lin(F^{\perp} + P) = F^{\perp}$. The application of decomposition of cones yields:

$$F + P = F + P_{F^{\perp}}(F + P) = F + P_{F^{\perp}}(P)$$

Because this last expression is unique, if $P_{F^{\perp}}(P)$ were not closed we would contradict that $F + P$ is closed. ∎

11. THE CONE CAPPING TOOL

There is a connection between the last Theorem and the theory of cone capping. These concepts are also instrumental in the sequel and so we briefly outline them. We follow [12], in which more details as well as references can be found.

In fact we have used a special instance of universal caps for pointed convex cones in locally convex spaces. These concepts have been introduced by Choquet, to the purpose of extending The Krein-Milman Theorem in a framework of Measure Theory.

If C is a convex pointed cone, then $K \subset C$ is a cap of C if it is compact and convex and $C \backslash K$ is convex. The cap is universal if $C = \cup\{nK : n = 1, 2, ..\}$. Once a cone is capped , *all its closed subcones are obviously capped as well.*

In our specific technique for P in $(l_2, \mathcal{X})$, we indeed used for P the universal cap:

$$N(\alpha, f) = K = \{y : (f, y) \leq \alpha\} \cap P$$

with some $\alpha > 0$ and $f \in P^{\vee}$. To avoid confusion, we change a little bit Phelp's terminology, and call the compact set

$$L(\alpha, f) = \{y : (f, y) = \alpha\} \cap P$$

the roof (at level α) of the capped cone. The fact that the hyperplane F is strictly tangent to P means that N is a cap and L is a roof. Note that we can also express the capping of the cone using the roofs instead. For example we can write:

$$P = N(\alpha, f) \cup (\cup\{L(\beta, f) : \beta > \alpha\})$$

We can rephrase Proposition 13.1 of [12] in our context:

Proposition 11.1. *A point y is an extreme point of a roof if and only if it lays on an extreme ray of the cone.*

Naturally, this result provides the internal description (in terms of extreme rays, for all the closed subcones of P, and in particular those of the form $F \cap P$, where F is a closed subspace, either internal or extern to P.

In the sequel we will use in a self-evident way the same terminology relative to cone capping for subset of P called cups, which are sum of an $\mathcal{X}$-compact subset of P plus a closed subcone of $F \cap P$.

We now use the cone capping standpoint to show the anticipated result 7.1 asserting that, if a subspace F is closed, then the cone $F + P$ is also closed. Here is a proof:

Proof. It is immediate that if we cap P at a certain level β and denote by P_β the capped cone, we can write:

$$F + P = F + P_\beta + P$$

Moreover $F + P_\beta$ is $\mathcal{X}$-closed, because it is the sum of an $\mathcal{X}$-compact set plus an $\mathcal{X}$-closed set. But $\mathcal{X}$- is also convex, and hence by what we stated regarding the $\mathcal{X}$ topology, we can affirm that $F + P_\beta$ is strongly closed. Consider a converging sequence $\{v^i\}$ in $F + P$. using the increments with respect to v^0, we can write:

$$v^i = v^0 + \delta^i = w + z + \delta^{i-} + \delta^{i+}$$

where $w \in F$, $z \in P$ and $\delta = \delta^{i-} + \delta^{i+}$ is the unique decomposition defined by the pair P and its polar cone $-P$. We choose β sufficiently large so that $z \in P_\beta$. Because the $\{\delta^{i+}\}$ is defined by a projection, which is continuous, it converges to a limit in P. Consequently the sequence $\{w+z+\delta^{i-}\}$ also converges. We claim that each vector $w+z+\delta^{i-}$ is in $F+P_\beta$. For if it were not so, it would evidently have some negative component of index in $in(\delta^{i-})$, which cannot compensated by the corresponding component of δ^{i+}, because this must be zero. But then the limit of this sequence is in $F + P_\beta$ and the Theorem is proved. ∎

12. THE INTERN CASE

We will prove that in the intern case the slack polyhedron $(v + F) \cap P$ has extreme points and is the sum of $\mathcal{X}$-compact proper subset containing $\mathcal{C}^-(ex((v + F) \cap P))$ plus the closed pointed cone $F \cap P$. This kind of polyhedron is called cup. The closed set is called the base of the cup.

Definition 12.1. A set $C \subset P$ is called a cup if $ex(C) \neq \phi$ and C is the sum of a proper $\mathcal{X}$-compact convex subset B containing $\mathcal{C}^-(ex(C))$ plus a closed pointed $F \cap P$. The set B called the base of the cup.

We start showing that in this definition $ex(C)$ cannot be void.

Lemma 12.1. *A cup C has extreme points.*

Proof. If $0 \in C$, then obviously 0 is an extreme point. Otherwise take a capping functional $(f, .)$ for P (so that $f \in P^\vee$). For some sufficiently high level $\alpha > 0$ the corresponding cap intersect the cup. The intersection is $\mathcal{X}$-compact and on such intersection the functional has a minimum. By a, by now usual, argument the minimum is attained on an extreme point of the roof corresponding to the minimum and such an extreme point is also extreme for the cup. ∎

First we examine the structure of the slack set in the case in which the subspace F is a closed hyperplane.

Theorem 12.2. *Suppose that $F = \mathcal{R}(G)$ is a (closed) hyperplane and that F is intern to P. Then $(v + F) \cap P$ is a non-void cup with $\mathcal{X}$-compact base $B = \mathcal{C}^-(\{\xi_j\})$ and cone $F \cap P$. More precisely, assuming without restriction of generality $v \in F^\perp$:*

$$(v + F) \cap P = \mathcal{C}^-(\{\xi_j\}) + (F \cap P) =$$

where:

$$\{\xi_j\} = \{\frac{\|v\|^2}{v_j} e_j \colon j \in ip(v)\}$$

Remark 12.1. We remind once more that the closure in the preceding formula can be taken in any of the three topologies we are working with.

Proof. As usual we can take $v \perp F$. The hyperplane has the form:

$$v + F = \{y : (v, y) = \|v\|^2\}$$

Then $\mathcal{L}(\{v\})$ must be strictly tangent and so v must have both positive and negative components. Let, for the present purposes, $\Omega = in(v) \cup iz(v)$. We claim:

$$[(v + F) \cap P] \cap \digamma_\Omega = \phi$$

This follows from the fact that for all vectors in $\digamma_\Omega$, the inner product in the expression of F is negative or zero. Thus all vectors in $(v + F) \cap P$ have either all zero components in Ω or, if not, they must have some positive components in $\overline{\Omega}$. Moreover, $v + F$ intersect all and only the coordinate axes with index $\overline{\Omega}$ in the points $\xi_j = \frac{\|v\|^2}{v_j} e_j$ and only once for each axis, as follows from the expression of $v + F$. Our remarks on components of vectors in $(v + F) \cap P$ imply that the ξ_j are all extreme points and that there are no other extreme points in $(v + F) \cap P$. The same argument used for the strictly tangent case proves that

$$B = \mathcal{C}^-(\{\xi_j\})$$

is $\mathcal{X}$-compact. At this point, in view of Lemma 8.8, we can claim that

$$\mathcal{C}^-(\{\xi_j\}) + (F \cap P) \subset (v + F) \cap P$$

Note that $B + (F \cap P)$ is convex and closed in the $\mathcal{X}$-topology (because B is compact). Next suppose $\exists z$ with $z \in (v+F) \cap P$ but $z \notin \mathcal{C}^-(\{\xi_j\}) + (F \cap P)$. Then $\{z\}$ (which is compact) can be strongly separated from $\mathcal{C}^-(\{\xi_j\}) + (F \cap P)$, by a continuous functional f. Let's look at this issue from the point of view of the $\mathcal{X}$-topology. The image of B is a closed bounded interval, and the image of $F \cap P$ is (without restriction of generality) $[0, +\infty)$, so that the resulting image has the form $[\delta, +\infty)$. It is easy to show that the value δ is attained in an extreme point ξ_k (in fact, briefly, $f^{-1}(\delta)$ is a closed and hence compact face of B and thus it has an extreme point). Next notice that, because the null space of the functional is closed, the functional is also strongly continuous so that $\exists! g \in l_2$ such that $f = (g, .)$. Moreover, it must be $g \in -(F \cap P)^p = F^\perp + P$ or $g = \alpha v + p$, with obvious meaning of symbols. But the component αv assumes the same value $\alpha^2 \|v\|^2$ on both ξ_k and $v + F$. Thus we may assume $g \in P$. Next we can add to g a vector g', obtained from a vector in $P^\vee$ zeroing the kth component. Clearly for this sum (still denoted by g), $\delta = (g, \xi_k)$ is again the infimum of the functional on $\mathcal{C}^-(\{\xi_j\}) + (F \cap P)$ and the perturbation can be taken small enough to ensure $(g, z) < \delta$. Note that now we are dealing with a capping functional for the set $(v + F) \cap P$. By the strictly tangent theory we have developed in the preceding Section, the intersection:

$$\Psi = \{x : (g, x) \leq (g, z)\} \cap (v + F) \cap P$$

is $\mathcal{X}$-compact and hence has extreme points. The functional $(g, .)$ attains its minimum (in Ψ and hence also on $(v + F) \cap P$) at an extreme point ζ. We claim that ζ is extreme for the whole $(v + F) \cap P$. For this cannot evidently contradicted neither using test points in the bottom roof $\{x : (g, x) = (g, z)\}$ and neither using points in roofs of higher level, because this case it would contradict minimality. But this new extreme point ζ cannot belong to $\{\xi_j\} = ex((v + F) \cap P)$. This contradiction concludes the proof. ∎

Theorem 12.3. *Suppose that $F = \mathcal{R}(G)$ is a (closed) subspace and that F is intern to P. Then $(v + F) \cap P$ is non-void, has extreme points and is a cup:*

$$(v + F) \cap P = \Gamma + (F \cap P)$$

where Γ (which contains $ex((v + F) \cap P)$ is defined as follows. Let L be a closed hyperplane with $L \supset F$ so that L is intern too. Thus we know that $(v + L) \cap P$ is a cup $B + (F \cap P)$.

Then:

$$\Gamma = B \cap ((v + F) \cap P)$$

Proof. First note that non-voidness is a consequence of the obvious condition $\mathcal{G} \neq \phi \Leftrightarrow v \in F+P$ and in the present case Theorem 8.1 ensures that $F+P = H$. Next consider a continuous linear functional $f \in F \cap P^{\vee}$. For some capping level α the set:

$$[(v + F) \cap P] \cap \{x : (f, x) \leq \alpha\}$$

must be non-void and it is $\mathcal{X}$-compact. (These sets are the cups for $(v + F) \cap P$ and the corresponding roofs are denoted still by $L(\alpha, f)$). Thus it has extreme points. Because $(f, .)$ is (by Lemma 3.3) continuous in the $\mathcal{X}$ topology, it has a minimum δ on this compact set (which is attained in an extreme point ζ). All the extreme points of the roof at level δ are clearly also extreme for $(v + F) \cap P$. Thus the first part of the statement is proved. Next bearing in mind Lemma 8.8 we can write:

$$\Gamma + (F \cap P) \subset (v + F) \cap P \subset B + (F \cap P)$$

From this we see that an extreme point of $(v + F) \cap P$ must belong to B and hence, also, such an extreme point must be in Γ by the very definition of Γ itself. In formula:

$$\Gamma \supset ex((v + F) \cap P)$$

Next suppose there is a vector in $(v + F) \cap P$ that is not in $\Gamma + (F \cap P)$. At this point a replication (mutatis mutandis) of the separation argument in the proof for the hyperplane case leads to a contradiction and so the proof is finished. ∎

13. THE WEAKLY TANGENT CASE

In the extern case there exists the maximal face M of P whose relative intern is met by F. The face M is determined by the corresponding subset Υ of $\mathfrak{N}$ and $F \cap P \subset \mathcal{L}^{-}(M) = \mathfrak{M}$. Deleting the rows of G in Υ (and doing the same on the components of v) we obtain a block G_2 out of G. This corresponds to redefine G as the column of two blocks:

$$Gx \leq v \rightleftharpoons \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} x \leq \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Recall that, by Lemma 8.4, $\mathcal{R}(G_1)$ and $\mathcal{R}(G_2)$ are closed. Moreover, by Theorem 8.5, the system $G_2 x \leq v_2$ is a strictly tangent. We call it the *strictly tangent relaxation*. The system $G_1 x \leq v_1$ is an intern system, by the very definition of M. We call it the *intern relaxation*. In dealing with these systems, if needed, we inject vectors of their range spaces suitably adding a zero column block and viceversa make projections , but without explicit notice.

In the extern case we have a results similar to the preceding one: we still have an $\mathcal{X}$-compact base plus the cone $F \cap P$. The expression of the base is derived from the bases of an internal and a strictly tangent case.

Theorem 13.1. *Suppose that $F = \mathcal{R}(G)$ is extern to P and that $S = (v + F) \cap P$ is non-void. Then S is a cup given by:*

$$(v + F) \cap P = \Sigma + (F \cap P)$$

$$\Sigma = (\Psi \times B) \cap [(v + F) \cap P]$$

where B is the (injected) polyhedron defined by the strictly tangent relaxation and Ψ is the (injected) base of the cup defined by the intern relaxation.

Proof. Consider a feasible slack vector $y \in (v+F) \cap P$. If we project y on the two coordinate spaces 1 and 2 we obtain two feasible coordinate vectors for system 1 and system 2. Thus the slack set of the system is contained in a product of the form:

$$(v+F) \cap P \subset [\Psi + (F \cap P)] \times B = (\Psi \times B) + (F \cap P)$$

with self-evident symbols that take into account the previous results on the strictly tangent and internal cases. Next we show that:

$$\Sigma = (\Psi \times B) \cap [(v+F) \cap P] \neq \phi$$

In fact if we take any vector y in $(v+F) \cap P$ it has the form:

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

with $y_2 \in B$, $y_1 = z_1 + \zeta$ with $z_1 \in \Psi$ and $\zeta \in F \cap P$. This follows from the fact that y_2 and y_1 are of slack vectors in their respective relaxations. Now can take an x in the domain such that

$$Gx = \begin{pmatrix} -\zeta \\ 0 \end{pmatrix}$$

and hence we can also obtain the slack vector:

$$\xi = \begin{pmatrix} z_1 \\ y_2 \end{pmatrix} \in \Sigma$$

as we wanted to show. At this point we know, by Lemma 8.8, that:

$$\Sigma + (F \cap P) \subset (v+F) \cap P$$

On the other hand we have just shown that a point in the rhs is the sum of a vector in Σ plus a vector in $F \cap P$ and hence the reverse inclusion holds as well. Clearly the Tychonof Theorem applies to show that Σ is $\mathcal{X}$-compact. Next, by a by now usual argument, because the cup is formed adding to the base a cone, the extreme points of S, if any, belong necessarily to Σ. That there are extreme points in S is deduced in the same way as in the previous case, taking a functional $(f,.)$ with $f \in P^{\vee}$ and showing that it has a minimum in S and that such minimum is necessarily attained at an extreme point of S. ∎

14. DUAL RANGE SPACE CONDITIONS

With the hypothesis that $\mathcal{R}(G)$ be closed in force, we easily generalize the finite dimensional dual feasibility condition as follows:

Theorem 14.1. *There exists a non negative matrix $Q \geq 0$ (inequality here intended entry-wise) such that*

$$v \in \mathcal{R}(G) + P \Leftrightarrow Qv \geq 0$$

Proof. Because, as we have proved, the cone $\mathcal{R}(G) + P$ is closed and hence a polyhedral cone, we can write for some matrix operator Q:

$$\mathcal{R}(G) + P = \{x : Qx \leq 0\}$$

at this point note that for any row q_i of Q it is true that $(q_i, x) \leq 0, \forall x \in \mathcal{R}(G) + P$ and hence:

$$q_i \in (\mathcal{R}(G) + P)^p = -[(\mathcal{R}(G))^p \cap P]$$

Therefore $q_i \in -P$. To complete the proof change Q in $-Q$ and, at the same time. change the sign of the inequality. ∎

Remark 14.1. If $\mathcal{R}(G)$ is not closed a similar condition holds for its closure. That is there exists a matrix $Q \geq 0$ such that:

$$v \in (\mathcal{R}(G) + P)^- = \mathcal{R}(G)^- + P \Leftrightarrow Qv \geq 0$$

15. APPLICATION TO INFINITE DIMENSIONAL LP

The linear optimization problem (LP) is defined as follows.

$$\max(f, x) \text{ on the polyhedron } \mathcal{G} = \{x : Gx \leq v\}$$

Note that, assuming $\mathcal{G} \neq \phi$, this is equivalent to look at the maximum of the set of reals $f(\mathcal{G})$, which is convex and hence an interval. If and only if the interval has a finite right extremum, there is a sup in $f(\mathcal{G})$. Thus three cases are possible. First, the polyhedron may be void (unfeasible problem). Second the polyhedron is non-void, but $\sup(f(\mathcal{G})) = \infty$ (feasible unbounded problem). Third the polyhedron is non-void, and $\sup(f(\mathcal{G})) < \infty$ (feasible bounded problem). These possibilities, and the value of the $\sup$, whenever it exists, are settled by our last Theorem below.

However, by contrast to the finite dimensional case, it is not assured that, when the $\sup$ exists, it is also attained. Therefore we give a sufficient condition for this to happen.

To begin with define:

$$\widehat{G} = \begin{pmatrix} -f \\ G \end{pmatrix}; \widehat{v}(h) = \begin{pmatrix} -h \\ v \end{pmatrix}$$

where clearly $-f$ is disposed as a row and h is a real parameter. So the problem LP becomes:

$$\max\{h : \widehat{\mathcal{G}}(\widehat{G}, \widehat{v}(h)) \neq \phi\}$$

Then we can state the following:

Theorem 15.1. *For a rl2 polyhedron, assume that $f(\mathcal{G})$ is bounded from above. Then if $\mathcal{R}(\widehat{G})$ is closed the maximum of the LP problem exists. Moreover, $\mathcal{R}(G)$ closed, and $f \in \mathcal{R}(G^*)^-$ imply that $\mathcal{R}(\widehat{G})$ is closed.*

Proof. Actually we have to look if there exists the:

$$\max\{h : \widehat{v}(h) \in \mathcal{R}(\widehat{G}) + P\}$$

but in this way, as the parameter h varies, we intersect a line:

$$\{-he_1 + \begin{pmatrix} 0 \\ v \end{pmatrix} : h \in R\}$$

with the set $\mathcal{R}(\widehat{G}) + P$. Note that if $\mathcal{R}(G)$ is not closed and so neither $\mathcal{R}(\widehat{G})$ is closed, the intersection of the line:

$$\{-he_1 + \begin{pmatrix} 0 \\ v \end{pmatrix} : h \in R\}$$

with the cone $\mathcal{R}(\widehat{G}) + P$ is anyway an interval, whose closure is the intersection of the same line with $\mathcal{R}(\widehat{G})^- + P$. If $\mathcal{R}(\widehat{G})$ is closed (which implies $\mathcal{R}(\widehat{G}) + P$ closed by Theorem 7.1) then the interval is closed and the maximum is attained. As to the second part, consider a net $\{\widehat{y}_\alpha\} = \{\widehat{G}x_\alpha\}$ in $\mathcal{R}(\widehat{G})$ such that $\{\widehat{y}_\alpha\} \to \widehat{y}$. Partitioning the $\widehat{y}_\alpha$ as above, the first block yields a net in R, $\{t_\alpha\} = \{(-f, x_\alpha)\} \to t$, and the second block a net in l_2, $\{y_\alpha\} = \{Gx_\alpha\} \to y$. Applying the strong topology part of the induced map Theorem 3.4, we know that the net $\{P_{\mathcal{R}(G^*)^-}x_\alpha\} = \{z_\alpha\}$ in $\mathcal{R}(G^*)^-$ is such that $\{z_\alpha\} \to z$, and $\{Gz_\alpha\} \to Gz = y$. Next $f \in \mathcal{R}(G^*)^- \Leftrightarrow \{f\}^\perp = \mathcal{N}((-f, .)) \supset \mathcal{N}(G)$. Thus, $\forall \alpha$, $(-f, x_\alpha) = (-f, z_\alpha)$ and therefore $\{(-f, x_\alpha)\} \to (-f, z) = t$. And this shows that $\mathcal{R}(\widehat{G})$ is closed. ∎

Using the dual conical condition we can now solve the LP problem. Let $\widehat{Q}$ be the matrix whose rows are the generators of $\mathcal{R}(\widehat{G})^{\perp} \cap P$, and partition the matrix as

$$\widehat{Q} = \begin{pmatrix} p & S \end{pmatrix}$$

Then we have to find:

$$\sup\{h : \widehat{Q}\widehat{v}(h) \geq 0\}$$

or equivalently:

$$\sup\{h : hp \leq b\}$$

where $b = Sv$. Note that the vector p is non-negative.

If $\mathcal{R}(\widehat{G}) + P$ is closed the Theorem below give us the $\max$ of the problem. If not it gives the $\sup$, and such sup may or may not be attained according to the circumstances.

At this point we can state the following Theorem, which is proved by direct inspection.

Theorem 15.2. *Define $J = \{i : p(i) = 0\}$. Then the following mutually exclusive and exhaustive cases are possible.*

- a) $J = \phi$. If $\inf\{\frac{b(i)}{p(i)}\} > -\infty$ then the problem is feasible and

$$\overline{h} = \sup\{(f, x) : x \in \mathcal{G}\} = \inf\{\frac{b(i)}{p(i)}\}$$

  Otherwise the problem is unfeasible.
- b) $J \neq \phi$ and $J \neq \mathfrak{N}$. This case is partitioned in:
- b1) $\exists j \in J$ such that $b(j) < 0$. In this case the problem is unfeasible.
- b2) $\nexists j \in J$ such that $b(j) < 0$. If $\inf\{\frac{b(i)}{p(i)} : i \notin J\} > -\infty$ then the problem is feasible and:

$$\overline{h} = \sup\{(f, x) : x \in \mathcal{G}\} = \inf\{\frac{b(i)}{p(i)} : i \notin J\}$$

  Otherwise the problem is unfeasible.
- c)$J = \mathfrak{N}$ This case is partitioned in:
- c1) $\exists j \in \mathfrak{N}$ such that $b(j) < 0$. In this case the problem is unfeasible.
- c2) $\forall j \in \mathfrak{N}$ such that $b(j) \geq 0$. In this case the problem is feasible unbounded.

16. CONCLUSIONS

We conclude with an important observation, that indicate a possible orientation of future research. As is clear the extension of theory of polyhedra is twofold: first the environment is infinite dimensional, but also we intersect countable families of closed semispaces in a separable space. Such an extension allows to encompass not only, as required, the positive cone but also any closed convex set, although along the way we chose to deal only with closed subspaces (and closed affine spaces). The fact of encompassing a large class of closed convex sets, suggests that the optimization results presented could be adapted to included a special class of Convex Programming Problems, by representing epigraphs as polyhedra. Moreover, in finite dimension, the polar conical conditions have been used to prove linear programming problems duality. If we extend this results in our setting, we could possibly prove a version of such duality in infinite dimension and even obtain a corresponding Convex Programming interpretation.

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