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THE BEST UPPER BOUND FOR JENSEN'S INEQUALITY

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ABSTRACT. In this paper we give the best upper bound for the weighted Jensen's discrete inequality applied to a convex function f defined on a closed interval I in the case when the bound depends on f, I and weights. In addition, we give a simpler expression of the upper bound, which is better than existing similar one.

Key words and phrases: Jensen's inequality, Best upper bound, Weighted AM-GM inequality.

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1. INTRODUCTION

Let $\tilde{x} = \{x_1, x_2, ..., x_n\}$ be a sequence of real numbers belonging to a given closed interval $I = [a, b]$, $a < b$, and let $\tilde{p} = \{p_1, p_2, ..., p_n\}$ be a sequence of given positive weights associated to $\tilde{x}$ and satisfying $p_1 + p_2 + ... + p_n = 1$. If f is a convex function on I, then the following inequality is well-known Jensen's discrete inequality [5]:

$$0 \le \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right). \tag{1.1}$$

An upper global bound (depending on f and I only) of Jensen's difference

$$\Delta(f, \tilde{p}, \tilde{x}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \tag{1.2}$$

is given by Dragomir in [4]:

Theorem 1.1. *If f is a differentiable convex function on I, then*

$$\Delta(f, \tilde{p}, \tilde{x}) \le \frac{1}{4}(b-a)(f'(b) - f'(a)) := D_f(a, b). \tag{1.3}$$

In [6], Simić gave an upper global bound without differentiability restriction on f:

Theorem 1.2. *If f is a convex function on I and $0 < p < 1$, then*

$$\Delta(f, \tilde{p}, \tilde{x}) \le \max_p [pf(a) + (1-p)f(b) - f(pa + (1-p)b)] := T_f(a, b). \tag{1.4}$$

Using Theorem 1.2, it is easy to show that

$$\Delta(f, \tilde{p}, \tilde{x}) \le f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) := S_f(a, b). \tag{1.5}$$

Indeed, (1.5) holds if

$$pf(a) + (1-p)f(b) - f(pa + (1-p)b) \le f(a) + f(b) - 2f\left(\frac{a+b}{2}\right)$$

for all $p \in (0, 1)$. This is equivalent to Jensen's inequality

$$(1-p)f(a) + pf(b) + f(pa + (1-p)b) \ge 2f\left(\frac{a+b}{2}\right).$$

In the present paper, we establish the best upper bound $C_{\tilde{p},f}(a, b)$ of $\Delta(f, \tilde{p}, \tilde{x})$, show that $T_f(a, b)$ is the best upper global bound depending on f and I only, determine $C_{\tilde{p},f}(a, b)$ in the case of the weighted AM-GM inequality, and give a simpler expression $V_{\tilde{p},f}(a, b)$ of the upper bound, which is better than $S_f(a, b)$.

2. MAIN RESULTS

Our main results rely on an old result in [1], in virtue of which if f is a differentiable convex function on I, then Jensen's difference $\Delta(f, \tilde{p}, \tilde{x})$ is maximal when all $x_i \in \{a, b\}$. The following theorem states that this property holds without differentiability restriction on f and establishes the best upper bound $C_{\tilde{p},f}(a, b)$ of Jensen's difference Δ.

Theorem 2.1. *Let $\tilde{p}$ and $\tilde{x}$ be defined as above, and let*

$$P = \{p_{i_1} + p_{i_2} + ... + p_{i_k}\}, \quad k = 1, 2, ..., n-1, \quad 1 \le i_1 < i_2 < ... < i_k \le n.$$

If f is a convex function on $I = [a, b]$, then

$$\Delta(f, \tilde{p}, \tilde{x}) \leq \max_{p \in P} [pf(a) + (1 - p)f(b) - f(pa + (1 - p)b)] \\ := C_{\tilde{p},f}(a, b), \tag{2.1}$$

with equality when some of x_i are equal to a, and the others x_i are equal to b.

The following theorem establishes the best upper bound of Jensen's difference Δ for the case when the bound depends on f and I only.

Theorem 2.2. *$T_f(a, b)$ is the best upper global bound (depending on f and I only) of Jensen's difference $\Delta(f, \tilde{p}, \tilde{x})$.*

For

$$p_1 = p_2 = ... = p_n = \frac{1}{n},$$

the set P contains the distinct elements $\frac{k}{n}$, $k = 1, 2, ..., n - 1$. Let us define

$$P_0 = \left\{ \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n} \right\}. \tag{2.2}$$

From Theorem 2.1, one gets

Corollary 2.3. *Let f be a convex function on $I = [a, b]$. If $x_1, x_2, ..., x_n \in I$, then*

$$\frac{f(x_1) + f(x_2) + ... + f(x_n)}{n} - f(\frac{x_1 + x_2 + ... + x_n}{n}) \\ \leq \max_{p \in P_0} [pf(a) + (1 - p)f(b) - f(pa + (1 - p)b)]. \tag{2.3}$$

Applying Theorem 2.1 for $f(x) = -lnx$, $x > 0$, we get

Corollary 2.4. *For $I = [a, b]$ with $0 < a < b$, let $\tilde{p}$, $\tilde{x}$ and P be defined as above. Then*

$$\frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq \max_{p \in P} \frac{p + (1 - p)u}{u^{1-p}}, \tag{2.4}$$

where $A(\tilde{p}, \tilde{x}) = \sum_{i=1}^{n} p_i x_i$, $G(\tilde{p}, \tilde{x}) = \prod_{i=1}^{n} x_i^{p_i}$ and $u = \frac{b}{a}$.

From Corollary 2.4, we get

$$\frac{x_1 + x_2 + ... + x_n}{n\sqrt[n]{x_1 x_2 ... x_n}} \leq \max_{p \in P_0} \frac{p + (1 - p)u}{u^{1-p}} := C_n(u). \tag{2.5}$$

In addition, for $b = 2a$, (2.5) becomes

$$\frac{x_1 + x_2 + ... + x_n}{n\sqrt[n]{x_1 x_2 ... x_n}} \leq \max_{p \in P_0} g(p) := C_n(2), \tag{2.6}$$

where

$$g(p) = \frac{2 - p}{2^{1-p}}. \tag{2.7}$$

Similarly, applying Theorem 1.2 for $f(x) = -lnx$, $x > 0$, we get [6]

$$\frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq \frac{(u - 1)u^{\frac{1}{u-1}}}{e \ln u}. \tag{2.8}$$

For $b = 2a$, (2.8) becomes

$$\frac{x_1 + x_2 + ... + x_n}{n\sqrt[n]{x_1x_2...x_n}} \le \frac{2}{e\ln 2} \approx 1.06147. \tag{2.9}$$

Logically, we have $C_n(2) \le \frac{2}{e\ln 2}$ for any integer $n \ge 2$. For instant, $C_2(2) = g(\frac{1}{2}) \approx 1.06066$, $C_3(2) = g(\frac{2}{3}) \approx 1.05826$, $C_4(2) = g(\frac{2}{4}) \approx 1.06066$, $C_5(2) = g(\frac{3}{5}) \approx 1.06100$, $C_{10}(2) = g(\frac{6}{10}) \approx 1.06100$, $C_{11}(2) = g(\frac{6}{11}) \approx 1.06144$.

The following theorem establishes a simpler formula $V_{\tilde{p},f}(a, b)$ for the upper bound of Jensen's difference Δ in the case when this bound depends on f, I and $\tilde{p}$.

Theorem 2.5. *Let $\tilde{p}$ and $\tilde{x}$ be defined as above. If f is a convex function on $I = [a, b]$, then*

$$\Delta(f, \tilde{p}, \tilde{x}) \le [1 - \min\{p_1, p_2, ..., p_n\}] \left[f(a) + f(b) - 2f\left(\frac{a+b}{2}\right)\right] \tag{2.10}$$

$$:= V_{\tilde{p},f}(a, b) \le S_f(a, b)).$$

In the particular case

$$p_1 = p_2 = ... = p_n = \frac{1}{n},$$

from Theorem 2.5 we get

Corollary 2.6. *Let f be a convex function on $I = [a, b]$. If $x_1, x_2, ..., x_n \in I$, then*

$$\frac{f(x_1) + f(x_2) + ... + f(x_n)}{n} - f\left(\frac{x_1 + x_2 + ... + x_n}{n}\right) \tag{2.11}$$

$$\le \left(1 - \frac{1}{n}\right)\left[f(a) + f(b) - 2f\left(\frac{a+b}{2}\right)\right].$$

Applying in succession Corollary 2.6 for $f(x) = \dfrac{1}{x}$, $f(x) = -lnx$ [3] and $f(x) = e^x$ [2], we obtain

Proposition 2.7. *If $0 < a < b$ and $x_1, x_2, ..., x_n \in [a, b]$, then*

$$\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n} - \frac{n^2}{x_1 + x_2 + ... + x_n} \le \frac{(n-1)(b-a)^2}{ab(a+b)}, \tag{2.12}$$

$$\frac{x_1 + x_2 + ... + x_n}{n\sqrt[n]{x_1x_2...x_n}} \le \left(\frac{\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}}{2}\right)^{2-\frac{2}{n}}, \tag{2.13}$$

$$x_1 + x_2 + ... + x_n - n\sqrt[n]{x_1x_2...x_n} \le (n-1)(\sqrt{b} - \sqrt{a})^2. \tag{2.14}$$

In addition, the following statement is true.

Proposition 2.8. *If $0 < a < b$ and $x_1, x_2, ..., x_n \in [a, b]$, then*

$$x_1 + x_2 + ... + x_n - n\sqrt[n]{x_1x_2...x_n} \le \frac{(n-1)(b-a)^2}{b + k_n a}, \tag{2.15}$$

where

$$k_n = \begin{cases} 7 - \frac{8}{n+1}, & n \text{ odd} \\ 7 - \frac{8}{n}, & n \text{ even} \end{cases}$$

is the best possible.

Remark 2.1. The inequality (2.15) is sharper than (2.14) if $\frac{b}{a} \leq (3 - \frac{4}{n+1})^2$ for odd n, and $\frac{b}{a} \leq (3 - \frac{4}{n})^2$ for even n.

3. PROOFS

Proof of Theorem 2.1. For given f and $\tilde{p}$, let $F(\tilde{x}) := \Delta(f, \tilde{p}, \tilde{x})$. It suffices to show that $F(\tilde{x})$ increases by replacing each $x_i \in (a, b)$ with either a or b. For convenience, consider that $i = 1$. For fixed $x_2, ..., x_n$, let us denote

$$s = \frac{p_2 x_2 + p_3 x_3 + ... + p_n x_n}{1 - p_1}$$

and

$$f_1(y) := p_1 f(y) + p_2 f(x_2) + ... + p_n f(x_n) - f(p_1 y + (1 - p_1)s).$$

If we prove that f_1 is decreasing on $[a, s]$ and increasing on $[s, b]$, then the proof is completed. We need to show that $f_1(y_1) \geq f_1(y_2)$ for $a \leq y_1 < y_2 \leq s$ and for $s \leq y_2 < y_1 \leq b$. Write the desired inequality $f_1(y_1) \geq f_1(y_2)$ as

$$p_1 f(y_1) + f(p_1 y_2 + (1 - p_1)s) \geq p_1 f(y_2) + f(p_1 y_1 + (1 - p_1)s).$$

This inequality follows by adding Jensen's inequalities

$$(p_1 - \alpha) f(y_1) + \alpha f(p_1 y_2 + (1 - p_1)s) \geq p_1 f(y_2)$$

and

$$\alpha f(y_1) + (1 - \alpha) f(p y_2 + (1 - p_1)s) \geq f(p_1 y_1 + (1 - p_1)s),$$

where

$$\alpha = \frac{p_1(y_1 - y_2)}{y_1 - p_1 y_2 - (1 - p_1)s}.$$

We see that $y_1 - p_1 y_2 - (1 - p_1)s < 0$ for $a \leq y_1 < y_2 \leq s$, and $y_1 - p_1 y_2 - (1 - p_1)s > 0$ for $s \leq y_2 < y_1 \leq b$. Therefore, we have $\alpha > 0$. In addition,

$$p_1 - \alpha = \frac{p_1(1 - p_1)(y_2 - s)}{y_1 - p_1 y_2 - (1 - p_1)s} \geq 0,$$

$$1 - \alpha = \frac{(1 - p_1)(y_1 - s)}{y_1 - p_1 y_2 - (1 - p_1)s} > 0,$$

$$(p_1 - \alpha) y_1 + \alpha(p_1 y_2 + (1 - p_1)s) = p_1 y_2,$$

$$\alpha y_1 + (1 - \alpha)(p y_2 + (1 - p_1)s) = p_1 y_1 + (1 - p_1)s.$$

∎

Proof of Theorem 2.2. For fixed a and b, let us denote

$$g(p) := p f(a) + (1 - p) f(b) - f(pa + (1 - p)b).$$

Since g is concave on $[0, 1]$ and satisfies $g(0) = g(1) = 0$, $g(p)$ attains its maximal value $T_f(a, b)$ for a $p_0 \in (0, 1)$. To complete the proof we only need to show that there exists a finite sequence $\tilde{x}$ and an associated sequence $\tilde{p}$ such that $\Delta(f, \tilde{p}, \tilde{x}) = T_f(a, b)$. Indeed, choosing $n = 2$, $\tilde{x} = \{a, b\}$ and $\tilde{p} = \{p_0, 1 - p_0\}$, this condition is fulfilled.

∎

Proof of Theorem 2.5. Let us denote $p_0 = \min\{p_1, p_2, ..., p_n\}$. In the nontrivial case $n \geq 2$, for any $p \in P$, we have $p \geq p_0$ and $p + p_0 \leq 1$.

Using Theorem 2.1, it suffices to show that

$$(1-p_0)\left[f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right] \geq pf(a)+(1-p)f(b)-f(pa+(1-p)b)$$

for any $p \in P$. Indeed, this inequality is equivalent to Jensen's inequality

$$(1-p-p_0)f(a)+(p-p_0)f(b)+f(pa+(1-p)b) \geq 2(1-p_0)f\left(\frac{a+b}{2}\right).$$

∎

Proof of Proposition 2.8. Applying Theorem 2.1 for $f(x)=-lnx$, $x>0$, and $p_1=p_2=...=p_n=\frac{1}{n}$, it suffices to show that

$$ka+(n-k)b-n\sqrt[n]{a^k b^{n-k}} \leq \frac{(n-1)(b-a)^2}{b+k_n a}$$

for all $k=1,2,...,n-1$. Due to homogeneity, we may assume that $b=1$ and $0<a<1$. Thus, we need to show that $g(a)\geq 0$, where

$$g(a)=(n-1)(a-1)^2-(k_n a+1)(ka+n-k-na^{\frac{k}{n}}).$$

We have

$$g'(a)=2(n-1)(a-1)-k_n(ka+n-k-na^{\frac{k}{n}})-k(k_n a+1)(1-a^{\frac{k}{n}-1}),$$

$$g''(a)=2(n-1)-2kk_n(1-a^{\frac{k}{n}-1})-\frac{k(n-k)}{n}(k_n a+1)a^{\frac{k}{n}-2},$$

$$g''(1)=2n-2-\frac{k(n-k)(k_n+1)}{n}$$

and

$$g'''(a)=\frac{k(n-k)}{n^2}a^{\frac{k}{n}-3}h(a),$$

where

$$h(a)=2n-k-k_n(n+k)a.$$

Since $k(n-k)\leq \frac{n^2}{4}$ for even n, and $k(n-k)\leq \frac{n^2-1}{4}$ for odd n, we get $g''(1)\geq 0$. From $h(0)=2n-k>0$ and $h(1)=2n-k-k_n(n+k)\leq 2n-k-3(n+k)<0$, it follows that there is $a_1\in(0,1)$ such that $g'''(a)>0$ for $a\in(0,a_1)$ and $g'''(a)<0$ for $a\in(a_1,1]$. Therefore, $g''(a)$ is strictly increasing on $(0,a_1]$ and strictly decreasing on $[a_1,1]$. Since $\lim_{a\to 0} g''(a)=-\infty$ and $g''(1)\geq 0$, there is $a_2\in(0,1)$ such that $g''(a)<0$ for $a\in(0,a_2)$, and $g''(a)>0$ for $a\in(a_2,1)$. Thus, $g'(a)$ is strictly decreasing on $(0,a_2]$ and strictly increasing on $[a_2,1]$. From $\lim_{a\to 0} g'(a)=\infty$ and $g'(1)=0$, it follows that there is $a_3\in(0,1)$ such that $g'(a)>0$ for $a\in(0,a_3)$, and $g'(a)<0$ for $a\in(a_3,1)$. Then, $g(a)$ is strictly increasing on $[0,a_3]$ and strictly decreasing on $[a_3,1]$. Since $g(0)=k-1\geq 0$ and $g(1)=0$, we have $g(a)\geq 0$ for $a\in[0,1]$.

To prove that the original value of k_n is the best possible, we see that $g''(1)=0$ for $k=\frac{n}{2}$ if n is even, and for $k=\frac{n-1}{2}$ if n is odd. Therefore, for this value of k and for any value of k_n greater than the original one, we have $g''(1)<0$. Then, there is $\varepsilon>0$ such that $g''(a)<0$ for $a\in(1-\varepsilon,1]$. Since $g'(a)$ is strictly decreasing on $(1-\varepsilon,1]$ and $g'(1)=0$, we have $g'(a)>0$ for $a\in(1-\varepsilon,1)$. Thus, $g(a)$ is strictly increasing on $(1-\varepsilon,1]$, and from $g(1)=0$ it follows that $g(a)<0$ for $a\in(1-\varepsilon,1)$. From this result, the conclusion follows.

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