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REFINEMENTS OF THE TRACE INEQUALITY OF BELMEGA, LASAULCE AND DEBBAH

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ABSTRACT. In this short paper, we show a certain matrix trace inequality and then give a refinement of the trace inequality proven by Belmega, Lasaulce and Debbah. In addition, we give an another improvement of their trace inequality.

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1. INTRODUCTION

Recently, E.-V. Belmega, S. Lasaulce and M. Debbah obtained the following elegant trace inequality for positive definite matrices.

Theorem 1.1. **([1])** *For positive definite matrices A, B and positive semidefinite matrices C, D, we have*

$$Tr[(A-B)(B^{-1}-A^{-1})+(C-D)\left\{(B+D)^{-1}-(A+C)^{-1}\right\}]\geq 0. \tag{1.1}$$

In this short paper, we first prove a certain trace inequality for products of matrices, and then as its application, we give a simple proof of (1.1). At the same time, our alternative proof gives a refinement and of Theorem 1.1. An another improvement of the Theorem 1.1 is also considered at the end of the paper.

2. MAIN RESULTS

In this section, we prove the following theorem.

Theorem 2.1. *For positive definite matrices A, B and positive semidefinite matrices C, D, we have*

$$\begin{aligned} &Tr[(A-B)(B^{-1}-A^{-1})+(C-D)\left\{(B+D)^{-1}-(A+C)^{-1}\right\}] \\ &\geq |Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}]|. \end{aligned} \tag{2.1}$$

To prove this theorem, we need a few lemmas.

Lemma 2.2. **([1])** *For positive definite matrices A, B and positive semidefinite matrices C, D, and Hermitian matrix X, we have*

$$Tr[XA^{-1}XB^{-1}]\geq Tr[X(A+C)^{-1}X(B+D)^{-1}].$$

Lemma 2.3. *For any matrices X and Y, we have*

$$Tr[X^*X]+Tr[Y^*Y]\geq 2|Tr[X^*Y]|.$$

Proof: Since $Tr[X^*X]\geq 0$, by the fact that the arithmetical mean is greater than the geometrical mean and Cauchy-Schwarz inequality, we have

$$\frac{Tr[X^*X]+Tr[Y^*Y]}{2}\geq\sqrt{Tr[X^*X]Tr[Y^*Y]}\geq |Tr[X^*Y]|.$$

∎

Theorem 2.4. *For Hermitian matrices X_1, X_2 and positive semidefinite matrices S_1, S_2, we have*

$$Tr[X_1S_1X_1S_2]+Tr[X_2S_1X_2S_2]\geq 2|Tr[X_1S_1X_2S_2]|.$$

Proof: Applying Lemma 2.3, we have

$$\begin{aligned} &Tr[X_1S_1X_1S_2]+Tr[X_2S_1X_2S_2] \\ &=Tr[(S_2^{1/2}X_1S_1^{1/2})(S_1^{1/2}X_1S_2^{1/2})]+Tr[(S_2^{1/2}X_2S_1^{1/2})(S_1^{1/2}X_2S_2^{1/2})] \\ &\geq 2|Tr[(S_2^{1/2}X_1S_1^{1/2})(S_1^{1/2}X_2S_2^{1/2})]| \\ &=2|Tr[X_1S_1X_2S_2]|. \end{aligned}$$

∎

Remark 2.5. *Theorem 2.4 can be regarded as a kind of the generalization of Proposition 1.1 in* [2].

Proof of Theorem 2.1: By Lemma 2.2, we have

$$\begin{aligned} Tr[(A-B)(B^{-1}-A^{-1})] &= Tr[(A-B)B^{-1}(A-B)A^{-1}] \\ &\geq Tr[(A-B)(A+C)^{-1}(A-B)(B+D)^{-1}] \\ &= Tr[(A-B)(B+D)^{-1}(A-B)(A+C)^{-1}]. \end{aligned}$$

Thus the left hand side of the inequality (2.1) can be bounded from below:

$$\begin{aligned} & Tr[(A-B)(B^{-1}-A^{-1})+(C-D)\left\{(B+D)^{-1}-(A+C)^{-1}\right\}] \\ & \geq Tr[(A-B)(B+D)^{-1}(A-B)(A+C)^{-1}+(C-D)(B+D)^{-1}(C-D)(A+C)^{-1}] \\ & \quad +Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}] \\ & \geq 2|Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}]| \\ & \quad +Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}] \qquad (2.2) \end{aligned}$$

Throughout the process of the above, Theorem 2.4 was used in the second inequality. Since we have the following equation,

$$\begin{aligned} & Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}] \\ & = Tr[(C-D)(B+D)^{-1}]-Tr[(C-D)(A+C)^{-1}] \\ & -Tr[(C-D)(B+D)^{-1}(C-D)(A+C)^{-1}] \end{aligned}$$

we have $Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}] \in \mathbb{R}$. Therefore we have

$$(2.2) \geq |Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}]|.$$

∎

3. AN ANOTHER IMPROVEMENT OF THE INEQUALITY (1.1)

In this section, we show the following trace inequality.

Theorem 3.1. *For positive definite matrices A, B and positive semidefinite matrices C, D, we have*

$$Tr[(A-B)(B^{-1}-A^{-1})+4(C-D)\left\{(B+D)^{-1}-(A+C)^{-1}\right\}] \geq 0. \qquad (3.1)$$

To prove this theorem, we use the following lemmas, which are proven by the similar way of Lemma 2.3 and Theorem 2.4 in the previous section.

Lemma 3.2. *For any matrices X and Y, any positive real numbers a and b, we have*

$$a \cdot Tr[X^*X]+b \cdot Tr[Y^*Y] \geq 2\sqrt{ab} \cdot |Tr[X^*Y]|.$$

Applying this lemma, we have the following lemma.

Lemma 3.3. *For Hermitian matrices X_1, X_2, positive semidefinite matrices S_1, S_2 and any positive real numbers a and b, we have*

$$a \cdot Tr[X_1S_1X_1S_2]+b \cdot Tr[X_2S_1X_2S_2] \geq 2\sqrt{ab} \cdot |Tr[X_1S_1X_2S_2]|.$$

Proof of Theorem 3.1: By the similar way to the proof of Theorem 2.1, applying Lemma 3.2 as $a=1$ and $b=4$, the left hand side of the inequality of (3.1) can be bounded from the below:

$$\begin{aligned} & Tr[(A-B)(B^{-1}-A^{-1})+4(C-D)\left\{(B+D)^{-1}-(A+C)^{-1}\right\}] \\ & \geq Tr[(A-B)(B+D)^{-1}(A-B)(A+C)^{-1}+4(C-D)(B+D)^{-1}(C-D)(A+C)^{-1}] \\ & \quad +Tr[4(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}] \\ & \geq 4|Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}]| \\ & \quad +4 \cdot Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}] \geq 0, \end{aligned}$$

since $Tr[(C-D)(B+D)^{-1}(A-B)(A+C)^{-1}] \in \mathbb{R}$. ∎

Remark 3.4. *Here we note that we have* $Tr[(A-B)(B^{-1}-A^{-1})] \geq 0$. *However we have the possibility that* $Tr[(C-D)\{(B+D)^{-1}-(A+C)^{-1}\}]$ *takes a negative value. Therefore Theorem 3.1 is an improvement of Theorem 1.1.*

Corollary 3.5. *For positive definite matrices* A, B, *positive semidefinite matrices* C, D *and positive real number* r, *we have*

$$Tr[(A-B)(B^{-1}-A^{-1})+4(C-D)\left\{(rB+D)^{-1}-(rA+C)^{-1}\right\}] \geq 0. \tag{3.2}$$

Proof. Put $A = rA_1$ and $B = rB_1$ for positive definite matrices A_1 and B_1, in Theorem 3.1. ∎

Remark 3.6. *In the case of* $r = 2$ *in Corollary 3.5, the inequality (3.2) corresponds to the scalar inequality:*

$$(\alpha-\beta)\left(\frac{1}{4\beta}-\frac{1}{4\alpha}\right)+(\gamma-\delta)\left(\frac{1}{2\beta+\delta}-\frac{1}{2\alpha+\gamma}\right) \geq 0$$

for positive real numbers α *and* β, *nonnegative real numbers* γ *and* δ.

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