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HARDY TYPE INEQUALITIES VIA CONVEXITY - THE JOURNEY SO FAR

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ABSTRACT. It is nowadays well-known that Hardy's inequality (like many other inequalities) follows directly from Jensen's inequality. Most of the development of Hardy type inequalities has not used this simple fact, which obviously was unknown by Hardy himself and many others. Here we report on some results obtained in this way mostly after 2002 by mainly using this fundamental idea.

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1. INTRODUCTION

The research into what is today called the classical Hardy inequality actually began in 1915 in an attempt by Hardy to find a new and more elementary proof of Hilbert's inequality. In the process Hardy [11] in a note published in 1920 stated that:

If $p > 1$ and $\{a_k\}_{k=1}^{\infty}$ is a sequence of nonnegative real numbers, then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{\infty} a_k\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p; \tag{1.1}$$

and announced (without proof) that if $p > 1$ and f is a nonnegative $p-$integrable function on $(0, \infty)$, then f is integrable over the interval $(0, x)$ for each positive x and that

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f^p(x)dx. \tag{1.2}$$

Obviously, (1.2) implies (1.1). Inequality (1.2) is usually called the *classical Hardy inequality* while inequality (1.1) is its discrete analogue. Nowadays a well-known simple fact is that (1.2) can equivalently (via the substitution $f(x) = h(x^{1-\frac{1}{p}})x^{-\frac{1}{p}}$), be rewritten in the form

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x h(t)dt\right)^p \frac{dx}{x} \le \int_0^{\infty} h^p(x)\frac{dx}{x} \tag{1.3}$$

and in this form it even holds with equality when $p = 1$. In this form we see that Hardy's inequality is a simple consequence of Jensen's inequality but this was not discovered in the dramatic period when Hardy discovered and finally proved inequality (1.2) in his famous paper [12] from 1925 (see [24] and [25]).

It is interesting to note that inequality (1.3) holds also for $p < 0$. This observation was first pointed out by Beesack and Heinig [2] (for further historical remarks see [25]). Moreover, in 1928 Hardy [13] (see also [14]) proved a generalized form of (1.2), namely

$$\int_0^{\infty} x^{-k}\left(\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{k-1}\right)^p \int_0^{\infty} x^{p-k} f^p(x)dx \ (p \ge 1, k > 1) \tag{1.4}$$

and also the dual form of this inequality

$$\int_0^{\infty} x^{-k}\left(\int_x^{\infty} f(t)dt\right)^p dx \le \left(\frac{p}{1-k}\right)^p \int_0^{\infty} x^{p-k} f^p(x)dx \ (p \ge 1, k < 1). \tag{1.5}$$

Over the last decades, many generalizations and refinements of (1.4) and (1.5) have been discovered and rediscovered (see e.g. [25], [26], [30], [31], [36], [37] and the references cited therein). For example, in 1971 Shum [37] obtained the following refinements of (1.4) and (1.5):

$$\int_0^{b} x^{-k}\left(\int_0^x f(t)dt\right)^p dx + \frac{p}{k-1}b^{1-k}\left(\int_0^b f(t)dt\right)^p \le \left(\frac{p}{k-1}\right)^p \int_0^{b} x^{p-k} f^p(x)dx \tag{1.6}$$

for $p \ge 1$, $k > 1$, $0 < b \le \infty$, and

$$\int_b^{\infty} x^{-k}\left(\int_x^{\infty} f(t)dt\right)^p dx + \frac{p}{1-k}b^{1-k}\left(\int_b^{\infty} f(t)dt\right)^p \le \left(\frac{p}{1-k}\right)^p \int_b^{\infty} x^{p-k} f^p(x)dx \tag{1.7}$$

for $p \geq 1$, $k < 1$, $0 \leq b < \infty$.

Our aim in this paper is to give a survey of Hardy's inequalities via the use of convexity argument. Hardy himself did obviously not discover this simple and natural argument even if Jensen's inequality (proved in 1906) was of course known to him. We strongly believe that if so both the history and the prehistory of what is today called Hardy type inequalities would had changed in a dramatic way. See our Section 5 for a further discussion on this topic, which is important for our paper.

2. HARDY TYPE INEQUALITIES

In 1965, Godunova [8] while studying inequalities with convex functions, initiated a simple direct way of obtaining Hardy's inequality via a convexity argument by proving that for an arbitrary function $\varphi(x)$ such that $\varphi^{-1}(x)$ is convex the following inequality holds:

$$\int_0^\infty \psi(x)\varphi^{-1}\left(\int_0^x k(x,\xi)\varphi(f(\xi)d\xi\right)dx \leq C\int_0^\infty f(x)dx, \tag{2.1}$$

where $k(x,\xi)$ and $\psi(x)$ are weight and kernel functions and C is a constant independent of the function $\varphi^{-1}(x)$. In fact, a direct application of Jensen's inequality

$$\varphi^{-1}\left(\int_0^x k(x,\xi)\varphi(f(\xi)d\xi\right)dx \leq \int_0^x k(x,\xi)f(\xi)d\xi$$

and Fubini's theorem show that the left hand side of (2.1) yields

$$\begin{aligned}\int_0^\infty \psi(x)\varphi^{-1}\left(\int_0^x k(x,\xi)\varphi(f(\xi)d\xi\right)dx &\leq \int_0^\infty \psi(x)\int_0^x k(x,\xi)f(\xi)d\xi dx \\ &= \int_0^\infty f(\xi)\int_\xi^\infty \psi(x)k(x,\xi)dxd\xi.\end{aligned}$$

By using this simple technique we in particular obtain the following useful result:

Theorem 2.1. *Let ϕ be a positive and convex function on $(0,\infty)$. Then*

$$\int_0^\infty \phi\left(\frac{1}{x}\int_0^x g(t)dt\right)\frac{dx}{x} \leq \int_0^\infty \phi(g(x))\frac{dx}{x}. \tag{2.2}$$

Remark 2.1. By choosing $\phi(x) = x^p$ inequality (2.2) yields Hardy's inequality in the following form

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(t)dt\right)^p\frac{dx}{x} \leq \int_0^\infty g^p(x)\frac{dx}{x}, \quad p \geq 1. \tag{2.3}$$

By using the substitution $f(x) = g(x^{\frac{p-1}{p}})x^{-\frac{1}{p}}$ in (2.3) with $p > 1$ yields the classical Hardy's inequality (1.2).

Remark 2.2. This result by Godunova seems to be fairly little referred to and almost unknown in the western literature. It was rediscovered in 2002 in the paper [20] by Kaijser et al. This was the starting point of a new development of the subject. For example, most of the results reported in this paper are influenced by Theorem 2.1 and Remark 2.1.

Remark 2.3. We also note that (2.2) also directly implies other classical inequalities. For example by using it with $\phi(u) = u^p$, replacing $g(x)$ by $\ln g(x)$ and making the substitution $h(x) = \frac{g(x)}{x}$ we obtain that

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln h(t)dt\right) dx \le e \int_0^\infty h(x)dx. \tag{2.4}$$

This is the classical Pólya-Knopp's inequality, which sometimes is called Knopp's inequality [21] but from the literature it is obvious that Pólya knew it before (from around 1925). Moreover, by restricting to step functions (2.4) implies another classical inequality, namely the famous Carleman inequality

$$\sum_{n=1}^\infty \sqrt{a_1 ... a_n} \le e \sum_{n=1}^\infty a_n, \tag{2.5}$$

from 1922, see [3]. Moreover, it is easy to see that (2.4) and (2.5) are limiting cases of (1.2) and (1.1), respectively, as $p \to \infty$.

The above remarks means that the simply proved inequality (1.3) by convexity argument in fact directly implies (1.1), (1.2), (2.4) and (2.5).

We also mention that in 1964, Levinson [27] proved that if Φ is a twice differenciable convex increasing function on $[0, \infty)$ with $\Phi(t)\Phi''(t) \ge \left(1 - \frac{1}{p}\right) \Phi'(t)^2$ for all $t > 0$, then

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x |f(t)|\, dt\right) dx \le (p')^p \int_0^\infty \Phi\left(|f(t)|\right) dx.$$

The above estimate is a consequence of the Jensen inequality used with the convex function $\Psi(u) = \Phi(u)^{\frac{1}{p}}$ and the classical Hardy inequality (1.2) used with the function $\Psi(|x)|)$ (cf. [26]). In 1999, Heinig [15] obtained the weighted extension of Levinson's result [27] by showing the following result:

Theorem 2.2. *(i) If $\varphi \in \Phi_p$, $p > 1$, and $\sup_{r>0} \left(\int_r^\infty \frac{u(t)}{t^p} dt\right)^{\frac{1}{p}} \left(\int_0^r v(t)^{1-p'} dt\right)^{\frac{1}{p'}} < \infty$, then*

$$\int_0^\infty u(x)\varphi\left(\frac{1}{x}\int_0^x f(t)dt\right) dx \le C \int_0^\infty v(x)\varphi\left(f(x)\right) dx. \tag{2.6}$$

(ii) If $\varphi \in \Phi$, and $v(x) = x^\alpha \int_x^\infty \frac{u(t)}{t^{\alpha+1}} dt$, $\alpha > 0$, then (2.6) holds with $C = e^\alpha$.

Moreover, by mainly using a convexity argument, Imoru [16] gave another proof of a generalized form of (1.6)-(1.7). In particular, Imoru [16] established the following result:

Theorem 2.3. *Let g be continuous and nondecreasing on $[0,\infty]$ with $g(0) = 0$, $g(x) > 0$ for $x > 0$ and $g(\infty) = \infty$. If $p \geq 1$, $k \neq 1$ and $f(x)$ is nonnegative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[0,b]$ or $[b,\infty]$ according to if $k > 1$ or $k < 1$. Then*

$$\int_0^b g(x)^{-k}\left(\int_0^x f(t)dg(t)\right)^p dg(x) + \frac{p}{k-1}g(b)^{1-k}\left(\int_0^b f(t)dg(t)\right)^p$$
$$\leq \left(\frac{p}{k-1}\right)^p \int_0^b g(x)^{p-k} f^p(x)dg(x) \tag{2.7}$$

for $p \geq 1$, $k > 1$, $0 < b \leq \infty$, and

$$\int_b^\infty g(x)^{-k}\left(\int_x^\infty f(t)dg(t)\right)^p dg(x) + \frac{p}{1-k}g(b)^{1-k}\left(\int_b^\infty f(t)dg(t)\right)^p$$
$$\leq \left(\frac{p}{1-k}\right)^p \int_b^\infty g(x)^{p-k} f^p(x)dg(x) \tag{2.8}$$

for $p \geq 1$, $k < 1$, $0 \leq b < \infty$. The inequalities sign in (2.7)-(2.8) are reversed if $0 < p \leq 1$, furthermore the constant on the right hand sides of (2.7) or (2.8) is the best possible.

In a recent paper, Persson and Oguntuase [39] presented another elementary proof of (1.6)-(1.7) (or more generally (2.7)-(2.8)) using Hölder's and reversed Hölder's inequalities, which are consequences of Jensen's inequality. In particular, Persson and Oguntuase [39] established the following results:

Theorem 2.4. *Let g be continuous and nondecreasing on $[0,\infty]$ with $g(0) = 0$, $g(x) > 0$ for $x > 0$ and $g(\infty) = \infty$. Let $p, k, b \in \mathbb{R}$, where $0 < b \leq \infty$ and such that one of the following holds:*

(i) $p \geq 1$ *and* $k > 1$,

(ii) $p < 0$ *and* $k < 1$.

If $f(x)$ is a nonnegative integrable function on $[0,b]$ such that

$$0 < \int_0^b g(x)^{p-k} f^p(x)dg(x) < \infty,$$

then

$$\int_0^b g(x)^{-k}\left(\int_0^x f(t)dg(t)\right)^p dg(x) + \frac{p}{k-1}g(b)^{1-k}\left(\int_0^b f(t)dg(t)\right)^p$$
$$\leq \left(\frac{p}{k-1}\right)^p \int_0^b g(x)^{p-k} f^p(x)dg(x). \tag{2.9}$$

(iii) *If $0 < p \leq 1$ and $k > 1$, then inequality (2.9) holds in the reversed direction.*

The constant $\left(\frac{p}{k-1}\right)^p$ on the right hand side of (2.9) is the best possible in all cases.

Theorem 2.5. *Let g be continuous and nondecreasing on $[0,\infty]$ with $g(0)=0$, $g(x)>0$ for $x>0$ and $g(\infty)=\infty$. Let $p,k,b\in\mathbb{R}$ with $0<b\leq\infty$ and such that one of the following holds:*

(iv) $p\geq 1$ *and* $k<1$,

(v) $p<0$ *and* $k>1$.

If $f(x)$ is a nonnegative integrable function on $[b,\infty]$ such that

$$0<\int_b^\infty g(x)^{p-k}f^p(x)dg(x)<\infty,$$

then

$$\int_b^\infty g(x)^{-k}\left(\int_x^\infty f(t)dg(t)\right)^p dg(x)+\frac{p}{1-k}g(b)^{1-k}\left(\int_b^\infty f(t)dg(t)\right)^p$$

$$\leq\left(\frac{p}{1-k}\right)^p\int_b^\infty g(x)^{p-k}f^p(x)dg(x). \tag{2.10}$$

(vi) *If $0<p\leq 1$ and $k<1$, then inequality (2.10) holds in the reversed direction.*
The constant $\left(\frac{p}{k-1}\right)^p$ on the right hand side of (2.10) is the best possible in all cases.

Remark 2.4. The simple proof in [39] shows that some versions of (1.6)-(1.7) (or more generally (2.7)-(2.8)) in fact holds also for $p<0$.

3. MULTIDIMENSIONAL HARDY TYPE INEQUALITIES

In 1968, Gudanova [9] gave the multidimensional analogue of the result in [8] by proving the following result:

Theorem 3.1. *Let $k(t_1,t_2,...,t_n)\geq 0$ for $0<t_i<\infty$, $i=1,2,...,n$;*

$$\int_{V_t}k(t_1,t_2,...,t_n)dV_t=1,$$

where V_t is a domain in the n-dimensional Eucleadian space such that $0<t_i<\infty$, $i=1,2,...,n$; V_x and V_y are defined analogously to V_t; $\varphi(u)$ is nonnegative convex function for $u\geq 0$; $k(y_1,y_2,...,y_n)\geq 0$ for $0<y_i<\infty$; $f\neq 0$; $\frac{\varphi(x_1,x_2,...,x_n)}{x_1,x_2,...,x_n}\in L_1(V_x)$, then

$$\int_{V_x}\frac{1}{x_1...x_n}\varphi\left(\frac{1}{x_1...x_n}\int_{V_y}k\left(\frac{y_1}{x_1},...,\frac{y_n}{x_n}\right)f(y_1,y_2,...,y_n)dV_y\right)dV_x$$

$$\leq\int_{V_x}\frac{\varphi(f(x_1,x_2,...,x_n))}{x_1...x_n}dV_x. \tag{3.1}$$

Proof. This follows almost directly by applying Jensen's inequality, Fubini's theorem and using the substitution $\frac{y_i}{x_i}=t_i,\ i=1,2,3,\cdots,n.$ ∎

Remark 3.1. We see that the case $n=1$ inequality (3.1) yields the classical Hardy's inequality (1.2) via (1.3).

In 1992, Pachpatte [38] obtained a natural n-dimensional generalization of the classical Hardy integral inequality (1.2) by using Fubini's theorem and Jensen's inequality. In particular, he proved that

$$\begin{aligned}&\int_0^\infty \dots \int_0^\infty \left(\prod_{i=1}^n x_i\right)^{-p} F^p(x_1, x_2, \dots, x_n)dx_1\dots dx_n\\ &\le \left(\frac{p}{p-1}\right)^{np}\int_0^\infty \dots \int_0^\infty f^p(x_1, x_2, \dots, x_n)dx_1\dots dx_n \end{aligned}\tag{3.2}$$

holds for $p > 1$ and all nonnegative function $f \in L^p(\mathbb{R}^n_+)$, where F is defined on $\mathbb{R}^n_+$ by

$$F(x_1, x_2, \dots, x_n) = \int_0^{x_1} \dots \int_0^{x_n} f(t_1, t_2, \dots, t_n)dt_1\dots dt_n,$$

and the constant $\left(\frac{p}{p-1}\right)^{np}$ is the best possible. The corresponding results for $p < 0$ and $0 < p < 1$ were recently obtained by Oguntuase *et al*. [31] as consequences of much more general inequalities for convex and concave functions (see also [30] for further details).

In 2005, Kaijser et. al. [19] used the notion of convexity to obtain the following multidimensional Hardy-type inequality:

Theorem 3.2. *Let $0 < b_i \le \infty$, $i = 1, 2, \dots, n$ $(n \in \mathbb{Z}^+)$, $-\infty \le a < c \le \infty$ and let Φ be a positive function on $[a, c]$, if Φ is convex, then*

$$\begin{aligned}&\int_0^{b_1} \dots \int_0^{b_n} \phi\left(\frac{1}{x_1\dots x_n}\int_0^{x_1} \dots \int_0^{x_n} f(t_1\dots t_n)dt_1\dots dt_n\right)\frac{dx_1\dots dx_n}{x_1\dots x_n}\\ &\le \int_0^{b_1} \dots \int_0^{b_n} \phi\left(f(x_1\dots x_n)\right)\left(1-\frac{x_1}{b_1}\right)\dots\left(1-\frac{x_n}{b_n}\right)\frac{dx_1\dots dx_n}{x_1\dots x_n},\end{aligned}\tag{3.3}$$

*for every function on **(0,b)** such that $a < f(\mathbf{x}) < c$. Furthermore, they showed that if Φ is concave, then the sign of (3.3) holds in the reversed direction.*

Remark 3.2. By choosing $\Phi(t) = t^p$ in the above result, some natural multidimensional forms of the classical and reversed Hardy type inequalities are obtained. Specifically, the following results were derived.

Corollary 3.3. *Let $0 < d_i \le \infty$, $i = 1, 2, \dots, n$ $(n \in \mathbb{Z}^+)$.*
(a) *If $p > 1$ or $p < 0$, then*

$$\begin{aligned}&\int_0^{d_1} \dots \int_0^{d_n} \left(\frac{1}{y_1\dots y_n}\int_0^{y_1} \dots \int_0^{y_n} g(s_1\dots s_n)ds_1\dots ds_n\right)^p dy_1\dots dy_n\\ &\le \left(\frac{p}{p-1}\right)^{pn}\int_0^{d_1} \dots \int_0^{d_n} g^p\left(y_1\dots y_n\right)\left(1-\left(\frac{y_1}{d_1}\right)^{\frac{p-1}{p}}\right)\dots\left(1-\left(\frac{y_n}{d_n}\right)^{\frac{p-1}{p}}\right)dy_1\dots dy_n,\end{aligned}$$

holds for each positive function g on $(\mathbf{0}, \mathbf{d})$.

(b) *If* $0 < p < 1$, *then*

$$\int_{d_1}^{\infty} \dots \int_{d_n}^{\infty} \left(\frac{1}{y_1 \dots y_n} \int_0^{y_1} \dots \int_0^{y_n} g(s_1 \dots s_n) ds_1 \dots ds_n \right)^p dy_1 \dots dy_n$$
$$\geq \left(\frac{p}{1-p} \right)^{pn} \int_{d_1}^{\infty} \dots \int_{d_n}^{\infty} g^p (y_1 \dots y_n) \left(1 - \left(\frac{d_1}{y_1} \right)^{\frac{1-p}{p}} \right) \dots \left(1 - \left(\frac{d_n}{y_n} \right)^{\frac{1-p}{p}} \right) dy_1 \dots dy_n,$$

for each positive function g *on* $(\mathbf{d}, \infty)$.

(c) *The constants in the inequalities above are sharp in all cases.*

Remark 3.3. By applying Corollary 3.3 (a) with $p > 1$ and $d_1 = d_2 = \cdots = d_n = \infty$ we obtain the inequality (3.2) by Pachpatte.

In a recent paper, Oguntuase *et al.* [32] established a class of more general integral multidimensional Hardy type inequalities for an almost everywhere positive function ϕ which is convex for $p > 1$ and $p < 0$, concave for $p \in (0, 1)$ and such that $Ax^p \leq \phi(x) \leq Bx^p$ holds on $\mathbb{R}_+$, for some positive constants $A \leq B$. Oguntuase *et al.* [32] obtained a class of general integral multidimensional Hardy type inequalities with power weights, whose left hand sides involve $\phi \left(\int_0^{x_1} \dots \int_0^{x_n} f(\mathbf{t}) d\mathbf{t} \right)$ instead of $\left(\int_0^{x_1} \dots \int_0^{x_n} f(\mathbf{t}) d\mathbf{t} \right)^p$, while the corresponding right hand sides remain as in the classical Hardy's inequality and have explicit constants in front of the integrals. In particular, for the case $p > 1$ the following strenghtened multidimensional Hardy type inequalities were obtained:

Theorem 3.4. *Let* $1 < p < \infty$ *and* $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{R}^n$ *be such that* $m_i \neq 1$, $i = 1, \dots, n$. *Let* $\phi : [0, \infty) \to \mathbb{R}$ *be a convex, almost everywhere positive function, such that* $Ax^p \leq \phi(x) \leq Bx^p$ *holds on* $[0, \infty)$ *for some constants* $0 < A \leq B < \infty$,

(i) *If* $\mathbf{b} \in (\mathbf{0}, \infty]$ *and* $\mathbf{m} > \mathbf{1}$, *then the inequality*

$$\int_0^{b_1} \dots \int_0^{b_n} \phi \left(\int_0^{x_1} \dots \int_0^{x_n} f(\mathbf{t}) d\mathbf{t} \right) \prod_{i=1}^n x_i^{-m_i} d\mathbf{x}$$
$$\leq \frac{B^2}{A} \left(\prod_{i=1}^n \frac{p}{m_i - 1} \right)^p \int_0^{b_1} \dots \int_0^{b_n} f^p(\mathbf{x}) d\mathbf{x} \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{x_i}{b_i} \right)^{\frac{m_i - 1}{p}} \right] d\mathbf{x} \tag{3.4}$$

holds for all nonnegative integrable functions $f : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$.

(ii) *If* $\mathbf{b} \in [\mathbf{0}, \infty)$ *and* $\mathbf{m} < \mathbf{1}$, *then the inequality*

$$\int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} \phi \left(\int_{x_1}^{\infty} \dots \int_{x_n}^{\infty} f(\mathbf{t}) d\mathbf{t} \right) \prod_{i=1}^n x_i^{-m_i} d\mathbf{x}$$
$$\leq \frac{B^2}{A} \left(\prod_{i=1}^n \frac{p}{1 - m_i} \right)^p \int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} f^p(\mathbf{x}) d\mathbf{x} \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{b_i}{x_i} \right)^{\frac{1 - m_i}{p}} \right] d\mathbf{x} \tag{3.5}$$

holds for all nonnegative integrable functions $f : (\mathbf{b}, \infty) \to \mathbb{R}$.

Furthermore, the related inequalities to those in Theorem 3.4 to the cases $p < 0$ and $0 < p < 1$ were also established in [32] as follows:

Theorem 3.5. *Suppose that* $-\infty < p < 0$ *and* $\mathbf{m} = (m_1, ..., m_n) \in \mathbb{R}^n$ *be such that* $m_i \neq 1$, $i = 1, ..., n$. *Let* $\phi : (0, \infty) \to \mathbb{R}$ *be a positive convex function, such that* $Ax^p \leq \phi(x) \leq Bx^p$ *holds on* $(0, \infty)$ *for some constants* $0 < A \leq B < \infty$. *If* $\mathbf{b} \in (\mathbf{0}, \infty]$ *and* $\mathbf{m} < \mathbf{1}$, *then inequality (3.4) holds for all positive integrable functions* f *on* $(\mathbf{0}, \mathbf{b})$. *If* $\mathbf{b} \in [\mathbf{0}, \infty)$ *and* $\mathbf{m} > \mathbf{1}$, *then the inequality (3.5) holds for all positive integrable functions* f *on* $(\mathbf{b}, \infty)$.

Theorem 3.6. *Let* $0 < p < 1$ *and* $\mathbf{m} = (m_1, ..., m_n) \in \mathbb{R}^n$ *be such that* $m_i \neq 1$, $i = 1, ..., n$. *Let* $\phi : [0, \infty) \to \mathbb{R}$ *be a convex, almost everywhere positive function, such that* $Bx^p \leq \phi(x) \leq Ax^p, x \in [0, \infty)$, *for some constants* $0 < B \leq A < \infty$. *For a function* f *let* F *be defined by*

$$F(\mathbf{x}) = \begin{cases} \int_0^{x_1} \ldots \int_0^{x_n} f(\mathbf{t}) d\mathbf{t} & \mathbf{m} > \mathbf{1} \\ \int_{x_1}^{\infty} \ldots \int_{x_n}^{\infty} f(\mathbf{t}) d\mathbf{t} & \mathbf{m} < \mathbf{1}. \end{cases}$$

(i) *If* $\mathbf{b} \in (\mathbf{0}, \infty]$ *and* $\mathbf{m} > \mathbf{1}$, *then*

$$\begin{aligned} &\int_0^{b_1} \ldots \int_0^{b_n} \phi\left(F(\mathbf{x})\right) \prod_{i=1}^n x_i^{-m_i} d\mathbf{x} \\ &\geq \frac{B^2}{A} \left(\prod_{i=1}^n \frac{p}{m_i - 1} \right)^p \int_0^{b_1} \ldots \int_0^{b_n} f^p(\mathbf{x}) d\mathbf{x} \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{x_i}{b_i} \right)^{\frac{m_i - 1}{p}} \right] d\mathbf{x} \end{aligned}$$

holds for all nonnegative integrable functions $f : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$.

(ii) *If* $\mathbf{b} \in [\mathbf{0}, \infty)$ *and* $\mathbf{m} < \mathbf{1}$, *then*

$$\begin{aligned} &\int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} \phi\left(F(\mathbf{x})\right) \prod_{i=1}^n x_i^{-m_i} d\mathbf{x} \\ &\geq \frac{B^2}{A} \left(\prod_{i=1}^n \frac{p}{1 - m_i} \right)^p \int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} f^p(\mathbf{x}) d\mathbf{x} \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{b_i}{x_i} \right)^{\frac{1 - m_i}{p}} \right] d\mathbf{x} \end{aligned}$$

4. Hardy Type Inequalities with General Kernels

As mentioned the first author which derived a Hardy type inequality with a general kernel was Godunova [8] (see also [9]). In a recent paper, Kaijser et. al. [19] obtained some new integral inequalities with general integral operators (without additional restrictions on the kernel). Specifically, the following results were stated and proved:

Theorem 4.1. *Let* u *be a weight function on* $(0, b)$, $0 < b \leq \infty$, *and let* $k(x, y) \geq 0$ *on* $(0, b) \times (0, b)$. *Assume that* $\frac{k(x,y)u(x)}{xK(x)}$ *is locally integrable on* $(0, b)$ *for each fixed* $y \in (0, b)$ *and define* v *by*

$$v(y) = y \int_y^b \frac{k(x, y)}{K(x)} u(x) \frac{dx}{x} < \infty, \quad y \in (0, b).$$

If Φ is a positive and convex function on (a,c), $-\infty \leq a < c \leq \infty$, then

$$\int_0^b \Phi(A_k f(x))u(x)\frac{dx}{x} \leq \int_0^b \Phi(f(x))v(x)\frac{dx}{x},$$

for all f with $a < f(x) < c$, $0 \leq x \leq b$, where

$$A_k f(x) = \frac{1}{K(x)}\int_0^x k(x,y)f(y)dy, \quad K(x) = \int_0^x k(x,y)dy < \infty. \tag{4.1}$$

In the same paper the dual operator $A_k^\star$, defined by

$$A_k^\star f(x) = \frac{1}{\widetilde{K}(x)}\int_x^\infty k(x,y)f(y)dy, \tag{4.2}$$

where $\widetilde{K}(x) = \int_x^\infty k(x,y)dy < \infty$ was studied and the following result was proved:

Theorem 4.2. *For $0 \leq b < \infty$, let u be a weight function such that $\frac{k(x,y)u(x)}{x\widetilde{K}(x)}$ is locally integrable on (b,∞) for each fixed $y \in (b,\infty)$. Let the function v be defined by*

$$v(y) = y\int_b^y \frac{k(x,y)}{\widetilde{K}(x)}u(x)\frac{dx}{x} < \infty, \quad y \in (b,\infty).$$

If Φ is a positive and convex function on (a,c), $-\infty \leq a < c \leq \infty$, then

$$\int_b^\infty \Phi(A_k^\star f(x))u(x)\frac{dx}{x} \leq \int_b^\infty \Phi(f(x))v(x)\frac{dx}{x},$$

for all f with $a < f(x) < c$, $0 \leq x \leq b$, where $A_k^\star$ is defined by (4.2).

The result for the operator A_k which involves the cases $p \neq q$ without additional restrictions on the kernel was derived by Kaijser et. al. [19] as follows:

Theorem 4.3. *Let $1 < p \leq q < \infty$, $0 < b \leq \infty$, $s \in (1,p)$, and let Φ be a convex and strictly monotone function on (a,c), $-\infty \leq a < c \leq \infty$. Let A_K be the general Hardy operator defined by (4.1). Then the inequality*

$$\left(\int_0^b [\Phi(A_K f(x))]^q u(x)\frac{dx}{x}\right)^{\frac{1}{q}} \leq C\left(\int_0^b \Phi^p(f(x))v(x)\frac{dx}{x}\right)^{\frac{1}{p}} \tag{4.3}$$

holds for all functions $f(x)$, $a < f(x) < c$, if

$$A(s) := \sup_{0<t<b}\left(\int_t^b \left[\frac{k(x,t)}{K(x)}\right]^q u(x)V(x)^{\frac{q(p-s)}{p}}\frac{dx}{x}\right)^{\frac{1}{q}} V(t)^{\frac{s-1}{p}} < \infty$$

holds, where

$$V(t)=\int_0^t \frac{v^{1-p'}(x)}{x^{1-p'}}dx.$$

Moreover, if C *is the best possible constant in (4.3), then*

$$C\le \inf_{1<s<p}\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}A(s).$$

In 2008, Oguntuase *et. al*. [34] obtained some new multidimensional Hardy-type inequalities involving arithmetic mean operators with general positive kernels using mainly a convexity argument. The results obtained improved some known results in the literature and in particular, some recent results of Kaijser et. al. [19] are generalized and complemented. By defining the general Hardy-type (arithmetic mean) operators A_K and its dual A_{K^*} as follows:

$$A_Kf(x_1,...,x_n):=\frac{1}{K(x_1,...,x_n)}\int_0^{x_1}...\int_0^{x_n}k(x_1,...,x_n,t_1,...,t_n)f(t_1,...,t_n)dt_1...dt_n \tag{4.4}$$

and

$$A_{K^*}f(x_1,...,x_n):=\frac{1}{\widetilde{K}(x_1,...,x_n)}\int_{x_1}^{\infty}...\int_{x_n}^{\infty}k(x_1,...,x_n,t_1,...,t_n)f(t_1,...,t_n)dt_1...dt_n \tag{4.5}$$

with $K(x_1,...,x_n)$ and $\widetilde{K}(x_1,...,x_n)$ given by

$$K(x_1,...,x_n):=\int_0^{x_1}...\int_0^{x_n}k(x_1,...,x_n,t_1,...,t_n)dt_1...dt_n \tag{4.6}$$

and

$$\widetilde{K}(x_1,...,x_n):=\int_{x_1}^{\infty}...\int_{x_n}^{\infty}k(x_1,...,x_n,t_1,...,t_n)dt_1...dt_n, \tag{4.7}$$

then Oguntuase *et. al*. [34] obtained the following results:

Theorem 4.4. *Let* $n\in\mathbb{N}_+$, $k(x_1,...,x_n,t_1,...,t_n)$ *and* $u(x_1,...,x_n)$ *be weight functions and assume that*

$$\frac{k(x_1,...,x_n,t_1,...,t_n)u(x_1,...,x_n)}{x_1...x_nK(x_1,...,x_n)}$$

is locally integrable and for each $(t_1,...,t_n)$, $t_i\in(0,b_i)$, *define* v *by*

$$v(x_1,...,x_n)=t_1...t_n\int_{t_1}^{b_1}...\int_{t_n}^{b_n}\frac{k(x_{1,}...,x_n,t_1,...,t_n)u(x_1,...,x_n)}{K(x_1,...,x_n)}\frac{dx_1...dx_n}{x_1...x_n}<\infty.$$

(i) *If* Φ *is positive and convex on* (a,c), $-\infty\le a<c\le\infty$, *then*

$$\int_0^{b_1} \dots \int_0^{b_n} \Phi\left(A_K f(x_1,...,x_n)\right) u(x_1,...,x_n) \frac{dx_1...dx_n}{x_1...x_n}$$
$$\leq \int_0^{b_1} \dots \int_0^{b_n} \Phi\left(f(x_1,...,x_n)\right) v(x_1,...,x_n) \frac{dx_1...dx_n}{x_1...x_n}$$

for all f with $a < f(x_1,...,x_n) < c$, $0 \leq x_i \leq b_i$, $i = 1,2,...,n$.

(ii) *If Φ is positive and concave on (a,c), $-\infty \leq a < c \leq \infty$, then*

$$\int_0^{b_1} \dots \int_0^{b_n} \Phi\left(A_K f(x_1,...,x_n)\right) u(x_1,...,x_n) \frac{dx_1...dx_n}{x_1...x_n}$$
$$\geq \int_0^{b_1} \dots \int_0^{b_n} \Phi\left(f(x_1,...,x_n)\right) v(x_1,...,x_n) \frac{dx_1...dx_n}{x_1...x_n}$$

for all f with $a < f(x_1,...,x_n) < c$, $0 \leq x_i \leq b_i$, $i = 1,2,...,n$. Here A_K and $K(x_1,...,x_n)$ are as defined by (4.4) and (4.6), respectively.

Furthermore, in the same paper Oguntuase *et. al.* [34] also obtained the dual of Theorem 4.4 as follows:

Theorem 4.5. *Let $n \in \mathbb{N}_+$, $0 \leq b_i < x_i, t_i \leq \infty$, $i = 1,2,...,n$, and let $k(x_1...x_n, t_1,...,t_n)$ and $u(x_1,...,x_n)$ be weight functions such that*

$$\frac{k(x_1,...,x_n,t_1,...,t_n)u(x_1,...,x_n)}{x_1...x_n\widetilde{K}(x_1,...,x_n)}$$

is locally integrable and for each $(t_1,...,t_n)$, $t_i \in (b_i,\infty)$, define v by

$$v(x_1,...,x_n) = t_1...t_1 \int_{b_1}^{t_1} \dots \int_{b_n}^{t_n} \frac{k(x_1,...,x_n,t_1,...,t_n)u(x_1,...,x_n)}{\widetilde{K}(x_1,...,x_n)} \frac{dx_1...dx_n}{x_1...x_n} < \infty.$$

(i) *If Φ is positive and convex on (a,c), $-\infty \leq a < c \leq \infty$ and A_{K^*} is the general dual Hardy operator defined by (4.5), then the inequality*

$$\int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} \Phi\left(A_{K^*} f(x_1,...,x_n)\right) u(x_1,...,x_n) \frac{dx_1...dx_n}{x_1...x_n}$$
$$\leq \int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} \Phi\left(f(x_1,...,x_n)\right) v(x_1,...,x_n) \frac{dx_1...dx_n}{x_1...x_n}$$

holds for all f with $a < f(x_1,...,x_n) < c$, $0 \leq x_i \leq b_i$, $i = 1,2,...,n$.

(ii) *If Φ is positive and concave on (a,c), $-\infty \leq a < c \leq \infty$ and A_{K^*} is the general dual Hardy operator defined by (4.5), then the inequality*

$$\int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} \Phi\left(A_{K^*} f(x_1, \dots, x_n)\right) u(x_1, \dots, x_n) \frac{dx_1 \dots dx_n}{x_1 \dots x_n}$$
$$\geq \int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} \Phi\left(f(x_1, \dots, x_n)\right) v(x_1, \dots, x_n) \frac{dx_1 \dots dx_n}{x_1 \dots x_n}$$

holds for all f with $a < f(x_1, ..., x_n) < c$, $0 \leq x_i \leq b_i$, $i = 1, 2, ..., n$, and $\widetilde{K}(x_1, ..., x_n)$ is as defined by (4.7).

Moreover, for the case $1 < p \leq q < \infty$ the following generalization of the Kaijser et. al. [19] result was proved in [34]:

Theorem 4.6. *Let $1 < p \leq q < \infty$, $0 < b_i \leq \infty$, $s_1, ..., s_n \in (1, p)$, $i = 1, 2, ..., n$ and let Φ be a convex function on (a, c), $-\infty \leq a < c \leq \infty$. Let A_K be the general Hardy operator defined by (4.4) and let $u(x_1, ..., x_n)$ and $v(x_1, ..., x_n)$ be weight functions, where $v(x_1, ..., x_n)$ is of product type i.e. $v(x_1, ..., x_n) = v(x_1).v(x_2)...v(x_n)$. Then the inequality*

$$\left(\int_0^{b_1} \dots \int_0^{b_n} \left[\Phi(A_K f(x_1, \dots, x_n))\right]^q u(x_1, \dots, x_n) \frac{dx_1 \dots dx_n}{x_1 \dots x_n}\right)^{\frac{1}{q}}$$
$$\leq C \left(\int_0^{b_1} \dots \int_0^{b_n} \Phi^p(f(x_1, \dots, x_n)) v(x_1, \dots, x_n) \frac{dx_1 \dots dx_n}{x_1 \dots x_n}\right)^{\frac{1}{p}} \tag{4.8}$$

holds for all functions $f(x_1, ..., x_n)$, $a < f(x_1, ..., x_n) < c$, if

$$A(s_1, \dots, s_n) := \sup_{0 < t_1 \dots t_n < b_1 \dots b_n} \left(\int_{t_1}^{b_1} \dots \int_{t_n}^{b_n} \left[\frac{k(x_1, \dots, x_n, y_1, \dots, y_n)}{K(x_1, \dots, x_n)}\right]^q u(x_1, \dots, x_n)\right.$$
$$\left. \cdot V_1^{\frac{q(p-s_1)}{p}}(x_1) \dots V_n^{\frac{q(p-s_n)}{p}}(x_n) \frac{dx_1 \dots dx_n}{x_1 \dots x_n}\right)^{\frac{1}{q}} V_1^{\frac{s_1-1}{p}}(t_1) \dots V_n^{\frac{s_n-1}{p}}(t_n) < \infty$$

holds, where

$$V_i(x_i) = \int_0^{x_i} v^{1-p'} dt_i, \ i = 1, 2, \dots, n,$$

and $p' = \frac{p}{p-1}$. Furthermore, if C is the best possible constant in (4.8), then

$$C \leq \inf_{1 < s_1 \dots s_n < p} \left(\frac{p-1}{p-s_1}\right)^{\frac{1}{p'}} \dots \left(\frac{p-1}{p-s_n}\right)^{\frac{1}{p'}} A(s_1, \dots, s_n).$$

Remark 4.1. For the case $n = 1$ (4.8) coincides with inequality (4.3) obtained by Kaijser et. al. in [19].

5. FURTHER RESULTS AND CONCLUDING REMARKS

Remark 5.1. In this paper we have pronounced that Jensen's inequality implies Hardy's inequality. We also pronounce that Jensen's inequality implies several of the classical inequalities e.g. those by Hölder, Young, Minkowski, Beckenbach-Dresher, Hilbert, Levin, Hardy-Littlewood-Pólya and the AG-mean inequality or, more generally, that the scale of Power means P_α or even general Gini means $G_{\alpha,\beta}$ increases in both α and β $(-\infty < \alpha, \beta < \infty)$. See the book of Niculescu-Persson [29] and the references given there.

Here we just give a simple proof of the fact that Jensen's inequality directly implies Hölder inequality

$$\int_\Omega |fg|\, d\mu \le \left(\int_\Omega |f|^p\, d\mu\right)^{\frac{1}{p}} \left(\int_\Omega |g|^q\, d\mu\right)^{\frac{1}{q}},\ \frac{1}{p}+\frac{1}{q}=1,\ p>1, \tag{5.1}$$

without going via Young's inequality as is usually done in most textbooks, however c.f. e.g. [28, p. 39-40].

Proof. Without loss of generality we assume that $0 < \int_\Omega |g|^q\, d\mu < \infty$ and put $g_1 = |g|^q$ and $f_1 = |f|\,|g|^{-\frac{1}{p-1}}$. Then, by Jensen's inequality applied with the convex function $\Phi(u) = u^p$, we obtain that

$$\left(\frac{1}{\int_\Omega g_1 d\mu}\int_\Omega f_1 g_1 d\mu\right)^p \le \frac{1}{\int_\Omega g d\mu}\int_\Omega f_1^p g_1 d\mu, \tag{5.2}$$

i.e.,

$$\int_\Omega f_1 g_1 d\mu \le \left(\int_\Omega f_1^p g_1 d\mu\right)^{\frac{1}{p}} \left(\int_\Omega g_1 d\mu\right)^{\frac{1}{q}} \tag{5.3}$$

which is (5.1). ∎

Note that our proof above also directly implies that Hölder's inequality holds in the reverse direction when $0 < p < 1$, which follows from the fact that the function $\Phi(u) = u^p$ is convex also for $p < 0$ so that (5.2) holds also for $p < 0$. This means that (5.3) and, thus, (5.1) holds in the reverse direction for $p < 0$, i.e. for $0 < q < 1$.

Next we give some historical remarks.

Remark 5.2. The almost dramatic period 1915-1925 until Hardy finally stated and proved his inequality (1.1) was recently described in detail in [24]. Obviously, the prehistory would had been completely changed if Hardy (or some collaborators from this time) had discovered that (1.1) can equivalently be rewritten in the form (1.3) and this inequality follows directly from Jensen's inequality (and Fubini's theorem). And Jensen's inequality from 1906 (see [18] was of course known to Hardy.

Remark 5.3. Also in the 1928 paper [13] Hardy was not aware of this simple technique since in the same way we can see that also (1.4) (=the first weighted version of Hardy's inequality) is in fact equivalent to (1.3). Just consider the relation

$$f(x) = h(x^{\frac{k-1}{p}})x^{\frac{k-1}{p}-1}$$

and make the obvious calculations. In particular, this means that in fact (1.4) is not more general than (1.1).

Remark 5.4. The starting point to begin to develop weighted Hardy's inequality to what is today called Hardy type inequalities was just (1.4). The history concerning the development of this theory was recently described in the book [25]. This development has not been so much influenced by the simple convexity arguments discussed in this paper, see e.g. the books [14], [25], [26], [22] and the references given there. We strongly beleive that severaral proofs also in these more general cases can be simplified by using this powerful convexity technique.

Next, we mention that in the cases we have considered also some generalizations with general measures can be done. In fact, very recently, Krulić et. al. [23] obtained some new weighted Hardy type inequalities with an integral operator A_k defined by

$$A_k f(x) = \frac{1}{K(x)} \int\limits_{\Omega_2} k(x,y) f(y) d\mu_2(y), \tag{5.4}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a general nonnegative kernel, (Ω_1, μ_1) and (Ω_2, μ_2) are measure spaces and

$$K(x) = \int\limits_{\Omega_2} k(x,y) d\mu_2(y), \ x \in \Omega_1. \tag{5.5}$$

Specifically, they obtained the following results:

Theorem 5.1. *Let u be a weight function, $k(x,y) \geq 0$. Assume that $\frac{k(x,y)}{K(x)} u(x)$ is locally integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v by*

$$v(y) = \int\limits_{\Omega_1} \frac{k(x,y)}{K(x)} u(x) d\mu_1(x) < \infty.$$

If Φ is convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int\limits_{\Omega_1} \Phi(A_k f(x)) u(x) d\mu_1(x) \leq \int\limits_{\Omega_2} \Phi(f(y) v(y) d\mu_2(y)$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $\operatorname{Im} f \subseteq I$, where A_k is defined by (5.4) and (5.5).

Krulić et. al. [23] further obtained a more general result than Theorem 4.3 of Kaijser et. al. [19]. More precisely, they obtained the following generalization of Theorem 5.1 as follows:

Theorem 5.2. *Let $0 < p \leq q < \infty$ and let the assumptions in Theorem 4.3 hold but now with*

$$v(y) = \left(\int\limits_{\Omega_1} \left[\frac{k(x,y)}{K(x)} \right]^{\frac{q}{p}} u(x) d\mu_1(x) \right)^{\frac{p}{q}} < \infty.$$

If Φ is a positive convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int\limits_{\Omega_1} \left[\Phi(A_k f(x)) \right]^{\frac{q}{p}} u(x) d\mu_1(x) \right)^{\frac{1}{q}} \leq \left(\int\limits_{\Omega_2} \Phi(f(y) v(y) d\mu_2(y) \right)^{\frac{1}{p}}$$

holds for all measurable function $f : \Omega_2 \to \mathbb{R}$, such that $\operatorname{Im} f \subseteq I$.

Proof. Follows almost directly by using Jensen's inequality and Minkowski's general integral inequality. ∎

Remark 5.5. Observe that the case $p = q$ yields Theorem 4.3 obtained by Kaijser et. al. [19].

Another type of results can be obtained by using this technique but instead of convex functions by using some closely related class of functions. Here we shall use a class of functions recently introduced by Abramovich, Jameson and Sinnamon [1] (see also [33]).

Definition 5.1. A function $\varphi : [0, \infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi\left(|y - x|\right) \geq C_x \left(y - x\right) \quad \text{for all } y \geq 0.$$

We say that f is subquadratic if $-f$ is superquadratic.

We also state the following refinement of Jensen's inequality.(see [1]):

Lemma 5.3. *Let (Ω, μ) be a probability measure space. Then the inequality*

$$\varphi\left(\int_\Omega f(s)d\mu(s)\right) \leq \int_\Omega \varphi(f(s))d\mu(s) - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)d\mu(s)\right|\right) d\mu(s) \tag{5.6}$$

holds for all probability measures μ and all nonnegative $\mu-$integrable functions f if and only if φ is superquadratic. Moreover, (5.6) holds in the reversed direction if and only if φ is subquadratic.

In fact, since this refined version of Jensen's inequality holds for superquadratic functions, so by using the technique presented in this paper, we can expect to obtain some new refined Hardy type inequalities. In fact, recently Oguntuase and Persson [33] obtained the following surprising results (see also Remarks (5.6) and (5.8)).

Theorem 5.4. *Let $1 < p < \infty$, $\boldsymbol{k} = (k_1,...k_n) \in \mathbb{R}^n$ be such that $k_i > 1$ $(i = 1, ..., n)$, $0 < \boldsymbol{b} \leq \infty$, and let the function f be locally integrable on $(\mathbf{0}, \boldsymbol{b})$ such that $0 < \int_0^{b_1} ... \int_0^{b_n} \prod_{i=1}^n x_i^{p-k_i} f^p(\boldsymbol{x})d\boldsymbol{x} < \infty$.*

(i) If $p \geq 2$, then

$$\begin{aligned}
&\int_0^{b_1} ... \int_0^{b_n} \prod_{i=1}^n x_i^{-k_i} \left(\int_0^{x_1} ... \int_0^{x_n} f(\boldsymbol{t})d\boldsymbol{t}\right)^p d\boldsymbol{x} \\
&+ \left(\prod_{i=1}^n \frac{k_i - 1}{p}\right) \int_0^{b_1} ... \int_0^{b_n} \int_{t_1}^{b_1} ... \int_{t_n}^{b_n} \left|\prod_{i=1}^n \frac{p}{k_i - 1} \left(\frac{t_i}{x_i}\right)^{1 - \frac{k_i - 1}{p}} f(\boldsymbol{t}) \right. \\
&\left. - \frac{1}{x_1...x_n} \int_0^{x_1} ... \int_0^{x_n} f(\boldsymbol{t})d\boldsymbol{t}\right|^p \prod_{i=1}^n x_i^{p-k_i-\frac{k_i-1}{p}} d\boldsymbol{x} \prod_{i=1}^n t_i^{\frac{k_i-1}{p}-1} d\boldsymbol{t} \\
&\leq \left(\prod_{i=1}^n \frac{p}{k_i - 1}\right)^p \int_0^{b_1} ... \int_0^{b_n} \prod_{i=1}^n \left(1 - \left[\frac{x_i}{b_i}\right]^{\frac{k_i-1}{p}}\right) x_i^{p-k_i} f^p(\boldsymbol{x})d\boldsymbol{x}.
\end{aligned} \tag{5.7}$$

(ii) If $1 < p \leq 2$, then inequality (5.7) holds in the reversed direction.

Remark 5.6. Equality holds in (5.7) for $p = 2$.

Remark 5.7. For the case $n = 1$ Theorem 5.4 coincides with the corresponding Theorem 3.1 in [33].

Theorem 5.5. *Let* $1 < p < \infty$, $\boldsymbol{k} = (k_1, ...k_n) \in \mathbb{R}^n$ *be such that* $k_i < 1$, $i = 1, 2, ..., n$, $0 \leq \boldsymbol{b} < \infty$, *and let* f *be locally integrable on* $(\boldsymbol{b}, \boldsymbol{\infty})$ *and such that*
$0 < \int_{b_1}^{\infty} ... \int_{b_n}^{\infty} \prod_{i=1}^{n} x_i^{p-k_i} f^p(\boldsymbol{x}) d\boldsymbol{x} < \infty$.
(iii) *If* $p \geq 2$, *then*

$$\begin{aligned}
&\int_{b_1}^{\infty} ... \int_{b_n}^{\infty} \prod_{i=1}^{n} x_i^{-k_i} \left(\int_{x_1}^{\infty} ... \int_{x_n}^{\infty} f(\boldsymbol{t}) d\boldsymbol{t} \right)^p d\boldsymbol{x} \\
&+ \left(\prod_{i=1}^{n} \frac{1-k_i}{p} \right) \int_{b_1}^{\infty} ... \int_{b_n}^{\infty} \int_{b_1}^{t_1} ... \int_{b_n}^{t_n} \left| \prod_{i=1}^{n} \frac{p}{1-k_i} \left(\frac{t_i}{x_i} \right)^{\frac{1-k_i}{p}+1} f(\boldsymbol{t}) \right. \\
&\left. - \frac{1}{x_1 ... x_n} \int_{x_1}^{\infty} ... \int_{x_n}^{\infty} f(\boldsymbol{t}) d\boldsymbol{t} \right|^p \prod_{i=1}^{n} x_i^{\frac{1-k_i}{p}+p-k_i} d\boldsymbol{x} \prod_{i=1}^{n} t_i^{\frac{k_i-1}{p}-1} d\boldsymbol{t} \\
&\leq \left(\prod_{i=1}^{n} \frac{p}{1-k_i} \right)^p \int_{b_1}^{\infty} ... \int_{b_1}^{\infty} \prod_{i=1}^{n} \left(1 - \left[\frac{b_i}{x_i} \right]^{\frac{1-k_i}{p}} \right) x_i^{p-k_i} f^p(\boldsymbol{x}) d\boldsymbol{x}.
\end{aligned} \tag{5.8}$$

(iv) *If* $1 < p \leq 2$, *then inequality (5.8) holds in the reversed direction.*

Remark 5.8. Equality holds in (5.8) for $p = 2$.

Remark 5.9. For the case $n = 1$ Theorem 5.5 reduces to Theorem 3.2 in [33].

Remark 5.10. Finally, we remark that this technique can also be used to prove some operator valued Hardy type inequalities, see Hansen et. al. [10] (see also [23]).

This is the journey so far, which mainly started in this new setting in the paper [20]. We strongly believe that this will signal a new direction in the development of Hardy type inequalities

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