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TRAPPING OF WATER WAVES BY UNDERWATER RIDGES

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ABSTRACT. As is well-known, underwater ridges and submerged horizontal cylinders can serve as waveguides for surface water waves. For large values of the wavenumber in the direction of the ridge, there is only one trapped wave (this was proved in Bonnet & Joly (1993, *SIAM J. Appl. Math.*, **53**, pp. 1507–1550)). We construct the asymptotics of these trapped waves and their frequencies at high frequency by means of reducing the initial problem to a pair of boundary integral equations and then by applying the method of Zhevandrov & Merzon (2003, *AMS Transl. (2)*, **208**, pp. 235–284), in order to solve them.

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1. INTRODUCTION

It is well-known that underwater ridges (horizontal "bumps" on the bottom) can trap water waves (see pioneering paper of Ursell [10] for the case of submerged horizontal cylinders as waveguides; see Garipov [7] for the case of underwater ridges). In [2], Bonnet-Ben Dhia and Joly proved that large values of the wavenumber k along the direction of the ridge, there is only one trapped mode. Some estimates for the frequency of this mode were also obtained. Our goal here is to construct the asymptotics of this frequency for large values of k to the case of n ridges. The general plan of the present paper is the same as in [8].In our case, we use a technique similar to that of [11], where a close analogy of the problem of water waves and small perturbations of the one-dimensional Schrödinger equation is established. The latter problem was studied by a number of authors (we mention, for example, [6, 9, 3], and, in the context of water waves, [5]). We note that the asymptotics turns out to be exponential, i.e., the distance of the trapped wave frequency to the cut-off frequency is exponentially small in k. Nevertheless, in fact we construct an exact convergent expansion, and no additional difficulties arise.

2. MATHEMATICAL FORMULATION AND MAIN RESULTS

We will be mainly occupied by the problem of an underwater ridge. Consider the water layer $\Omega = \{-h(x) < y < 0\}$, where x is the horizontal coordinate orthogonal to the direction of the ridge, y is the vertical coordinate and the bottom satisfies $y = -h(x)$. With the velocity potential in the form $\Phi(x, y)e^{i(\omega t - kz)}$, where z is the horizontal coordinate along the ridge and ω is the frequency, we come to the problem

$$\Phi_y = \lambda\Phi, \qquad y = 0, \tag{2.1}$$

$$\Phi_{xx} + \Phi_{yy} - k^2\Phi = 0, \qquad -h(x) < y < 0, \tag{2.2}$$

$$\partial\Phi/\partial n = 0, \qquad y = -h, \tag{2.3}$$

for the function Φ; here $\lambda = \omega^2/g$. Trapped waves are the solutions of this problem from the Sobolev space $H_1(\Omega)$ and exist only for certain values of λ for k fixed.

We assume that $h(x) = h_0$ for $|x| \geq R > 0$ and h is a C^∞-function that has exactly n nondegenerate local minima at $x = 0, 1, 2, \cdots, n < R$, say, $h''(0) > 0, h''(1) > 0, h''(2) > 0, \cdots, h''(n) > 0$ (the last condition is for simplicity only, refer to Figure 2.1). The continuous spectrum of (2.1)-(2.3) coincides with that for the flat bottom and represents the ray $\lambda \in [k\tanh(kh_0), \infty)$. From the results of [2] it follows that there is only one eigenfrequency λ below the continuous spectrum for large values of k (with dimensionless coordinates choosing). We will construct an asymptotics of this frequency. The main result consists in the following statement.

Theorem 2.1. *The unique eigenvalue $\lambda(k)$ of (2.1)-(2.3) has the form*

$$\lambda(k) = k\tanh kh_0 - \beta^2, \tag{2.4}$$

where

$$\beta = \sum_{j=0}^{n} k\sqrt{\frac{\pi}{2h''(j)}}e^{-2kh(j)}\left(1 + O\left(\frac{1}{k}\right)\right). \tag{2.5}$$

From now on we will devote to the proof of the statement and the construction of the corresponding eigenfunction.

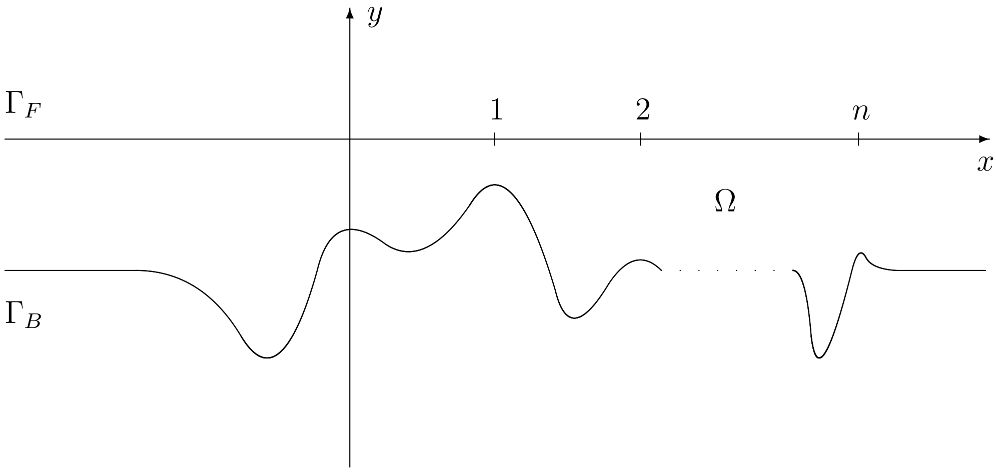

Figure 2.1

3. REDUCTION TO A SYSTEM OF INTEGRAL EQUATIONS

As a first step, we reduce (2.1)-(2.3) to a pair of integral equations on Γ_F and Γ_B for the function $\varphi = \Phi|_{y=0}$ and $\theta = \Phi|_{y=-h}$. To this end, we apply the Green formula to $\Phi(\xi, \eta)$ and $-(1/2\pi)K_0(kr)$, where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$ and K_0 is the Macdonald function (so that $-(1/2\pi)K_0(kr)$ is the fundamental solution of the operator $\Delta - k^2$). We obtain

$$\begin{aligned}\pi\varphi(\xi) =& \lambda \int_{-\infty}^{\infty} K_0(k|x-\xi|)\varphi(x)dx \\ &+ k \int_{-\infty}^{\infty} \frac{K_0'(k\sqrt{(x-\xi)^2 + h(x)^2})}{\sqrt{(x-\xi)^2 + h(x)^2}}[h'(x)(x-\xi) - h(x)]\theta(x)dx \end{aligned} \tag{3.1}$$

$$\begin{aligned}\pi\theta(\xi) =& \lambda \int_{-\infty}^{\infty} K_0(k\sqrt{(x-\xi)^2 + h(\xi)^2})\varphi(x)dx \\ &- kh(\xi) \int_{-\infty}^{\infty} \frac{K_0'(k\sqrt{(x-\xi)^2 + h(\xi)^2})}{\sqrt{(x-\xi)^2 + h(\xi)^2}}\varphi(x)dx \\ &+ k \int_{-\infty}^{\infty} \frac{K_0'(k\sqrt{(x-\xi)^2 + (h(x)-h(\xi))^2})}{\sqrt{(x-\xi)^2 + (h(x)-h(\xi))^2}}[h'(x)(x-\xi) - (h(x)-h(\xi))]\theta(x)dx. \end{aligned} \tag{3.2}$$

In order to apply the technique of [11] to (3.1), (3.2) it is necessary to pass to the Fourier transform $\tilde{\varphi}$ of the function φ,

$$\mathcal{F}_{\xi\to p}[\varphi(\xi)](p) \equiv \tilde{\varphi}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ip\xi}\varphi(\xi)d\xi.$$

Using the formulas (see [1])

$$K_0'(z) = -K_1(z), \quad \mathcal{F}_{\xi\to p}[K_0(k|\xi|)](p) = \frac{\pi}{\sqrt{k^2+p^2}},$$

$$\mathcal{F}_{\xi\to p}\left[\frac{K_1(k\sqrt{\xi^2+h_0^2})}{\sqrt{\xi^2+h_0^2}}\right](p) = \frac{\pi}{kh_0}e^{-h_0\sqrt{k^2+p^2}},$$

$$\mathcal{F}_{\xi\to p}\left[K_0(k\sqrt{\xi^2+h_0^2})\right](p) = \frac{\pi}{\sqrt{k^2+p^2}}e^{-h_0\sqrt{k^2+p^2}},$$

we come to the following system for $\tilde{\varphi}(p),\ \theta(\xi)$:

$$\left(1-\frac{\lambda}{\tau(p)}\right)\tilde{\varphi}(p) = \int_{-\infty}^{\infty} e^{ipx-h(x)\tau(p)}\left(1+\frac{iph'(x)}{\tau(p)}\right)\theta(x)dx, \tag{3.3}$$

$$\begin{aligned}\theta(x) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx-h(x)\tau(p)}(\frac{\lambda}{\tau(p)}+1)\tilde{\varphi}(p)dp \\ &+ k\int_{-\infty}^{\infty}\frac{K_0'(k\sqrt{\varrho(\xi,x)})}{\sqrt{\varrho(\xi,x)}}[h'(\xi)(\xi-x)-(h(\xi)-h(x))]\theta(\xi)d\xi,\end{aligned} \tag{3.4}$$

where

$$\tau(p) =: \sqrt{k^2+p^2},$$
$$\varrho(\xi,x) =: (\xi-x)^2+(h(\xi)-h(x))^2.$$

Rewrite system (3.3) - (3.4) as

$$\left(1-\frac{\lambda}{\tau(p)}\right)\tilde{\varphi}(p) =(\hat{M}_1\theta)(p), \tag{3.5}$$

$$[(1-\hat{M}_3)\theta](x) =(\hat{M}_2\tilde{\varphi})(x), \tag{3.6}$$

where

$$(\hat{M}_1\theta)(p) = \int_{-\infty}^{\infty} M_1(p,x)\theta(x)dx,$$

$$(\hat{M}_2\tilde{\varphi})(x) = \int_{-\infty}^{\infty} M_2(x,p)\tilde{\varphi}(p)dp,$$

$$(\hat{M}_3\theta)(x) = \int_{-\infty}^{\infty} M_3(x,\xi)\theta(\xi)d\xi,$$

with

$$M_1(p,x) = e^{ipx-h(x)\tau(p)}\left(1+\frac{iph'(x)}{\tau(p)}\right),$$
$$M_2(x,p) = \frac{1}{2\pi}e^{ipx-h(x)\tau(p)}\left(1+\frac{\lambda}{\tau(p)}\right),$$
$$M_3(x,\xi) = \frac{k}{\pi}\frac{K_0'(k\sqrt{\varrho(\xi,x)})}{\sqrt{\varrho(\xi,x)}}[h'(\xi)(\xi-x)-(h(\xi)-h(x))].$$

Obviously, a solution of (3.5), (3.6) gives via the standard formulas of the potential theory a solution of (2.1)-(2.3).

4. SOLUTION OF THE SYSTEM OF INTEGRAL EQUATIONS

Consider equation (3.6). Recall the following (see, e.g., [4])

Lemma 4.1. *Let*

$$\int_{-\infty}^{\infty}|K(x,y)|dx < M, \qquad \int_{-\infty}^{\infty}|K(x,y)|dy < M.$$

Then

$$\|\hat{K}u\|_{L_2} \le M\|\, u\|_{L_2},$$

where

$$\hat{K}u = \int_{-\infty}^{\infty} K(x,y)u(y)dy.$$

It is not hard to see, using the asymptotics of $K_1(x)$ for small and large x, that the kernel M_3 in (3.6) satisfies the conditions of Lemma 4.1 with $M = \mathrm{Const.}\, k^{-1/2}$. Hence we can invert the operator $(1-\hat{M}_3)$ in (3.6) using the Neumann series and obtain

$$\theta(x) = [(1-\hat{M}_3)^{-1}\hat{M}_2\tilde{\varphi}](x), \tag{4.1}$$

where $(1-\hat{M}_3)^{-1} = \sum_{n=0}^{\infty}\hat{M}_3^n$. Substituting (4.1) in (3.5) we finally come to

$$\left(1-\frac{\lambda}{\tau(p)}\right)\tilde{\varphi}(p) = [\hat{M}_1(1-\hat{M}_3)^{-1}\hat{M}_2\tilde{\varphi}](p). \tag{4.2}$$

We apply the reasoning of [11] to (4.2). Indeed, we know that λ is given by (2.4), where β is exponentially small in k [2]. Hence the first factor in the left-hand side of (4.2),

$$L(p) := 1-\frac{\lambda}{\tau(p)} = 1-\frac{k-\beta^2}{\sqrt{k^2+p^2}} + O(e^{-2kh_0}), \tag{4.3}$$

is exponentially small in k for $p=0$. In fact, the roots of $L(p)=0$ which tend to zero as $k\to\infty$, as it is not hard to see, are simple and given by

$$p = p_\pm = \pm\frac{i\sqrt{2}\beta}{\sqrt{\varepsilon}} + O(\varepsilon^{1/2}\beta^3), \qquad \varepsilon = \frac{1}{k}. \tag{4.4}$$

For this reason, the heuristic considerations of section 2 of [11] are applicable to (4.2). Following these arguments, we look for $\tilde{\varphi}$ in the form $\tilde{\varphi}(p) = A(p)/L(p)$. As we shall see (see formula (4.6) below), $A(p)$ and $M_2(x,p)$ are analytic in a strip containing the real axis, and we can change the contour of integration in the integral

$$\int_{-\infty}^{\infty} M_2(x,p)\frac{A(p)}{L(p)}dp$$

to that given by

$$C := (-\infty, -a] \cup \{p + iq : p^2 + q^2 = a^2,\ q > 0\} \cup [a, \infty)$$

in the complex plane, with a suitable $a > 0$ such that in the disc $|p| < a$ there are no zeros of $L(p)$ apart from $p_\pm$.

We have, by the residue theorem,

$$\int_{-\infty}^{\infty} M_2(x,p)\frac{A(p)}{L(p)}dp = \int_C M_2(x,p)\frac{A(p)}{L(p)}dp + 2\pi i\frac{M_2(x,p_+)A(p_+)}{(dL(p)/dp)|_{p=p_+}}. \tag{4.5}$$

Thus (4.2) transforms into

$$A(p) = [\hat{M}_1(1-\hat{M}_3)^{-1}\hat{M}_4 A](p) + [\hat{M}_1(1-\hat{M}_3)^{-1}f(x)]A(p_+), \tag{4.6}$$

where

$$[\hat{M}_4 A](x) = \int_C M_2(x,p)\frac{A(p)}{L(p)}dp, \qquad f(x) = 2\pi i\frac{M_2(x,p_+)A(p_+)}{(dL(p)/dp)|_{p=p_+}}.$$

Note that now the operator $\hat{M}_5 = \hat{M}_1(1-\hat{M}_3)^{-1}\hat{M}_4$ is small in ε since $|L(p)| \geq const\ k^{-2}$ along C and $M_2(x,p)$ is exponentially small. Indeed, on the arc we have up to $O(k^{-\infty})$

$$|L(p)| = \left|1 - \frac{1}{\sqrt{1+p^2/k^2}}\right| = \frac{a^2}{2k^2} + O(k^{-4}),$$

and on the part of the contour which lies on the real axis the minimum of $|L(p)|$ is attained at the points $p = \pm a$, hence, the above estimate still holds. Rewriting (4.6) as

$$(1-\hat{M}_5)A(p) = g(p)A(p_+), \tag{4.7}$$

where $g(p) = \hat{M}_1(1-\hat{M}_3)^{-1}f(x)$, we see that $(1-\hat{M}_5)$ is invertible and $A(p) = (1-\hat{M}_5)^{-1}g(p)A(p_+)$. Putting $p = p_+$ in the last equality and dividing by $A(p_+)$, we obtain an equation for β:

$$1 = (1-\hat{M}_5)^{-1}g(p)|_{p=p_+}. \tag{4.8}$$

A standard application of the Laplace method of asymptotic evaluation of integrals to the leading term in (4.8) yields formula (2.5).

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