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ASYMPTOTIC INEQUALITIES FOR THE MAXIMUM MODULUS OF THE DERIVATIVE OF A POLYNOMIAL

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ABSTRACT. Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be an algebraic polynomial of degree $\leq n$, and let $\|p\| = \max\{|p(z)| \colon |z| = 1\}$. We study the asymptotic behavior of the best possible constant $\varphi_{n,k}(R)$, for $k = 0$ and $k = 1$, in the inequality $\|p'(Rz)\| + \varphi_{n,k}(R)|a_k| \leq nR^{n-1}\|p\|$, $R \to \infty$.

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1. INTRODUCTION AND STATEMENTS OF THE RESULTS

We denote by $\mathcal{P}_n$ the class of all polynomials p, of degree $\leq n$, with complex coefficients:

$$p(z) = \sum_{j=0}^{n} a_j z^j. \tag{1.1}$$

Let $\|p\| := \max_{|z|=1} |p(z)|$. The classical Bernstein's inequality [3, p. 508]

$$\|p'\| \leq n\|p\| \tag{1.2}$$

admits several extensions and refinements. It is known that the best possible constant $c_{n,0}$ in the inequality

$$\|p'\| + c_{n,0}|a_0| \leq n\|p\| \tag{1.3}$$

is ([1, p. 70]; [3, p. 518]) $c_{1,0} = 1$ and $c_{n,0} = \frac{2n}{n+2}$ for $n \geq 2$. By "best possible" we mean that, for every $\varepsilon > 0$, there exists a polynomial $p_\varepsilon(z) = \sum_{j=0}^{n} a_j(\varepsilon) z^j$ such that

$$\|p'_\varepsilon\| + (c_{n,0} + \varepsilon)|a_0(\varepsilon)| > n\|p_\varepsilon\|.$$

A similar meaning holds for the other best possible constants appearing in this paper.

The best possible constant $c_{n,1}$ in the inequality

$$\|p'\| + c_{n,1}|a_1| \leq n\|p\| \tag{1.4}$$

is [1, p. 77] $c_{1,1} = 0$, $c_{2,1} = \sqrt{2} - 1$, $c_{3,1} = \frac{1}{\sqrt{2}}$ and $c_{n,1} = \frac{2n}{n+4}\left(\sqrt{\frac{2(n+2)}{n}} - 1\right)$ for $n \geq 4$.

Another classical inequality is ([4], [3, p. 405])

$$\|p(Rz)\| \leq R^n\|p\|, \quad R \geq 1. \tag{1.5}$$

A direct consequence of (1.2) and (1.5) is

$$\|p'(Rz)\| \leq nR^{n-1}\|p\|, \quad R \geq 1. \tag{1.6}$$

It is natural to ask for the best possible constants $\varphi_{n,0}(R)$ and $\varphi_{n,1}(R)$, $R \geq 1$, in the inequalities

$$\|p'(Rz)\| + \varphi_{n,0}(R)|a_0| \leq nR^{n-1}\|p\| \tag{1.7}$$

and

$$\|p'(Rz)\| + \varphi_{n,1}(R)|a_1| \leq nR^{n-1}\|p\|. \tag{1.8}$$

We have $\varphi_{n,0}(1) = c_{n,0}$ and $\varphi_{n,1}(1) = c_{n,1}$. As we shall see, the exact values of $\varphi_{n,0}(R)$ and $\varphi_{n,1}(R)$ are not easy to find in explicit form. The aim of this paper is to prove the following asymptotic results.

Theorem 1.1. *Let $\varphi_{n,0}(R)$ be the best possible constant in the inequality* (1.7). *We have, for $n \geq 3$,*

$$\lim_{R\to\infty} \frac{\varphi_{n,0}(R) - nR^{n-1}}{R^{n-3}} = \frac{-(n-1)^2}{n}. \tag{1.9}$$

Theorem 1.2. *Let $\varphi_{n,1}(R)$ be the best possible constant in the inequality* (1.8). *We have, for $n \geq 5$,*

$$\lim_{R\to\infty} \frac{\varphi_{n,1}(R) - nR^{n-1}}{R^{n-2}} = -(n-1). \tag{1.10}$$

We also give some explicit values for $\varphi_{n,0}(R)$.

2. THE METHOD OF CONVOLUTION

The preceding theorems will be proved with the so called method of convolution [3, Chapter 12]. We give some details for the sake of completeness.

The Hadamard product of two analytic functions

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad g(z) = \sum_{j=0}^{\infty} b_j z^j \qquad (|z| \le K)$$

is the function

$$(f * g)(z) = \sum_{j=0}^{\infty} a_j b_j z^j \qquad (|z| \le K).$$

Let $\mathcal{B}_n$ be the class of polynomials Q in $\mathcal{P}_n$ such that

$$\|Q * p\| \le \|p\| \quad \text{for every } p \in \mathcal{P}_n.$$

We have

$$Q \in \mathcal{B}_n \iff \widetilde{Q} \in \mathcal{B}_n,$$

where $\widetilde{Q}(z) := z^n \overline{p(\frac{1}{\bar{z}})}$.

Let us denote by $\mathcal{B}_n^0$ the subclass of $\mathcal{B}_n$ consisting of polynomials R in $\mathcal{B}_n$ for which $R(0) = 1$.

Lemma 2.1. [3, p. 414] *The polynomial $R(z) = \sum_{j=0}^{n} b_j z^j$, where $b_0 = 1$, belongs to $\mathcal{B}_n^0$ if and only if the matrix*

$$M(b_0, b_1, \dots, b_n) := \begin{pmatrix} b_0 & b_1 & \dots & b_{n-1} & b_n \\ \bar{b}_1 & b_0 & \dots & b_{n-2} & b_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \bar{b}_{n-1} & \bar{b}_{n-2} & \dots & b_0 & b_1 \\ \bar{b}_n & \bar{b}_{n-1} & \dots & \bar{b}_1 & b_0 \end{pmatrix}$$

is positive semi-definite.

The definiteness of the matrix $M(b_0, b_1, \dots, b_n)$ is studied with the following well-known result.

Lemma 2.2. [2, p. 274] *The hermitian matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad a_{ij} = \bar{a}_{ji},$$

is positive definite if and only if its leading principal minors are all positive.

3. PROOF OF THEOREM 1.1

We observe that

$$\frac{1}{nR^{n-1}}(\|zp'(Rz)\|+\varphi_{n,0}(R)|a_0|)=\sup_{|\alpha|<\varphi_{n,0}(R)}\frac{1}{nR^{n-1}}\|\bar{\alpha}a_0+zp'(Rz)\|=\sup_{|\alpha|<\varphi_{n,0}(R)}\|Q_\alpha * p\|,$$

where

$$Q_\alpha(z)=\frac{\bar{\alpha}}{nR^{n-1}}+\sum_{j=1}^{n}\frac{jz^j}{nR^{n-j}}$$

and

$$R_\alpha(z):=\widetilde{Q}_\alpha(z)=\sum_{j=0}^{n-1}\frac{(n-j)z^j}{nR^j}+\frac{\alpha z^n}{nR^{n-1}}.$$

That leads us to study the definiteness of the matrix

$$M\left(n,\ \tfrac{n-1}{R},\ \tfrac{n-2}{R^2}\ ,\ldots,\ \tfrac{1}{R^{n-1}},\ \tfrac{\alpha}{R^{n-1}}\right).$$

Since the inequality (1.6) is known to be valid, we can assert, in view of Lemmas 2.1 and 2.2, that all its leading principal minors of order k, $1\le k\le n$, are non-negative. The leading principal minor of order $(n+1)$,

$$D_{n+1}(\alpha):=\det\left(M\left(n,\ \tfrac{n-1}{R},\ \tfrac{n-2}{R^2}\ ,\ldots,\ \tfrac{1}{R^{n-1}},\ \tfrac{\alpha}{R^{n-1}}\right)\right), \tag{3.1}$$

can be written in the form

$$D_{n+1}(\alpha)=A+B\alpha+B\bar{\alpha}+C|\alpha|^2, \tag{3.2}$$

where $A=D_{n+1}(0)$, C is a determinant of order $(n-1)$, and B is the determinant of order n

$$B=\frac{(-1)^n}{R^{n-1}}\begin{vmatrix} \frac{(n-1)}{R} & n & \frac{(n-1)}{R} & \cdots & \frac{3}{R^{n-3}} & \frac{2}{R^{n-2}} \\ \frac{(n-2)}{R^2} & \frac{(n-1)}{R} & n & \cdots & \frac{4}{R^{n-4}} & \frac{3}{R^{n-3}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{2}{R^{n-2}} & \frac{3}{R^{n-3}} & \frac{4}{R^{n-4}} & \cdots & \frac{(n-1)}{R} & n \\ 0 & \frac{1}{R^{n-1}} & \frac{2}{R^{n-2}} & \cdots & \frac{(n-2)}{R^2} & \frac{(n-1)}{R} \end{vmatrix}. \tag{3.3}$$

The decomposition (3.2) is easily obtained by using the linearity property of determinants. We also can replace α by u, $\bar{\alpha}$ by v in (3.1) and expand the resulting function (polynomial) of u and v as a Taylor's expansion about $u=0$, $v=0$.

Let $\alpha=ae^{it}$, $a=|\alpha|$, so that

$$D_{n+1}(\alpha)=A+2B\cos(t)a+Ca^2=:f_R(a,t). \tag{3.4}$$

We will show that the minimum value of $f_R(a,t)$, as a function of t, is attained for $t=\pi$. In order to do that, we evaluate the determinant B by performing the following operations:

(i) multiply all the elements by R^n;
(ii) L_i-RL_{i+1}, $1\le i\le n-1$ (where L_i is the i-th line);
(iii) L_i-RL_{i+1}, $1\le i\le n-2$ (where L_i also denotes the new i-th line).

We readily obtain

$$B=\frac{\left((n+1)R^2-(n-1)\right)^{n-2}}{R^{4n-5}},\quad n\ge 2. \tag{3.5}$$

Hence, the minimum value of $f_R(a,t)$ is

$$f_R(a,\pi) = A - 2Ba + Ca^2. \tag{3.6}$$

From (3.6) and the lemmas of Section 2, we see that $\varphi_{n,0}(R)$, $R \geq 1$, is the least positive root $a = a(R)$ of the equation

$$A - 2Ba + Ca^2 = 0. \tag{3.7}$$

It is not difficult to find some values of $\varphi_{n,0}(R)$ with (3.7). We have $\varphi_{1,0}(R) = 1$, $\varphi_{2,0}(R) = \dfrac{2R^2-1}{R}$, $\varphi_{3,0}(R) = \dfrac{9R^4+6R^3-4R^2-4R-1}{R(3R+2)}, \dots$. However, these values become rapidly complicated; for example,

$$\varphi_{7,0}(R) = \frac{\begin{aligned}&2401R^{14}+2058R^{13}-5292R^{12}-4592R^{11}\\&+3837R^{10}+3340R^9-944R^8-832R^7-13R^6\\&+106R^5+92R^4-56R^3-49R^2\end{aligned}}{\begin{aligned}&343R^8+294R^7-504R^6-440R^5+178R^4\\&+154R^3-4R^2+8R+7\end{aligned}}.$$

We study their asymptotic behavior as $R \to \infty$. The above discussion shows that the asymptotic value of the least positive root $a = a(R)$ of the equation $f_R(a,\pi) = 0$ needs to be examined. Let

$$a = nR^{n-1} + bR^{n-3}, \tag{3.8}$$

so that

$$f_R(a,\pi) = \begin{vmatrix} n & \frac{(n-1)}{R} & \frac{(n-2)}{R^2} & \dots & \frac{1}{R^{n-1}} & -n-\frac{b}{R^2} \\ \frac{(n-1)}{R} & n & \frac{(n-1)}{R} & \dots & \frac{2}{R^{n-2}} & \frac{1}{R^{n-1}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{R^{n-1}} & \frac{2}{R^{n-2}} & \frac{3}{R^{n-3}} & \dots & n & \frac{(n-1)}{R} \\ -n-\frac{b}{R^2} & \frac{1}{R^{n-1}} & \frac{2}{R^{n-2}} & \dots & \frac{(n-1)}{R} & n \end{vmatrix}. \tag{3.9}$$

On $f_R(a,\pi)$ we perform the operations

(i) $R(C_{n+1} + C_1)$ (where C_j is the j-th column);
(ii) $R(L_1 + L_{n+1})$.

We obtain

$$(3.10)\quad R^2 f_R(a,\pi)$$

$$= \begin{vmatrix} \frac{-b}{R} & (n-1)+\frac{1}{R^{n-2}} & \frac{(n-2)}{R}+\frac{2}{R^{n-3}} & \dots & (n-1)+\frac{1}{R^{n-2}} & -2b \\ \frac{(n-1)}{R} & n & \frac{(n-1)}{R} & \dots & \frac{2}{R^{n-2}} & (n-1)+\frac{1}{R^{n-2}} \\ \frac{(n-2)}{R^2} & \frac{(n-1)}{R} & n & \dots & \frac{3}{R^{n-3}} & \frac{(n-2)}{R}+\frac{2}{R^{n-3}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{R^{n-1}} & \frac{2}{R^{n-2}} & \frac{3}{R^{n-3}} & \dots & n & (n-1)+\frac{1}{R^{n-2}} \\ -n-\frac{b}{R^2} & \frac{1}{R^{n-1}} & \frac{2}{R^{n-2}} & \dots & \frac{(n-1)}{R} & -\frac{b}{R} \end{vmatrix},$$

whence

$$(3.11)\qquad \lim_{R\to\infty} R^2 f_R(a,\pi) = \begin{vmatrix} 0 & (n-1) & 0 & \dots & (n-1) & -2b \\ 0 & n & 0 & \dots & 0 & (n-1) \\ 0 & 0 & n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n & (n-1) \\ -n & 0 & 0 & \dots & 0 & 0 \end{vmatrix}$$

$$= n^{n-2}\begin{vmatrix} (n-1) & (n-1) & -2b \\ n & 0 & (n-1) \\ 0 & n & (n-1) \end{vmatrix}$$

$$= -2n^{n-1}\big((n-1)^2+nb\big),$$

with $(n-1)^2+nb=0$ for $b=\dfrac{-(n-1)^2}{n}$.

By Hurwitz' theorem, the roots $b(R)$ of $R^2 f_R(a,\pi)=0$ (see (3.7) and (3.8)) tend, as $R\to\infty$, towards the root b of $\lim\limits_{R\to\infty} R^2 f_R(a,\pi)$. We thus have

$$(3.12)\qquad \lim_{R\to\infty}\frac{a(R)-nR^{n-1}}{R^{n-3}} = -\frac{(n-1)^2}{n},$$

which is equivalent to (1.9).

4. PROOF OF THEOREM 1.2

We have

$$\frac{1}{nR^{n-1}}(\|zp'(Rz)\|+\varphi_{n,1}(R)|a_1| = \sup_{|\alpha|<\varphi_{n,1}(R)}\|Q_\alpha * p\|,$$

where

$$Q_\alpha(z)=\frac{(1+\bar{\alpha})z}{nR^{n-1}}+\sum_{j=2}^{n}\frac{jz^j}{nR^{n-j}}$$

and

$$R_\alpha(z) := \widetilde{Q}_\alpha(z) = \sum_{j=0}^{n-1} \frac{(n-j)z^j}{nR^j} + \frac{\alpha z^{n-1}}{nR^{n-1}}.$$

In this manner, we study the definiteness of the matrix

$$M\left(n,\ \tfrac{(n-1)}{R},\ \tfrac{(n-2)}{R^2}\ \ ,\ldots,\ \tfrac{2}{R^{n-2}},\ \tfrac{(1+\alpha)}{R^{n-1}},\ 0\right).$$

The lemmas of Section 2 and (1.6) show that its leading principal minors of order k, $1 \le k < n$ are non-negative. Its leading principal minors of order n and $(n+1)$ depend on the parameter α.

As in (3.2), the leading principal minor of order n can be written in the form (we use the same letters to simplify the notation)

$$A + 2B\cos(t)a + Ca^2 =: f_R(a,t), \tag{4.1}$$

where $\alpha = ae^{it}$, $a = |\alpha|$, and where A is a determinant of order n, C is a determinant of order $(n-2)$ and

$$B = \frac{(-1)^{n+1}}{R^{n-1}} \begin{vmatrix} \frac{(n-1)}{R} & n & \frac{(n-1)}{R} & \cdots & \frac{4}{R^{n-4}} & \frac{3}{R^{n-3}} \\ \frac{(n-2)}{R^2} & \frac{(n-1)}{R} & n & \cdots & \frac{5}{R^{n-5}} & \frac{4}{R^{n-4}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{2}{R^{n-2}} & \frac{3}{R^{n-3}} & \frac{4}{R^{n-4}} & \cdots & \frac{(n-1)}{R} & n \\ \frac{1}{R^{n-1}} & \frac{2}{R^{n-2}} & \frac{3}{R^{n-3}} & \cdots & \frac{(n-2)}{R^2} & \frac{(n-1)}{R} \end{vmatrix}. \tag{4.2}$$

We evaluate B by performing the following operations:

(i) multiply all the elements by R^n;
(ii) $L_i - RL_{i+1}$, $1 \le i \le n-2$;
(iii) $L_i - RL_{i+1}$, $1 \le i \le n-3$.

We find that

$$B = \frac{1}{R^{4n-8}}\big((n+1)R^2 - (n-1)\big)^{n-3}, \quad n \ge 3. \tag{4.3}$$

Hence, the minimum value of $f_R(a,t)$ (in (4.1)), as a function of t, is

$$f_R(a,\pi) = A - 2Ba + Ca^2 = \det\left(M\left(\tfrac{(n-1)}{R},\ \tfrac{(n-2)}{R^2}\ \ ,\ldots,\ \tfrac{2}{R^{n-2}},\ \tfrac{(1-a)}{R^{n-1}}\right)\right). \tag{4.4}$$

We now examine the leading principal minor of order $(n+1)$,

$$E_{n+1}(\alpha) := \det\left(M\left(n,\ \tfrac{(n-1)}{R},\ \tfrac{(n-2)}{R^2}\ \ ,\ldots,\ \tfrac{2}{R^{n-2}},\ \tfrac{(1+\alpha)}{R^{n-1}},\ 0\right)\right). \tag{4.5}$$

Using the property of linearity of determinant (in rows and columns), or an appropriate Taylor's expansion, we see that $E_{n+1}(\alpha)$ can be written in the form

$$E_{n+1}(\alpha) = A + B\alpha + B\bar{\alpha} + C\alpha^2 + C(\bar{\alpha})^2 + D|\alpha|^2 + E\alpha|\alpha|^2 + E\bar{\alpha}|\alpha|^2 + F|\alpha|^4 \tag{4.6}$$

where $A = E_{n+1}(0)$,

$$(4.7)\qquad B = \frac{2(-1)^{n+1}}{R^{n-1}} \begin{vmatrix} \frac{(n-1)}{R} & n & \frac{(n-1)}{R} & \cdots & \frac{4}{R^{n-4}} & \frac{3}{R^{n-3}} & \frac{1}{R^{n-1}} \\ \frac{(n-2)}{R^2} & \frac{(n-1)}{R} & n & \cdots & \frac{5}{R^{n-5}} & \frac{4}{R^{n-4}} & \frac{2}{R^{n-2}} \\ \frac{(n-3)}{R^3} & \frac{(n-2)}{R^2} & \frac{(n-1)}{R} & \cdots & \frac{6}{R^{n-6}} & \frac{5}{R^{n-5}} & \frac{3}{R^{n-3}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \frac{1}{R^{n-1}} & \frac{2}{R^{n-2}} & \frac{3}{R^{n-3}} & \cdots & \frac{(n-2)}{R^2} & \frac{(n-1)}{R} & \frac{(n-1)}{R} \\ 0 & \frac{1}{R^{n-1}} & \frac{2}{R^{n-2}} & \cdots & \frac{(n-3)}{R^3} & \frac{(n-2)}{R^2} & n \end{vmatrix},$$

$$(4.8)\qquad C = \frac{1}{R^{2n-2}} \begin{vmatrix} \frac{(n-2)}{R^2} & \frac{(n-1)}{R} & n & \cdots & \frac{5}{R^{n-5}} & \frac{4}{R^{n-4}} \\ \frac{(n-3)}{R^3} & \frac{(n-2)}{R^2} & \frac{(n-1)}{R} & \cdots & \frac{6}{R^{n-6}} & \frac{5}{R^{n-5}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{R^{n-1}} & \frac{2}{R^{n-2}} & \frac{3}{R^{n-3}} & \cdots & \frac{(n-2)}{R^2} & \frac{(n-1)}{R} \\ 0 & \frac{1}{R^{n-1}} & \frac{2}{R^{n-2}} & \cdots & \frac{(n-3)}{R^3} & \frac{(n-2)}{R^2} \end{vmatrix},$$

D is a determinant of order $(n-1)$,

$$(4.9)\qquad E = \frac{2(-1)^{n-1}}{R^{3n-3}} \begin{vmatrix} \frac{(n-1)}{R} & n & \frac{(n-1)}{R} & \cdots & \frac{5}{R^{n-5}} & \frac{4}{R^{n-4}} \\ \frac{(n-2)}{R^2} & \frac{(n-1)}{R} & n & \cdots & \frac{6}{R^{n-6}} & \frac{5}{R^{n-5}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{3}{R^{n-3}} & \frac{4}{R^{n-4}} & \frac{5}{R^{n-5}} & \cdots & \frac{(n-1)}{R} & n \\ \frac{1}{R^{n-1}} & \frac{2}{R^{n-2}} & \frac{3}{R^{n-3}} & \cdots & \frac{(n-3)}{R^3} & \frac{(n-2)}{R^2} \end{vmatrix}$$

and F is a determinant of order $(n-2)$.

On the determinant B (in (4.7)), of order n, we perform the following operations:

(i) multiply all the elements by R^n;
(ii) $L_i - RL_{i+1}$, $1 \le i \le n-1$;
(iii) $L_i - RL_{i+1}$, $1 \le i \le n-2$.

We obtain, for $n \ge 3$,

$$(4.10)\qquad B = \frac{2(R^2-1)}{R^{4n-4}}\big((n+1)R^2 - (n-1)\big)^{n-3}\big(nR^2 - (n-2)\big).$$

On the determinant C (in (4.8)), of order $(n-1)$, we perform the operations

(i) $L_i - RL_{i+1}$, for $i = n-3$ and $i = n-2$;
(ii) RL_{n-2}.

We obtain a determinant where L_{n-2} and L_{n-3} are two identical lines. We thus have $C = 0$.

We evaluate E (in (4.9)), a determinant of order $(n-2)$, with the following operations:

(i) multiply all the elements by R^n;
(ii) $L_i - RL_{i+1}$, $1 \le i \le n-4$;
(iii) $L_{n-3} - R^2L_{n-2}$;
(iv) factor the number 2 from L_{n-3};

(v) $L_i - RL_{i+1}$, $1 \le i \le n-4$.

We obtain, for $n \ge 4$,

$$E = -\frac{4}{R^{6n-12}}\big((n+1)R^2-(n-1)\big)^{n-4}. \tag{4.11}$$

Hence the coefficient of $\cos(t)$ in (see (4.6))

$$g_R(a,t) := E_{n+1}(ae^{it}) = A + 2Ba\cos(t) + Da^2 + 2Ea^3\cos(t) + Fa^4 \tag{4.12}$$

is equal, for $n \ge 4$, to

$$\begin{aligned} &2Ba + 2Ea^3 \\ &= -\frac{8a}{R^{6n-12}}\big((n+1)R^2-(n-1)\big)^{n-4}\bigg(a^2 - \frac{n(n+1)}{2}R^{2n-2} + \frac{(3n+2)(n-1)}{2}R^{2n-4} \\ &\qquad - \frac{n(3n-5)}{2}R^{2n-6} + \frac{(n-1)(n-2)}{2}R^{2n-8}\bigg). \end{aligned} \tag{4.13}$$

Observe that

$$\lim_{R\to\infty} E_{n+1}(\alpha R^{n-1}) = \begin{vmatrix} n & 0 & 0 & \dots & 0 & ae^{it} & 0 \\ 0 & n & 0 & \dots & 0 & 0 & ae^{it} \\ 0 & 0 & n & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n & 0 & 0 \\ ae^{-it} & 0 & 0 & \dots & 0 & n & 0 \\ 0 & ae^{-it} & 0 & \dots & 0 & 0 & n \end{vmatrix} = n^{n-3}(a^2-n^2)^2,$$

which implies that $a(R) \sim nR^{n-1}$, $R \to \infty$, where $a(R)$ is a positive root of the equation $g_R(a,t) = 0$. It follows from (4.13) that the coefficient of $\cos(t)$ in $g_R(a,t)$ is negative if R is large enough. The minimum value of $g_R(a,t)$, as a function of t, is thus attained for $t = 0$ if R is large enough. That value is

$$g_R(a,0) = \det\left(M\left(n,\ \tfrac{(n-1)}{R},\ \tfrac{(n-2)}{R^2}\ ,\dots,\ \tfrac{2}{R^{n-2}},\ \tfrac{(1+a)}{R^{n-1}},\ 0\right)\right). \tag{4.14}$$

It remains to examine and compare the least positive roots of $f_R(a,\pi)$ and $g_R(a,0)$, as $R \to \infty$. Let

$$a = nR^{n-1} + bR^{n-2}, \tag{4.15}$$

so that

$$f_R(a,\pi) = \begin{vmatrix} n & \frac{(n-1)}{R} & \frac{(n-2)}{R^2} & \dots & \frac{2}{R^{n-2}} & \frac{1}{R^{n-1}} - n - \frac{b}{R} \\ \frac{(n-1)}{R} & n & \frac{(n-1)}{R} & \dots & \frac{3}{R^{n-3}} & \frac{2}{R^{n-2}} \\ \frac{(n-2)}{R^2} & \frac{(n-1)}{R} & n & \dots & \frac{4}{R^{n-4}} & \frac{3}{R^{n-3}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{2}{R^{n-2}} & \frac{3}{R^{n-3}} & \frac{4}{R^{n-4}} & \dots & n & \frac{(n-1)}{R} \\ \frac{1}{R^{n-1}} - n - \frac{b}{R} & \frac{2}{R^{n-2}} & \frac{3}{R^{n-3}} & \dots & \frac{(n-1)}{R} & n \end{vmatrix} \tag{4.16}$$

and

$$g_R(a,0) \tag{4.17}$$
$$= \begin{vmatrix} n & \frac{(n-1)}{R} & \frac{(n-2)}{R^2} & \cdots & \frac{2}{R^{n-2}} & \frac{1}{R^{n-1}}+n+\frac{b}{R} & 0 \\ \frac{(n-1)}{R} & n & \frac{(n-1)}{R} & \cdots & \frac{3}{R^{n-3}} & \frac{2}{R^{n-2}} & \frac{1}{R^{n-1}}+n+\frac{b}{R} \\ \frac{(n-2)}{R^2} & \frac{(n-1)}{R} & n & \cdots & \frac{4}{R^{n-4}} & \frac{3}{R^{n-3}} & \frac{2}{R^{n-2}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \frac{1}{R^{n-1}}+n+\frac{b}{R} & \frac{2}{R^{n-2}} & \frac{3}{R^{n-3}} & \cdots & \frac{(n-1)}{R} & n & \frac{(n-1)}{R} \\ 0 & \frac{1}{R^{n-1}}+n+\frac{b}{R} & \frac{2}{R^{n-2}} & \cdots & \frac{(n-2)}{R^2} & \frac{(n-1)}{R} & n \end{vmatrix}.$$

We perform the operation $R(C_n+C_1)$ on $f_R(a,\pi)$. Afterwards, we see that

$$\lim_{R\to\infty} Rf_R(a,\pi) = -2n^{n-1}b, \tag{4.18}$$

which implies, in view of (4.15), that

$$\lim_{R\to\infty} \frac{a_1(R)-nR^{n-1}}{R^{n-2}} = 0, \tag{4.19}$$

where $a_1(R)$ is a positive root of the equation $Rf_R(a,\pi)=0$.

Finally, we perform the operations

(i) $R(C_{n+1}-C_2)$;
(ii) $R(C_n-C_1)$

on the determinant $g_R(a,0)$. We get

$$\lim_{R\to\infty} R^2 g_R(a,0) = n^{n-3}\begin{vmatrix} n & 0 & b & -(n-1) \\ 0 & n & -(n-1) & b \\ n & 0 & -b & (n-1) \\ 0 & n & (n-1) & -b \end{vmatrix} = 4n^{n-1}(b+n-1)(b-n+1), \tag{4.20}$$

which implies that

$$\lim_{R\to\infty} \frac{a_2(R)-nR^{n-1}}{R^{n-2}} = (n-1) \quad \text{or} \quad -(n-1), \tag{4.21}$$

where $a_2(R)$ is a positive root of the equation $R^2g_R(a,0)=0$. From the lemmas of Section 2, and from (4.19), (4.21), we conclude that

$$\lim_{R\to\infty} \frac{\varphi_{n,1}(R)-nR^{n-1}}{R^{n-2}} = -(n-1). \tag{4.22}$$

5. OPEN PROBLEMS

Empirical computations indicate that the asymptotic result of Theorem 1.1 can be written in a more precise form. The Taylor's expansion of $\varphi_{n,0}(R)$, as a function of R, about the point at infinity, seems to be

$$\begin{aligned} \varphi_{n,0}(R) = nR^{n-1} &- \frac{(n-1)^2}{n}R^{n-3} - \frac{1}{n^3}R^{n-5} - \frac{2(n^2+1)}{n^5}R^{n-7} \\ &- \frac{(n^2+1)(3n^2+5)}{n^7}R^{n-9} - \frac{2(n^2+1)(2n^4+8n^2+7)}{n^9}R^{n-11} + \cdots \end{aligned} \tag{5.1}$$

for $n \geq 11$ (for $n \geq 9$ if we stop with the term in $R^{n-9}, \dots$).

Similarly, we should have the following improvement of (1.10):

$$\varphi_{n,1}(R) = nR^{n-1} - (n-1)R^{n-2} - \frac{(n-1)^2}{2n}R^{n-3} + \frac{(n-1)(n-2)}{n}R^{n-4} + \dots \tag{5.2}$$

for $n \geq 6$ (for $n \geq 5$ if we stop with the term in R^{n-3}, ...).

Also, it would be interesting to determine the asymptotic behavior of the best possible constant $\varphi_{n,k}(R)$, $R \to \infty$, $k = 2, 3, \dots$, in the inequality

$$\|zp'(Rz)\| + \varphi_{n,k}(R)|a_k| \leq nR^{n-1}\|p\|. \tag{5.3}$$

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