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SOME HOMOGENEOUS CYCLIC INEQUALITIES OF THREE VARIABLES OF DEGREE THREE AND FOUR

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ABSTRACT. We shall show that the three variable cubic inequality

$$\begin{aligned} t^2(a^3+b^3+c^3)+(t^4-2t)(ab^2+bc^2+ca^2) \\ \geq (2t^3-1)(a^2b+b^2c+c^2a)+(3t^4-6t^3+3t^2-6t+3)abc \end{aligned}$$

holds for non-negative a, b, c, and for any real number t. We also show some similar three variable cyclic quartic inequalities.

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1. INTRODUCTION

We denote cyclic or symmetric polynomials of three variables by

$$S_{i,j,k} := a^i b^j c^k + b^i c^j a^k + c^i a^j b^k, \quad S_{i,j} := S_{i,j,0}, \quad S_i := S_{i,0,0} = a^i + b^i + c^i,$$
$$T_{i,j,k} := S_{i,j,k} + S_{j,i,k}, \quad T_{i,j} := T_{i,j,0} = S_{i,j} + S_{j,i}, \quad T_i := T_{i,0,0} = 2S_i,$$
$$U := \frac{1}{3}S_{1,1,1} = \frac{1}{6}T_{1,1,1} = abc.$$

About symmetric polynomials, it is well known that Muirhead's inequalities

$$2S_3 = T_3 \geq T_{2,1} \geq T_{1,1,1} = 6U,$$
$$2S_4 = T_4 \geq T_{3,1} \geq T_{2,2} \geq T_{2,1,1} = 2US_1$$

hold for non-negative real numbers a, b, c. Schur's inequality and Cirtoaje's inequality ([3, p.73])

$$S_3 + 3U \geq T_{2,1}, \quad S_4 + (t^2 + 2t)S_{2,2} \geq (t+1)T_{3,1} + (t^2 - 1)US_1 \tag{1.1}$$

are rather non-trivial, here t is any real number. Any of the above quartic inequalities except $T_{3,1} \geq T_{2,2}$ hold even if some of a, b, c are negative. About cyclic polynomials, similar inequalities

$$S_3 \geq S_{2,1} \geq S_{1,1,1} = 3U, \quad S_4 \geq S_{3,1} \geq S_{2,1,1} = US_1, \quad S_4 \geq S_{2,2} \geq S_{2,1,1}$$

hold for non-negative a, b, c. Here, $S_4 \geq S_{3,1}$ and $S_4 \geq S_{2,2} \geq S_{2,1,1}$ are true for any real a, b, c. But $S_{3,1} \geq S_{2,2}$ does not hold even if a, b, c are non-negative. Schur type inequality $S_3 + 3U \geq 2S_{2,1}$ does not also hold. But an weaker inequality

$$S_{2,1} \leq \frac{\sqrt[3]{4}}{3}S_3 + \left(3 - \sqrt[3]{4}\right)U \leq 0.529134S_3 + 0.470866U \tag{1.2}$$

holds for non-negative a, b, c. This will be proved as a corollary of the following Theorem.

Theorem 1.1. *For non-negative real numbers a, b, c and t, the following inequality holds:*

$$t^2S_3 + (t^4 - 2t)S_{2,1} \geq (2t^3 - 1)S_{1,2} + (3t^4 - 6t^3 + 3t^2 - 6t + 3)U. \tag{1.3}$$

The equality holds if and only if $a = b = c$ or $a : b : c = t : 0 : 1$ or any cyclic permutation thereof.

This inequality will be proved in Section 2. Note that when $t = \dfrac{1}{\sqrt[3]{2}}$ we obtain (1.2), and when $t = \dfrac{1 + \sqrt{2} + \sqrt{2\sqrt{2} - 1}}{2} = 1.88320350591352586416\cdots$, we obtain

$$S_3 + \alpha S_{2,1} \geq (\alpha + 1)S_{1,2},$$

here $\alpha = \dfrac{\sqrt{16\sqrt{2} + 13} - 1}{2} = 2.48443533176585687519\cdots$ which is a root of $\alpha^4 + 2\alpha^3 - 5\alpha^2 - 6\alpha - 23 = 0$.

Next, we consider quartic cyclic inequalities. It is well known the inequalities

$$S_4 + S_{1,3} \geq 2S_{3,1}, \quad S_4 + (3t^2 - 1)S_{2,2} \geq 3tS_{3,1} + 3t(t-1)US_1 \tag{1.4}$$

hold for any real numbers t, a, b, c. Recently, Cirtoaje proved very nice inequality which includes (1.1) and (1.4).

Theorem 1.2. ([4, Theorem 2.1]). *Let p, q, r be any real numbers. The cyclic inequality*

$$S_4 + rS_{2,2} + (p + q - r - 1)US_1 \geq pS_{3,1} + qS_{1,3}$$

holds for any real numbers a, b, c if and only if

$$3(1 + r) \geq p^2 + pq + q^2.$$

In Section 3, we shall prove the following.

Theorem 1.3. *(1) For any real numbers a, b, c and t such that $|t| \geq 2$, the following inequality holds:*

$$S_4 + \frac{t^2 + 8}{4} S_{2,2} + \frac{t(t-2)}{2} US_1 \geq tT_{3,1}. \tag{1.5}$$

Here, the equality holds if and only if

$$2(a^2 + b^2 + c^2) = t(ab + bc + ca).$$

(2) For any non-negative real numbers a, b, c and t, the following inequality holds:

$$2t^3 S_4 \geq t^2(3 - t^4)S_{3,1} + (3t^4 - 1)S_{1,3} + (1 - 3t^2 + 2t^3 - 3t^4 + t^6)US_1. \tag{1.6}$$

The equality holds if and only if $a = b = c$ or $a : b : c = t : 0 : 1$ or any cyclic permutation thereof.

As a special case of (1.6), when $t = 1/\sqrt[4]{3}$ we obain

$$S_4 + \left(\frac{4\sqrt[4]{3}}{3} - 1 \right) US_1 \geq \frac{4\sqrt[4]{3}}{3} S_{3,1}.$$

When $t = 0.602755925581141815 26 \cdots$ is a root of $t^6 - 3t^4 + 2t^3 - 3t^2 + 1 = 0$, we obtain

$$S_4 + \beta S_{1,3} \geq (\beta + 1)S_{3,1},$$

here $\beta = 1.37907443362539958016 \cdots$ which is a root of

$$4\beta^6 + 12\beta^5 - 48\beta^4 - 116\beta^3 + 24\beta^2 + 84\beta + 229 = 0.$$

Note that Theorem 1.2 does not included in Theorem 1.3, and Theorem 1.3(1) does not conflict to Theorem 1.2:

$$S_4 + (p^2 - 1)S_{2,2} + p(2 - p)US_1 \geq pT_{3,1}.$$

2. PROOF OF THEOREM 1.1

In this section, we shall show that

$$\begin{aligned} t^2(a^3 + b^3 + c^3) &+ (t^4 - 2t)(ab^2 + bc^2 + ca^2) \\ &\geq (2t^3 - 1)(a^2b + b^2c + c^2a) + (3t^4 - 6t^3 + 3t^2 - 6t + 3)abc. \end{aligned} \tag{1.2}$$

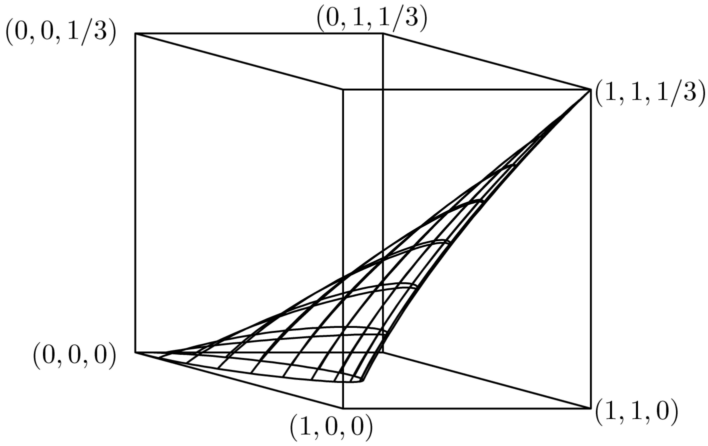
Let $u := a/c$, $v := b/c$, and

$$\begin{aligned} x &:= \frac{S_{2,1}}{S_3} = \frac{a^2b + b^2c + c^2a}{a^3 + b^3 + c^3} = \frac{u^2v + v^2 + u}{u^3 + v^3 + 1}, \\ y &:= \frac{S_{1,2}}{S_3} = \frac{ab^2 + bc^2 + ca^2}{a^3 + b^3 + c^3} = \frac{uv^2 + v + u^2}{u^3 + v^3 + 1}, \\ z &:= \frac{U}{S_3} = \frac{abc}{a^3 + b^3 + c^3} = \frac{uv}{u^3 + v^3 + 1}. \end{aligned}$$

Eliminate u, v from x, y, z, we obtain the relation

$$x^3 + y^3 + 9z^3 - 6xyz - xy + 3z^2 + z = 0, \tag{2.1}$$

here $x \geq 0$, $y \geq 0$, $z \geq 0$. (2.1) determines the rational algebraic surface S as the following figure.



Let C be the intersection of S and the (x, y)-plane. C is the rational cubic curve defined by

$$x^3 + y^3 - xy = 0.$$

The part of C in $x \geq 0$, $y \geq 0$ is a convex closed curve. Note that any point Q on C can be represented as

$$x = \frac{u}{u^3+1} =: f(u), \quad y = \frac{u^2}{u^3+1} =: g(u) \quad (u \geq 0).$$

S have a rational double point of the type A_1 at $P = (1, 1, 1/3)$ as the unique singularity. In fact, let

$$X := 1 - x, \quad Y := 1 - y, \quad Z := \frac{1}{3} - x.$$

Then (2.1) is equivalent to

$$3X^2 + 3Y^2 + 12Z^2 - 3XY - 6YZ - 6XZ - X^3 - Y^3 - 9Z^3 + 6XYZ = 0.$$

With parameters $A := X/Z$ and $B := Y/Z$, we can represent X, Y, Z as

$$X = \frac{3(4A - 2A^2 - 2AB + A^3 - A^2B + AB^2)}{9 - 6AB + A^3 + B^3},$$
$$Y = \frac{3(4B - 2B^2 - 2AB + A^2B - AB^2 + B^3)}{9 - 6AB + A^3 + B^3},$$
$$Z = \frac{3(4 - 2A - 2B + A^2 + B^2 - AB)}{9 - 6AB + A^3 + B^3}.$$

This means that every line passing though P meet again with S at only one point. We shall call such surface to be a 'concave cone'.

Let $Q := \big(f(u), g(u)\big)$ be a point on C, and let $\mathbb{R}^3_+$ be the first quadrant of (x, y, z)-space defined by $x \geq 0$, $y \geq 0$, $z \geq 0$. Let H be the plane which tangents to C at Q, and which passes through P. Since $PQ \cap S = \{P, Q\}$, and since C is a convex closed curve, we conclude that $H \cap S \cap \mathbb{R}^3_+ = \{P, Q\}$. The equation of H is

$$0 = \begin{vmatrix} x & y & z & 1 \\ 1 & 1 & \frac{1}{3} & 1 \\ f(u) & g(u) & 0 & 1 \\ f(u) + \frac{d}{du}f(u) & g(u) + \frac{d}{du}g(u) & 0 & 1 \end{vmatrix}$$
$$= \frac{-u^2 - (u^4 - 2u)x + (2u^3 - 1)y + (3u^4 - 6u^3 + 3u^2 - 6u + 3)z}{3(u^3 + 1)^2}.$$

The numerator of the above gives (1.2), after we observe which side of H there is S.

3. PROOF OF THEOREM 1.2

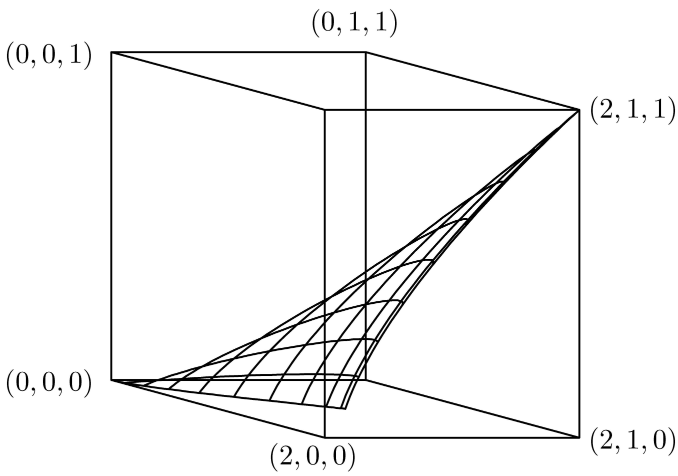
Proof of (1.5). Let a, b, c be any real numbers, $u := a/c$, $v := b/c$, and let

$$x := \frac{T_{3,1}}{S_4} = \frac{u^3v + v^3u + u^3 + v^3 + u + v}{u^4 + v^4 + 1} =: p(u, v),$$
$$y := \frac{S_{2,2}}{S_4} = \frac{u^2v^2 + u^2 + v^2}{u^4 + v^4 + 1} =: q(u, v),$$
$$z := \frac{US_1}{S_4} = \frac{uv(u + v + 1)}{u^4 + v^4 + 1} =: r(u, v).$$

These parametrization determines a rational quadric surface S defined by

$$(x + z)^2 - (2y + 1)(y + 2z) = 0.$$

Note that S is an elliptic cone with vertex $(-1/4, -1/2, 1/4)$.



The above figure is the part of S determined by $0 \le u \le 1$, $0 \le v \le 1$. This looks like a triangular sail of a yacht. Note that since the parametrization is not one to one, the (x, y, z) can move limited area of S.

Now we calculate the tangent plane of S. Let

$$A_0 := (u + v + uv)^2$$
$$A_1 := 2(u + v + uv)(1 + u^2 + v^2)$$
$$A_2 := 1 + 4u^2 + 4uv + 4v^2 + 4u^2v + 4uv^2 + u^4 + 4u^2v^2 + v^4$$
$$A_3 := 2(1 + u^2 + v^2)(1 - u - v + u^2 - uv + v^2).$$

Since

$$\begin{vmatrix} x & y & z & 1 \\ p(u,v) & q(u,v) & r(u,v) & 1 \\ p(u,v)+\frac{\partial}{\partial u}p(u,v) & q(u,v)+\frac{\partial}{\partial u}q(u,v) & r(u,v)+\frac{\partial}{\partial u}r(u,v) & 1 \\ p(u,v)+\frac{\partial}{\partial v}p(u,v) & q(u,v)+\frac{\partial}{\partial v}q(u,v) & r(u,v)+\frac{\partial}{\partial v}r(u,v) & 1 \end{vmatrix}$$
$$= \frac{(u-1)(v-1)(u-v)(u+v+1)^2}{(1+u^4+v^4)^3}\big(A_0 - A_1x + A_2y + A_3z\big),$$

the tangent plane H of S at $\big(p(u,v), q(u,v), r(u,v)\big)$ is given by

$$A_0 - A_1x + A_2y + A_3z = 0. \tag{3.1}$$

By the way, since

$$4\frac{A_2}{A_0} = \left(\frac{A_1}{A_0}\right)^2 + 8, \quad 2\frac{A_3}{A_0} = \left(\frac{A_1}{A_0}\right)^2 - 2\frac{A_1}{A_0},$$

we can represent (3.1) by the parameter

$$t := \frac{A_1}{A_0} = \frac{2(1+u^2+v^2)}{u+v+uv} = \frac{2(a^2+b^2+c^2)}{ab+bc+ca}$$

as

$$H : 1 - tx + \frac{t^2+8}{4}y + \frac{t^2-2t}{2}z = 0. \tag{3.2}$$

Sinc S is a cone, S is contained in a half space whose boundary is H. But since

$$2(1+u^2+v^2) - 2(u+v+uv) = (u-v)^2 + (u-1)^2 + (v-1)^2 \geq 0,$$

t can only take any real numbers such that $|t| \geq 2$. Thus we complete the proof. ∎

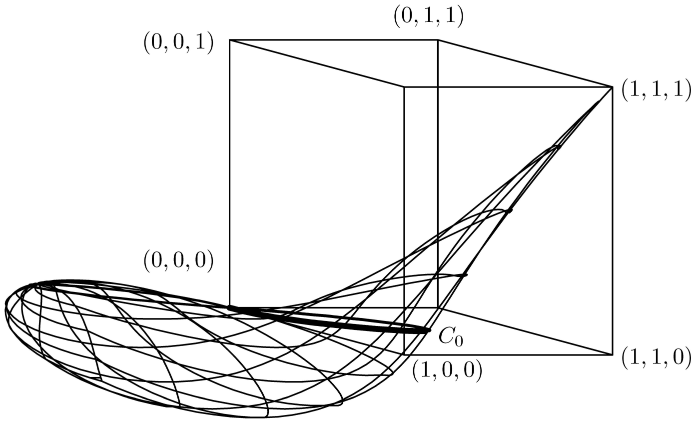
Note that H does not pass through $(2, 1, 1)$, unless $t = 2$.

Proof of (1.6). We may assume that a, b, c be real numbers. Let $u := a/c$, $v := b/c$, and let

$$x := \frac{S_{3,1}}{S_4} = \frac{u^3v + v^3 + u}{u^4 + v^4 + 1} =: p(u,v),$$
$$y := \frac{S_{1,3}}{S_4} = \frac{uv^3 + u^3 + v}{u^4 + v^4 + 1} =: q(u,v),$$
$$z := \frac{US_1}{S_4} = \frac{uv(u+v+1)}{u^4 + v^4 + 1} =: r(u,v).$$

These parametrization determines a rational quadric surface S defined by

$$x^4 + y^4 + 3z^4 + 2x^2y^2 + 4y^2z^2 + 4x^2z^2 - 4xz^3 - 4xyz^2 - 4yz^3$$
$$+ 5z^3 - 4xz^2 - 4yz^2 + x^2z - 4xyz + y^2z - xy - xz - yz + 4z^2 + z = 0.$$



S has a rational double point at $P = (1, 1, 1)$. Let C be the intersection of S and the (x, y)-plane. C is a rational curve with a parametrization

$$x = \frac{u}{u^4+1} =: p_0(u), \quad y = \frac{u^3}{u^4+1} =: q_0(u). \tag{3.3}$$

The part of C in $x \geq 0$, $y \geq 0$ is a convex closed curve.

Let Q be the point on C defined by (3.3), and let H be the tangent plane of C at Q which passes through P. The equation of H is obtained by

$$\begin{aligned} 0 &= \begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 1 & 1 \\ p_0(u) & q_0(u) & 0 & 1 \\ p_0(u) + \frac{d}{du}p_0(u) & q_0(u) + \frac{d}{du}q_0(u) & 0 & 1 \end{vmatrix} \\ &= \frac{u^2(3-u^4)x - (1-3u^4)y + (1-3u^2+2u^3-3u^4+u^6)z - 2u^3}{(1+u^4)^2}. \end{aligned}$$

If we show that $H \cap S \cap \mathbb{R}^3_+ = \{P, Q\}$, we know that S must be contained in the half space defined by

$$u^2(3-u^4)x - (1-3u^4)y + (1-3u^2+2u^3-3u^4+u^6)z - 2u^3 \leq 0.$$

Then we obtain (1.6).

Let's show that $H \cap S \cap \mathbb{R}^3_+ = \{P, Q\}$, Note that the intersection number PQ and S at P is not less than 2. Since S is quartic, and since S is closed surface, we have $PQ \cap S = \{P, P, Q, Q'\}$, here Q' is the third intersection of PQ and S. It is enough to show that the z-coordinate of Q' is not positive.

Assume that the z-coordinate of Q' is positive. Then, we can find a point $Q'' \in S \cap \mathbb{R}^3_+$ such that the line PQ'' tangents to S.

Let H be the tangent plane of S at $Q'' = \big(p(u,v), q(u,v), r(u,v)\big)$. Let

$$\begin{aligned} A_0 := &-2u^4v^4 + u^6v + u^5v^2 - u^4v^3 - u^3v^4 + u^2v^5 + uv^6 \\ &+ u^5v - 2u^4v^2 + 3u^3v^3 - 2u^2v^4 + uv^5 \\ &- u^4v + 3u^3v^2 + 3u^2v^3 - uv^4 \\ &- 2u^4 - u^3v - 2u^2v^2 - uv^3 - 2v^4 + u^2v + uv^2 + uv, \\ A_1 := &\, 3u^5v^3 - uv^7 - u^7 - u^6v + u^5v^2 + 2u^4v^3 - 3u^3v^4 + u^2v^5 - uv^6 \\ &+ u^5v - 2u^4v^2 - 2u^2v^4 + uv^5 - 3u^4v + 2uv^4 + 3v^5 \\ &+ 2u^3v - 2u^2v^2 - 3uv^3 + 3u^3 + u^2v + uv^2 - uv - v, \end{aligned}$$

$$
\begin{aligned}
A_2 :=\ & -u^7v+3u^3v^5-u^6v+u^5v^2-3u^4v^3+2u^3v^4+u^2v^5-uv^6+v^7\\
&+u^5v-2u^4v^2-2u^2v^4+uv^5+3u^5+2u^4v-3uv^4\\
&-3u^3v-2u^2v^2+2uv^3+u^2v+uv^2+3v^3-uv-u,\\
A_3 :=\ & u^8-u^7v+u^6v^2-u^5v^3-2u^4v^4-u^3v^5+u^2v^6-uv^7+v^8\\
&-u^7+u^4v^3+2u^6v+u^3v^4+2uv^6-v^7\\
&+u^6-9u^4v^2-9u^2v^4+8u^3v^3+v^6\\
&-u^5+u^4v+8u^3v^2+8u^2v^3+uv^4-v^5\\
&-2u^4+u^3v-9u^2v^2+uv^3-2v^4\\
&-u^3-v^3+u^2+2uv+v^2-u-v+1.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\begin{vmatrix}
x & y & z & 1\\
p(u,v) & q(u,v) & r(u,v) & 1\\
p(u,v)+\frac{\partial}{\partial u}p(u,v) & q(u,v)+\frac{\partial}{\partial u}q(u,v) & r(u,v)+\frac{\partial}{\partial u}r(u,v) & 1\\
p(u,v)+\frac{\partial}{\partial v}p(u,v) & q(u,v)+\frac{\partial}{\partial v}q(u,v) & r(u,v)+\frac{\partial}{\partial v}r(u,v) & 1
\end{vmatrix}\\
&= \frac{1+u+v}{(1+u^4+v^4)^3}\big(A_0+A_1x+A_2y+A_3z\big),
\end{aligned}
$$

the equation of the tangent plane of S at $Q=\big(p(u,v), q(u,v), r(u,v)\big)$ is

$$A_0+A_1x+A_2y+A_3z=0. \tag{3.4}$$

Since H passes through $P=(1,1,1)$, we have a relation

$$
\begin{aligned}
&A_0+A_1+A_2+A_3\\
&=(1+u+v)(1-u-v+u^2+v^2-uv)^2\\
&\quad\times(1-u-v-u^2-v^2+3uv+u^3-u^2v-uv^2+v^3)=0.
\end{aligned}
$$

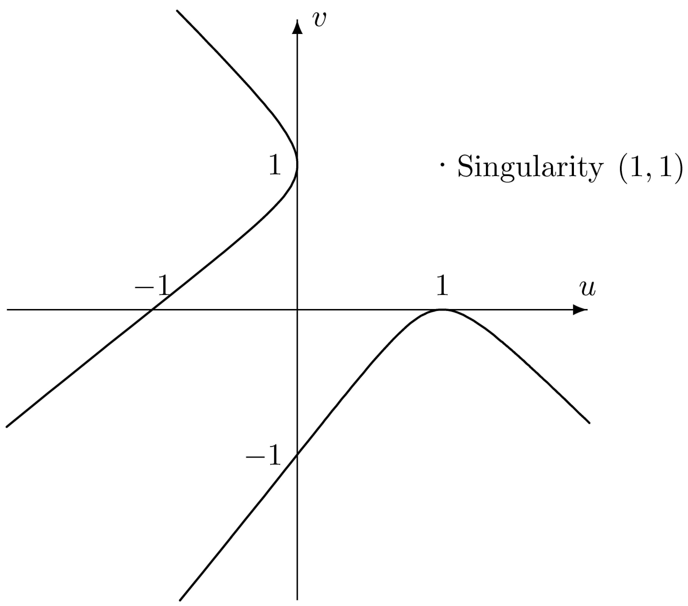
Note that

$$2(1-u-v+u^2+v^2-uv)=(u-v)^2+(u-1)^2+(v-1)^2>0,$$

if $Q''\neq P$. If $u+v+1=0$, then $Q''=(-1/2,-1/2,0)$. Thus we consider a relation

$$1-u-v-u^2-v^2+3uv+u^3-u^2v-uv^2+v^3=0.$$

This equation determines a rational cubic curve Γ in (u,v)-plane with a singularity at $(1,1)$.



Let $t := (1-v)/(1-u)$. Then Γ have a parametrization

(3.5) $$u = \frac{t^2(t-2)}{1-t-t^2+t^3}, \quad v = \frac{1-2t}{1-t-t^2+t^3}.$$

Thus we see that

$$\big\{(u,\, v) \in \Gamma \;\big|\; u > 0, v > 0\big\} = \{(1,1)\}.$$

Note that $(u,\, v) = (1,\, 1)$, if $t^3 = -1$ and $t \neq -1$. Thus $u \leq 0$ or $v \leq 0$, and $r(u,\, v) \leq 0$. This is a contradiction. ∎

By the above observation, we obtain another inequality. Assume that a, b, c are any real numbers. Substitute (3.5) for (3.4), and multiply

$$\begin{aligned}(1-t)^{16}(1+t)^8(1-t+t^2)^{-3}\\ \times \big(2 - 12t + 18t^2 + 20t^3 - 51t^4 - 48t^5 + 144t^6\\ - 48t^7 - 51t^8 + 20t^9 + 18t^{10} - 12t^{11} + 2t^{12}\big)^{-1},\end{aligned}$$

then we obtain

$$\begin{aligned}&(1 - 6t + 6t^2 + 8t^3 - 9t^4 + t^6)x + (1 - 9t^2 + 8t^3 + 6t^4 - 6t^5 + t^6)y\\ &= (1 - 3t - 3t^2 + 11t^3 - 3t^4 - 3t^5 + t^6)z + (1 - 3t + 5t^3 - 3t^5 + t^6).\end{aligned}$$

Since S is contained in a half space divided by H, thus we have the following:

$$\begin{aligned}&(1 - 3t + 5t^3 - 3t^5 + t^6)S_4 + (1 - 3t + t^3)(1 - 3t^2 + t^3)US_1\\ &\geq (1 - 6t + 6t^2 + 8t^3 - 9t^4 + t^6)S_{3,1}\\ &\quad + (1 - 9t^2 + 8t^3 + 6t^4 - 6t^5 + t^6)S_{1,3}.\end{aligned}$$

But this is a special case of Theorem 1.2.

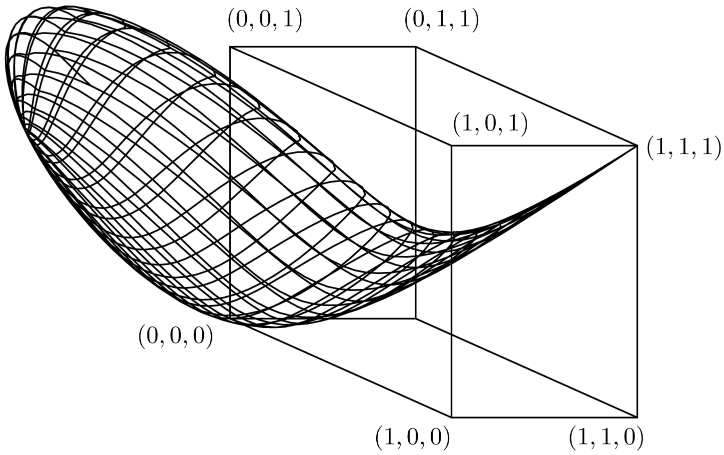
It is haeder to obtain inequalities about S_4, $S_{2,2}$, $S_{3,1}$ and $S_{1,3}$.

Let a, b, c be any real numbers, $u := a/c$, $v := b/c$, and let

$$x := \frac{S_{3,1}}{S_4} = \frac{u^3v + v^3 + u}{u^4 + v^4 + 1} =: p(u, v),$$

$$y := \frac{S_{1,3}}{S_4} = \frac{uv^3+u^3+v}{u^4+v^4+1} =: q(u,v),$$
$$z := \frac{S_{2,2}}{S_4} = \frac{u^2v^2+u^2+v^2}{u^4+v^4+1} =: r(u,v).$$

This determines a rational quartic surface S as the following figure.



S has an unique singular point at $P = (1, 1, 1)$. Note that $S \cap \mathbb{R}^3_+$ is not concave cone. Thus we can not use the above method.

Anyway, we calculate the equation of the tangent plane H of S at $Q = \big(p(u, v), q(u,, v), r(u, v)\big)$. Let

$$\begin{aligned}
B_0 :=\, & u^6v^2 - u^5v^3 + u^4v^4 - u^3v^5 + u^2v^6 \\
& - u^5v^2 - 2u^4v^3 - 2u^3v^4 - u^2v^5 \\
& + u^6 - u^5v + 5u^4v^2 + 5u^2v^4 - uv^5 + v^6 \\
& - u^5 - 2u^4v - 2uv^4 - v^5 + u^4 - 2u^3v + 5u^2v^2 - 2uv^3 + v^4 \\
& - u^3 - u^2v - uv^2 - v^3 + u^2 + v^2, \\
B_1 :=\, & -2(u^2+v^2+1)\big(u^5v - u^4v^2 - u^4v - u^3v^2 + u^2v^3 - uv^4 + v^5 \\
& + u^3v + 3u^2v^2 - uv^3 - v^4 - u^2v + uv^2 - u^2 - uv + u, \\
B_2 :=\, & -2(u^2+v^2+1)\big(-u^2v^4 + uv^5 + u^5 - u^4v + u^3v^2 - u^2v^3 - uv^4 \\
& - u^4 - u^3v + 3u^2v^2 + uv^3 + u^2v - uv^2 - uv - v^2 + v, \\
B_3 :=\, & u^8 - u^7v + u^6v^2 - u^5v^3 - 2u^4v^4 - u^3v^5 + u^2v^6 - uv^7 + v^8 \\
& - u^7 + 2u^6v + u^4v^3 + u^3v^4 + 2uv^6 - v^7 \\
& + u^6 - 9u^4v^2 + 8u^3v^3 - 9u^2v^4 + v^6 \\
& - u^5 + u^4v + 8u^3v^2 + 8u^2v^3 + uv^4 - v^5 \\
& - 2u^4 + u^3v - 9u^2v^2 + uv^3 - 2v^4 \\
& - u^3 - v^3 + u^2 + 2uv + v^2 - u - v + 1.
\end{aligned}$$

Since

$$\begin{vmatrix} x & y & z & 1 \\ p(u,v) & q(u,v) & r(u,v) & 1 \\ p(u,v)+\frac{\partial}{\partial u}p(u,v) & q(u,v)+\frac{\partial}{\partial u}q(u,v) & r(u,v)+\frac{\partial}{\partial u}r(u,v) & 1 \\ p(u,v)+\frac{\partial}{\partial v}p(u,v) & q(u,v)+\frac{\partial}{\partial v}q(u,v) & r(u,v)+\frac{\partial}{\partial v}r(u,v) & 1 \end{vmatrix}$$
$$= \frac{1+u+v}{(1+u^4+v^4)^3}\big(B_0+B_1x+B_2y+B_3z\big),$$

H is defined by

$$B_0+B_1x+B_2y+B_3z=0. \tag{3.6}$$

We assume that H passes through $P=(1,1,1)$. Then we have a relation

$$\begin{aligned} &B_0+B_1+B_2+B_3 \\ &=(1-u-v+u^2-uv+v^2)^2 \\ &\qquad\times\big(u^4-u^3v-u^2v^2-uv^3+v^4 \\ &\qquad\quad -u^3+2u^2v+2uv^2-v^3-u^2+2uv-v^2-u-v+1\big)=0 \end{aligned}$$

Since $1-u-v+u^2-uv+v^2>0$ if $(x,y,z)\neq P$, we only consider the case

$$u^4-u^3v-u^2v^2-uv^3+v^4-u^3+2u^2v+2uv^2-v^3-u^2+2uv-v^2-u-v+1=0$$

This define a rational quartic curve which can be parameterized by $t:=(1-v)/(1-u)$ as

$$u=\frac{-1+2t+2t^2-3t^3+t^4}{1-t-t^2-t^3+t^4},\quad v=\frac{1-3t+2t^2+2t^3-t^4}{1-t-t^2-t^3+t^4}. \tag{3.7}$$

Substitute (3.7) for (3.6), and multiply

$$-\frac{(1-t-t^2-t^3+t^4)^8}{2(2-5t+2t^2+2t^3-5t^4+2t^5)^4(1-3t+5t^3-3t^5+t^6)},$$

we obtain

$$\begin{aligned} &(1-3t+5t^3-3t^5+t^6)+3t(1-t)(1-2t-t^2+t^3)x \\ &\quad -3t(1-t)(1-t-2t^2+t^3)y-(1-3t+t^3)(1-3t^2+t^3)z=0. \end{aligned}$$

Since S is contained in a half space divided by H, we have the following:

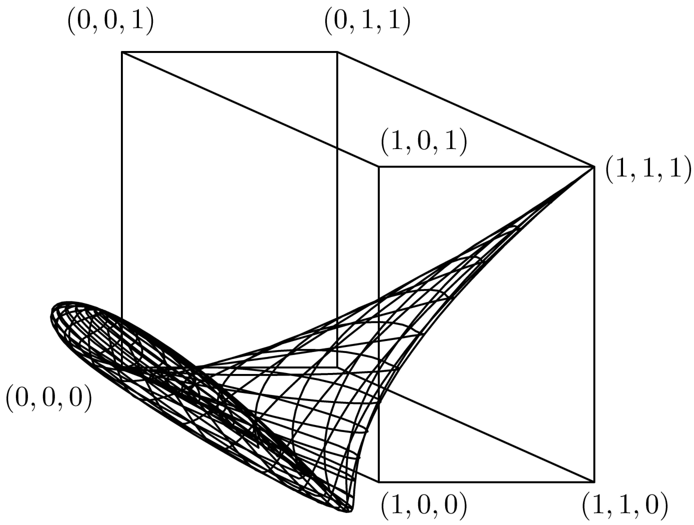
$$\begin{aligned} &(1-3t+5t^3-3t^5+t^6)S_4+3t(1-t)(1-2t-t^2+t^3)S_{3,1} \\ &\quad\geq 3t(1-t)(1-t-2t^2+t^3)S_{1,3}+(1-3t+t^3)(1-3t^2+t^3)S_{2,2}. \end{aligned} \tag{1.7}$$

But this is also a special case of Theorem 1.2.

Some reader may want to obtain some inequalities with S_4, $S_{3,1}$, $S_{2,2}$, US_1 by the same way. Let $u:=a/c$, $v:=b/c$ and

$$x:=\frac{S_{3,1}}{S_3}=\frac{u^3v+v^3+u}{u^4+v^4+1},\quad y:=\frac{S_{2,2}}{S_3},\quad z:=\frac{US_1}{S_4}.$$

This determines a rational quartic surface S as the following figure.



The singularities of S are not only the point $P = (1,\, 1,\, 1)$ but also a curve as in the figure. Moreover $S \cap \mathbb{R}^3_+$ is not concave cone. This observation will not succeed. But see Remark 4.1.

4. APPENDIX: NOTE FOR THE ELEMENTARY SYMMETRIC POLYNOMIALS

For the elementary symmetric polynomials $S_1 = a+b+c$, $S_{1,1} = ab+bc+ca$ and $U = abc$, the following theorem holds.

Theorem 4.1. *For non-negative real numbres a, b, c,*

$$4S_1^3U - S_1^2S_{1,1}^2 + 4S_{1,1}^3 - 18S_1S_{1,1}U + 27U^2 \le 0 \tag{4.1}$$

holds.

Proof. Let $x := \dfrac{U}{S_1^3}$, $y := \dfrac{S_{1,1}}{S_1^2}$, and $t := \dfrac{c}{S_1}$. Since $S_{1,1} = ab + c(a+b) = \dfrac{U}{c} + c(S_1 - c)$, we have

$$x - ty + t^2 - t^3 = 0. \tag{4.2}$$

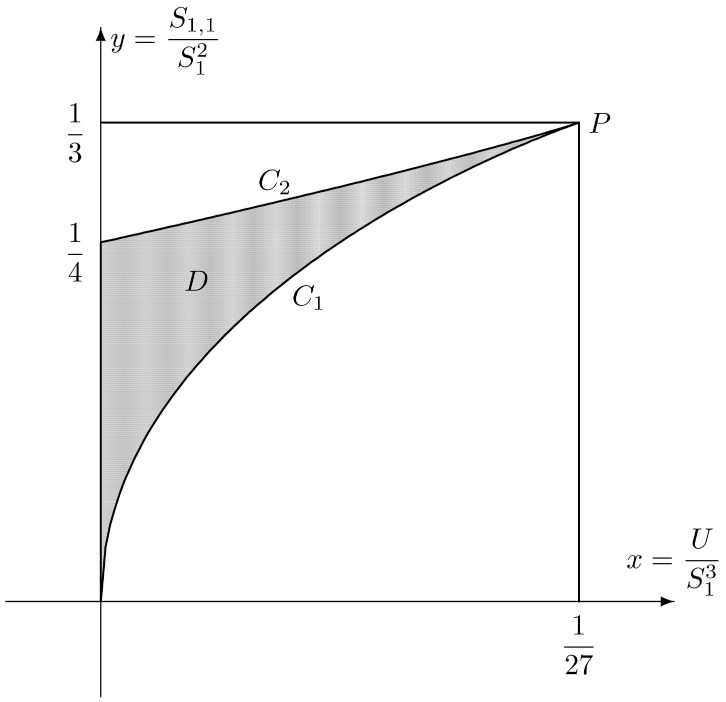
Let L_t be the line in $(x,\, y)$-plane defined by (4.2). The envelope of the family L_t is the curve C defined by

$$4y^3 - y^2 - 18xy + 27x^2 + 4x = 0$$

Note that L_t tangents to C at

$$(x,\, y) = \bigl(t^2(1-2t),\, t(2-3t)\bigr) =: \bigl(p(t),\, q(t)\bigr),$$

and this is a parametrization of C. C has a singular point at $P = \left(\dfrac{1}{27},\, \dfrac{1}{3}\right)$ when $t = 1/3$.



Let C_1 be the part of C with $0 \leq t \leq 1/3$, and C_2 be the part with $1/3 \leq t \leq 1/2$. We shall prove that when a, b, c vary all non-negative real numbers, (x, y) varies all the points of the closed domain D defined by

$$4y^3 - y^2 - 18xy + 27x^2 + 4x \leq 0, \quad x \geq 0, \quad y \geq 0.$$

We may assume $c \leq a$, $c \leq b$ and $a + b + c = 1$. Then $0 \leq t \leq 1/3$. Thus the point $(p(t), q(t))$ moves all the point on C_1. Since C_1 is concave, L_t exists above C_1. Since

$$x = tab \leq t\left(\frac{a+b}{2}\right)^2 = t\left(\frac{1-t}{2}\right)^2,$$

x varies $0 \leq x \leq t(1-t)^2/4$. By (4.2), when $x = t(1-t)^2/4$, then $y = 1 + 2t - 3t^2$. This point (x, y) lies on C_2. ∎

Since C_2 is convex, we have $y \leq \dfrac{9x+1}{4}$. Thus $S_1^3 + 9U \geq 4S_1S_{1,1}$. This is equivalent to $S_3 + 3U \geq T_{2,1}$.

Remark 4.1. We obtained a more strict inequality than (1.6), during this article is under publishing process. We only present its statement:

For non-negative real numbers a, b, c and s, and for $t \geq 1$, the following inequality holds:

$$S_4 - \left(2s - \frac{t}{s}\right) S_{3,1} - \left(\frac{2}{s} - st\right) S_{1,3} + \left(s^2 + \frac{1}{s^2} - 2t\right) S_{2,2} + \left(1 - (s-1)^2\left(1 + \frac{t}{s} + \frac{1}{s^2}\right)\right) US_1 \geq 0. \tag{4.3}$$

The equality holds if and only if $a = b = c$ or $a : b : c = 0 : s : 1$ or any cyclic permutation thereof. Conversely, if $t < 1$ and $s > 0$, there exists non-negative a, b, c for which (4.3) does not hold.

Remark 4.2. We also announce the following: Let

$$F_t := t^2 S_3 + (t^4 - 2t)S_{2,1} - (2t^3 - 1)S_{1,2} - (3t^4 - 6t^3 + 3t^2 - 6t + 3)U,$$
$$F_\infty := S_{2,1} - 3U.$$

If $f(a, b, c)$ is a cubic homogeneous cyclic polynomial such that $f(a, b, c) \geq 0$ for any $a \geq 0$, $b \geq 0$, $c \geq 0$, and that $f(1, 1, 1) = 0$, then there exists $\lambda_i \geq 0$ and $s_i \in [0, \infty]$ ($i = 1, 2, 3$) such that $f = \lambda_1 F_{s_1} + \lambda_2 F_{s_2} + \lambda_3 F_{s_3}$.

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