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APPROXIMATION OF COMMON FIXED POINTS OF A FINITE FAMILY OF ASYMPTOTICALLY DEMICONTRACTIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. By virtue of new analytic techniques, we analyze and study several strong convergence theorems for the approximation of common fixed points of asymptotically demicontractive mappings via the multistep iterative sequence with errors in Banach spaces. Our results improve and extend the corresponding ones announced by Osilike , Osilike and Aniagbosor, Igbokwe, Cho et al., Moore and Nnoli, Hu and all the others.

Key words and phrases: Asymptotically demicontractive mappings, Asymptotically pseudocontractive mappings, Common fixed points.

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1. INTRODUCTION AND PRELIMINARIES

Let X be a real normed space and $J : X \to 2^{X^*}$ denotes the normalized duality mapping defined by:

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, x \in X,$$

where X^* denotes the dual space of X and $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing of X and X^*. If X is a smooth Banach space, then J is single-valued. In the sequel, we shall use j denote the single-valued duality mapping and $F(T)$ denote the set of fixed points of a mapping T, i.e., $F(T) = \{x \in X : Tx = x\}$.

In 1996, Liu [1] introduced the notion of k-strictly asymptotically pseudocontractive and asymptotically demicontractive mappings in Hilbert spaces as follows:

Definition 1.1. ([1]) Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ is said to be

(i) k-strictly asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$, if there exists a constant $k \in [0, 1)$ such that

$$\|T^n x - T^n y\|^2 \le k_n^2\|x - y\|^2 + k\|(x - T^n x) - (y - T^n y)\|^2, \tag{1.1}$$

for all $x, y \in C$ and $n \ge 1$.

(ii) asymptotically demicontractive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$, if $F(T) \neq \emptyset$ and there exists a constant $k \in [0, 1)$ such that

$$\|T^n x - p\|^2 \le k_n^2\|x - p\|^2 + k\|x - T^n x\|^2, \tag{1.2}$$

for all $x \in C$, $p \in F(T)$ and $n \ge 1$.

Moreover, he proved several strong convergence theorems for approximating the fixed points of k-strictly asymptotically pseudocontractive and asymptotically demicontractive mappings in Hilbert spaces via the modified Mann iterative sequence introduced by Schu [2, 3].

By virtue of the normalized duality mapping, Osilike [4] first extended the concepts of k-strictly asymptotically pseudocontractive and asymptotically demicontractive mappings from Hilbert spaces to general Banach spaces.

Definition 1.2. ([4]) A mapping $T : C \to C$ is said to be

(i) k-strictly asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$, if there exists $k \in [0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle (I - T^n)x - (I - T^n)y, j(x - y)\rangle \ge \frac{1}{2}(1-k)\|(I-T^n)x-(I-T^n)y\|^2-\frac{1}{2}(k_n^2-1)\|x-y\|^2, \tag{1.3}$$

for all $x, y \in C$ and $n \ge 1$.

(ii) asymptotically demicontractive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$, if $F(T) \neq \emptyset$ and there exists a constant $k \in [0, 1)$ and $j(x - p) \in J(x - p)$ such that

$$\langle x - T^n x, j(x - p)\rangle \ge \frac{1}{2}(1 - k)\|x - T^n x\|^2 - \frac{1}{2}(k_n^2 - 1)\|x - p\|^2, \tag{1.4}$$

for all $x \in C$, $p \in F(T)$ and $n \ge 1$.

Furthermore, T is said to be uniformly L-lipschitizian mapping, if there exists a constant $L \ge 1$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|,$$

for all $x, y \in C$ and $n \ge 1$.

Remark 1.1. (i) In the Hilbert spaces, (1.1) and (1.2) are equivalent to (1.3) and (1.4), respectively.

(ii) If T is k-strictly asymptotically pseudocontractive mapping, then T is uniformly $L-$Lipschitzian mapping(cf. [6], [9]).

The following theorem is due to Osilike and Aniagbosor [5].

Theorem 1.1. ([5]) *Let $q > 1$ and X be a real $q-$uniformly smooth Banach space. Let C be a closed convex subset of X and $T : C \to C$ a completely continuous uniformly $L-$lipschitizian asymptotically demicontractive mapping with a sequence $k_n \subseteq [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}$ and $\{c_n^{'}\}$ be real sequences in $[0, 1]$ satisfying the conditions:*

(i) $a_n + b_n + c_n = 1 = a_n^{'} + b_n^{'} + c_n^{'}$;

(ii) $0 < \varepsilon \leq c_q(b_n^{'})^{q-1}(1 + Lb_n)^q \leq \frac{1}{2}[q(1 - k)(1 + L)^{-(q-2)}] - \varepsilon$, *for all* $n \geq 1$ *and for some* $\varepsilon > 0$;

(iii) $\sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} c_n^{'} < \infty$.

Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in C and let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in C$ by

$$
\begin{aligned}
y_n &= a_n x_n + b_n T^n x_n + c_n u_n, n \geq 1, \\
x_{n+1} &= a_n^{'} x_n + b_n^{'} T^n y_n + c_n^{'} v_n, n \geq 1,
\end{aligned} \tag{1.5}
$$

converges strongly to a fixed point of T.

In 2002, Igbokwe [6] extended Theorem 1.1 of Osilike and Aniagbosor [5] from real $q-$uniformly smooth Banach spaces to arbitrary real Banach spaces. More precisely, he proved the following main results.

Theorem 1.2. ([6]) *Let X be a real Banach space and C a nonempty closed convex subset of X. Let $T : C \to C$ be a completely continuous uniformly L-lipschitizian asymptotically demicontractive mapping with sequence $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let the sequence $\{x_n\}$ be defined by (1.5) and $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}$ and $\{c_n^{'}\}$ be real sequences in $[0, 1]$ satisfying*

(i) $a_n + b_n + c_n = 1 = a_n^{'} + b_n^{'} + c_n^{'}$;

(ii) $\sum_{n=1}^{\infty} b_n^{'} = \infty, \sum_{n=1}^{\infty}(b_n^{'})^2 < \infty$;

(iii) $\sum_{n=1}^{\infty} c_n^{'} < \infty, \sum_{n=1}^{\infty} b_n < \infty$, *and* $\sum_{n=1}^{\infty} c_n < \infty$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

By using new analysis technique, Cho et al. [7] established several strong convergence theorems for asymptotically demicontractive mapping in arbitrary real normed linear spaces and Banach spaces which generalized Theorem 1.2 of Igbokwe [6]. They proved the following theorems.

Theorem 1.3. ([7]) *Let X be a real normed linear space, C be a nonempty closed convex subset of X and $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let the sequence $\{x_n\}$ be defined by (1.5) with the restrictions*

$$
\sum_{n=1}^{\infty} b_n^{'} = \infty, \quad \sum_{n=1}^{\infty}(b_n^{'})^2 < \infty, \quad \sum_{n=1}^{\infty} c_n^{'} < \infty, \quad \sum_{n=1}^{\infty} c_n < \infty.
$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to satisfy Condition(A)[12] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(t) > 0$ for all $t > 0$ such that

$$\|x - Tx\| \geq f(d(x, F(T))),$$

for all $x \in C$, where $d(x, F(T)) = \inf_{p \in F(T)} \|x - p\|$.

Theorem 1.4. ([7]) *Let X be a real Banach space, C be a nonempty closed convex subset of X and $T : C \to C$ be a uniformly L-Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let the sequence $\{x_n\}$ be defined by (1.5) with the restrictions*

$$\sum_{n=1}^{\infty} b_n^{'} = \infty, \quad \sum_{n=1}^{\infty} (b_n^{'})^2 < \infty, \quad \sum_{n=1}^{\infty} c_n^{'} < \infty, \quad \sum_{n=1}^{\infty} c_n < \infty.$$

Suppose in addition that T satisfies the Condition(A), then $\{x_n\}$ converges strongly to a fixed point of T.

In 2005, Moore and Nnoli [8] proved a necessary and sufficient condition for approximating fixed point of asymptotically demicontractive mapping. More precisely, they got the following results.

Theorem 1.5. ([8]) *Let X a real Banach space and $T : X \to X$ be a uniformly $L-$Lipschitzian asymptotically demicontractive mapping with sequence $\{k_n\} \subseteq [1,\infty)$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in X$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1.$$

Let $\{x_n\}_{n\geq 1} \subset [0, 1]$ be a real sequence such that $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n(k_n^2 - 1) < \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

In 2008, Hu [9] proved several strong convergence theorems for asymptotically demicontractive mapping via the Noor [10] iterative sequences with errors, which improved the results of Igbokwe [6], Moore and Nnoli [8] and Cho et al. [7].

Theorem 1.6. ([9]) *Let X be a real normed linear space and C a nonempty closed convex subset of X. Let $T : C \to C$ be a demicompact and uniformly $L-$Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}, \{c_n^{'}\}, \{a_n^{''}\}, \{b_n^{''}\}$ and $\{c_n^{''}\}$ be real sequences in $[0, 1]$ satisfying*

(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$;
(ii) $\sum_{n=1}^{\infty} b_n^2 < \infty, \sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} b_n b_n^{'} < \infty, \sum_{n=1}^{\infty} b_n = \infty$;
(iii) $\sum_{n=1}^{\infty}(b_n^{'})^2 < \infty, \sum_{n=1}^{\infty} b_n^{'} c_n^{''} < \infty, \sum_{n=1}^{\infty} b_n^{'} b_n^{''} < \infty, \sum_{n=1}^{\infty} c_n^{'} < \infty$;
(iv) $\sum_{n=1}^{\infty} b_n(k_n^2 - 1) < \infty, \sum_{n=1}^{\infty} b_n^{'}(k_n^2 - 1) < \infty$.

Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be bounded sequences in C and let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in C$ by

$$\begin{aligned} z_n &= a_n^{''} x_n + b_n^{''} T^n x_n + c_n^{''} w_n, n \geq 1, \\ y_n &= a_n^{'} x_n + b_n^{'} T^n z_n + c_n^{'} v_n, n \geq 1, \\ x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n, n \geq 1, \end{aligned} \tag{1.6}$$

converges strongly to a fixed point of T.

Theorem 1.7. ([9]) *Let X be a real Banach space and C a nonempty closed convex subset of X. Let $T : C \to C$ be a demicompact and uniformly $L-$Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$. Let the sequence*

$\{x_n\}$ be as in Theorem 1.6. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

The multistep iterative sequence with errors which studied by Huang [11] is defined as follows : Let $\{T_i\}_{i=1}^r$ be a finite family of asymptotically mappings on C. For any $x_1 \in C$, $\{x_n\}$ is defined by

$$(1.7)\qquad \begin{cases} x_1 \in C, \\ x_{n+1} = a_{1n}x_n + b_{1n}T_1^n y_{1n} + c_{1n}u_{1n}, \\ y_{jn} = a_{(j+1)n}x_n + b_{(j+1)n}T_{(j+1)}^n y_{(j+1)n} + c_{(j+1)n}u_{(j+1)n}, n \geq 1, \end{cases}$$

where $\{a_{in}\}, \{b_{in}\}$ and $\{c_{in}\}$ are real sequences in $[0, 1]$ satisfying $a_{in}+b_{in}+c_{in} = 1$, for any $i = 1, 2, \cdots, r$ and $n \geq 1$. $\{u_{in}\}, i = 1, 2, \cdots, r$ are bounded sequences in C, $j = 1, 2, \cdots, r-1$, $y_{rn} = x_n$.

Remark 1.2. (i) If $r = 2$, $T_1 = T_2 = T$, then (1.7) reduces to (1.5).

(ii) If $r = 3$, $T_1 = T_2 = T_3 = T$, then (1.7) reduces to the Noor iterative sequences studied by Hu [9].

Motivated and inspired by theses results, the purpose of this paper is to establish several strong convergence theorems for the multistep iterative sequence with errors for a finite family of asymptotically demicontractive mappings in arbitrary normed linear spaces and Banach spaces. Our results improve and generalize the corresponding results of Osilike [4], Osilike and Aniagbosor [5], Igbokwe [6], Cho et al. [7], Moore and Nnoli [8], Hu [9] and others.

In the sequel, we shall need the following definitions and results.

A mapping $T : C \to C$ is said to be demicompact if for any bounded sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in C$. T is said to be completely continuous if for every bounded sequence $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that the sequence $\{Tx_{n_j}\}$ converges to some element of the range of T.

Remark 1.3. The following results are due to [12]. A mapping T is completely continuous, then it must be demicompact, and if T is continuous and demicompact, then it must satisfy the Condition(A).

Lemma 1.8. ([13]) *Let $\{a_n\}, \{b_n\}, \{\lambda_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \lambda_n)a_n + b_n, \qquad n \geq 1.$$

If $\sum_{n=1}^{\infty} \lambda_n < +\infty, \sum_{n=1}^{\infty} b_n < +\infty$, then (i) $\lim_{n\to\infty} a_n$ exists. (ii) If $\liminf_{n\to\infty} a_n = 0$, we have $\lim_{n\to\infty} a_n = 0$.

2. MAIN RESULTS

In this section, we shall use the following notations: Let $\{T_i\}_{i=1}^r$ be a finite family of asymptotically demicontractive self mappings on C with sequences $\{k_{in}\} \subseteq [1, \infty)$ and $\lim_{n\to\infty} k_{in} = 1$, for all $i = 1, 2, \cdots, r$, if $F(T_i) \neq \emptyset$ and there exist constants $k_i \in [0, 1)$ and $j(x - p) \in J(x - p)$ such that

$$\langle x - T_i^n x, j(x - p)\rangle \geq \frac{1}{2}(1 - k_i)\|x - T^n x\|^2 - \frac{1}{2}(k_{in}^2 - 1)\|x - p\|^2,$$

for all $x \in C$, $p \in F(T_i)$ and $i = 1, 2, \cdots, r$.

Let $k = \max_{1\leq i\leq r}\{k_i\}$, $k_n = \max_{1\leq i\leq r}\{k_{in}\}$, then

$$\langle x - T_i^n x, j(x-p)\rangle \geq \frac{1}{2}(1-k)\|x - T^n x\|^2 - \frac{1}{2}(k_n^2 - 1)\|x-p\|^2,$$

First, we prove the following lemma.

Lemma 2.1. *Let X be a real normed space and C a nonempty convex subset of X. Let $T_i : C \to C, i = 1, 2, \cdots, r$ be a finite family of uniformly L_i-Lipschitzian asymptotically demicontractive mappings with a sequence $\{k_{in}\} \subseteq [1,\infty)$ and $\sum_{n=1}^{\infty}(k_{in}^2 - 1) < \infty$. Let $F := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Suppose the sequence $\{x_n\}$ is defined by (1.7) and satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} b_{1n} = \infty, \sum_{n=1}^{\infty} b_{1n}^2 < \infty$;
(ii) $\sum_{n=1}^{\infty} b_{2n} < \infty, \sum_{n=1}^{\infty} c_{in} < \infty, i = 1, 2, \cdots, r.$

Then
(i) $\lim_{n\to\infty}\|x_n - p\|$ *exists for all* $p \in F$;
(ii) $\liminf_{n\to\infty}\|x_n - T_1 x_n\| = 0.$

Proof. Let $p \in F$ and $L = \max_{1\leq i\leq r} L_i$. Since $\{u_{in}, i = 1, 2, \cdots, r\}$ are bounded sequences in C, so there exists a constant $M > 0$ such that

$$M = \max_{1\leq i\leq r}\left\{\sup_{n\geq 1}\|u_{in} - p\|\right\}.$$

By (1.7), we have

$$\begin{aligned}\|y_{(r-1)n-p}\| &= \|a_{rn}(x_n - p) + b_{rn}(T_r^n x_n - p) + c_{rn}(u_{rn} - p)\| \\ &\leq a_{rn}\|x_n - p\| + b_{rn}L\|x_n - p\| + c_{rn}M \\ &\leq (1 - b_{rn})\|x_n - p\| + b_{rn}L\|x_n - p\| + c_{rn}M \\ &\leq L\|x_n - p\| + c_{rn}M,\end{aligned}$$

and

$$\begin{aligned}\|y_{(r-2)n} - p\| &\leq a_{(r-1)n}\|x_n - p\| + b_{(r-1)n}L\|y_{(r-1)n} - p\| + c_{(r-1)n}M \\ &\leq a_{(r-1)n}\|x_n - p\| + b_{(r-1)n}L^2\|x_n - p\| + c_{rn}LM + c_{(r-1)n}M \\ &\leq L^2\|x_n - p\| + \sum_{i=r-1}^{r} c_{in}L^{i-(r-1)}M.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|y_{(r-3)n} - p\| &\leq a_{(r-2)n}\|x_n - p\| + b_{(r-2)n}L\|y_{(r-2)n} - p\| + c_{(r-2)n}M \\ &\leq a_{(r-2)n}\|x_n - p\| + b_{(r-2)n}L^3\|x_n - p\| + c_{rn}L^2M + c_{(r-1)n}LM + c_{(r-2)n}M \\ &\leq L^3\|x_n - p\| + \sum_{i=r-2}^{r} c_{in}L^{i-(r-2)}M.\end{aligned}$$

By induction, we obtain

$$\|y_{1n} - p\| \leq L^{r-1}\|x_n - p\| + \sum_{i=2}^{r} c_{in}L^{i-2}M. \tag{2.1}$$

On the other hand, we get from (1.7) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|a_{1n}x_n + b_{1n}T_1^n y_{1n} + c_{1n}u_{1n} - p\|^2 \\
&= \|x_n - p + b_{1n}(T_1^n y_{1n} - x_n) + c_{1n}(u_{1n} - x_n)\|^2 \\
&= \langle x_n - p + b_{1n}(T_1^n y_{1n} - x_n) + c_{1n}(u_{1n} - x_n), j(x_{n+1}-p)\rangle \\
&\le \|x_n - p\| \cdot \|x_{n+1}-p\| - b_{1n}\langle x_n - T_1^n y_{1n}, j(x_{n+1}-p)\rangle \\
&\quad + c_{1n}\langle u_{1n} - x_n, j(x_{n+1}-p)\rangle \\
&\le \frac{1}{2}\|x_n - p\|^2 + \frac{1}{2}\|x_{n+1}-p\|^2 - b_{1n}\langle x_{n+1} - T_1^n x_{n+1}, j(x_{n+1}-p)\rangle \\
&\quad + b_{1n}\langle x_{n+1} - x_n, j(x_{n+1}-p)\rangle + b_{1n}\langle T_1^n y_{1n} - T_1^n x_{n+1}, j(x_{n+1}-p)\rangle \\
&\quad + c_{1n}\langle u_{1n} - x_n, j(x_{n+1}-p)\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \|x_n - p\|^2 - 2b_{1n}\langle x_{n+1} - T_1^n x_{n+1}, j(x_{n+1}-p)\rangle \\
&\quad + 2b_{1n}\langle x_{n+1} - x_n, j(x_{n+1}-p)\rangle + 2b_{1n}\langle T_1^n y_{1n} - T_1^n x_{n+1}, j(x_{n+1}-p)\rangle \\
&\quad + 2c_{1n}\langle u_{1n} - x_n, j(x_{n+1}-p)\rangle \\
&\le \|x_n - p\|^2 - b_{1n}(1-k)\|x_{n+1} - T_1^n x_{n+1}\|^2 + b_{1n}(k_n^2 - 1)\|x_{n+1}-p\|^2 \\
&\quad + 2b_{1n}\langle x_{n+1} - x_n, j(x_{n+1}-p)\rangle + 2b_{1n}\langle T_1^n y_{1n} - T_1^n x_{n+1}, j(x_{n+1}-p)\rangle \\
&\quad + 2c_{1n}\langle u_{1n} - x_n, j(x_{n+1}-p)\rangle \\
&= \|x_n - p\|^2 - b_{1n}(1-k)\|x_{n+1} - T_1^n x_{n+1}\|^2 + b_{1n}(k_n^2 - 1)\|x_{n+1}-p\|^2 \\
&\quad + 2b_{1n}^2\langle T_1^n y_{1n} - x_n, j(x_{n+1}-p)\rangle + 2b_{1n}\langle T_1^n y_{1n} - T_1^n x_{n+1}, j(x_{n+1}-p)\rangle \\
&\quad + 3c_{1n}\langle u_{1n} - x_n, j(x_{n+1}-p)\rangle \\
&\le \|x_n - p\|^2 - b_{1n}(1-k)\|x_{n+1} - T_1^n x_{n+1}\|^2 + b_{1n}(k_n^2 - 1)\|x_{n+1}-p\|^2 \\
&\quad + 2b_{1n}^2\|T_1^n y_{1n} - x_n\| \cdot \|x_{n+1}-p\| + 2b_{1n}\|T_1^n y_{1n} - T_1^n x_{n+1}\| \cdot \|x_{n+1}-p\| \\
&\quad + 3c_{1n}\|u_{1n} - x_n\| \cdot \|x_{n+1}-p\| \\
&\le \|x_n - p\|^2 - b_{1n}(1-k)\|x_{n+1} - T_1^n x_{n+1}\|^2 + b_{1n}(k_n^2 - 1)\|x_{n+1}-p\|^2 \\
&\quad + 2b_{1n}^2\|T_1^n y_{1n} - x_n\| \cdot \|x_{n+1}-p\| + 2b_{1n}L\|y_{1n} - x_{n+1}\| \cdot \|x_{n+1}-p\| \\
&\quad + 3c_{1n}(M + \|x_n - p\|) \cdot \|x_{n+1}-p\|. \qquad (2.2)
\end{aligned}$$

Next, we have the following estimations.

$$\begin{aligned}
\|y_{1n} - x_n\| &= \|b_{2n}(T_2^n - x_n) + c_{2n}(u_{2n} - x_n)\| \\
&\le b_{2n}\|T_2^n y_{2n} - x_n\| + c_{2n}\|u_{2n} - x_n\| \\
&\le b_{2n}(\|T_2^n y_{2n} - p\| + \|x_n - p\|) + c_{2n}(\|u_{2n} - p\| + \|p - x_n\|) \\
&\le b_{2n}L\|y_{2n} - p\| + b_{2n}\|x_n - p\| + c_{2n}M + c_{2n}\|x_n - p\| \\
&\le b_{2n}L^{r-1}\|x_n - p\| + b_{2n}\sum_{i=3}^{r} c_{in}L^{i-2}M + b_{2n}\|x_n - p\| + c_{2n}M + c_{2n}\|x_n - p\|, \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}\|x_{n+1}-y_{1n}\| &= \|a_{1n}(x_n-y_{1n})+b_{1n}(T_1^n y_{1n}-y_{1n})+c_{1n}(u_{1n}-y_{1n})\|\\ &\le a_{1n}\|x_n-y_{1n}\|+b_{1n}\|T_1^n y_{1n}-y_{1n}\|+c_{1n}\|u_{1n}-y_{1n}\|\\ &\le \|x_n-y_{1n}\|+b_{1n}(L+1)\|y_{1n}-p\|+c_{1n}M+c_{1n}\|y_{1n}-p\|\\ &\le \|x_n-y_{1n}\|+b_{1n}(L+1)L^{r-1}\|x_n-p\|+\sum_{i=2}^{r}c_{in}(L+1)L^{i-2}M+c_{1n}L^{r-1}\|x_n-p\|\\ &\quad +c_{1n}\sum_{i=2}^{r}c_{in}L^{i-2}M+c_{1n}M, \end{aligned} \tag{2.4}$$

$$\begin{aligned}\|T_1^n y_{1n}-x_n\| &\le \|T_1^n y_{1n}-p\|+\|x_n-p\|\\ &\le L\|y_{1n}-p\|+\|x_n-p\|\\ &\le (1+L^r)\|x_n-p\|+\sum_{i=2}^{r}c_{in}L^{i-1}M, \end{aligned} \tag{2.5}$$

$$\begin{aligned}\|x_{n+1}-p\| &= \|a_{1n}(x_n-p)+b_{1n}(T_1^n y_{1n}-p)+c_{1n}(u_{1n}-p)\|\\ &\le a_{1n}\|x_n-p\|+b_{1n}\|T_1^n y_{1n}-p\|+c_{1n}M\\ &\le (1-b_{1n})\|x_n-p\|+b_{1n}L\|y_{1n}-p\|+c_{1n}M\\ &\le (1-b_{1n})\|x_n-p\|+b_{1n}L^r\|x_n-p\|+\sum_{i=2}^{r}c_{in}L^{i-1}M+c_{1n}M\\ &\le L^r\|x_n-p\|+\sum_{i=1}^{r}c_{in}L^{i-1}M. \end{aligned} \tag{2.6}$$

Substituting (2.4)-(2.6) into (2.2) and noticing the inequality $\|x_n-p\|\le 1+\|x_n-p\|^2$. Then we get

$$\|x_{n+1}-p\|^2 \le (1+r_n)\|x_n-p\|^2+s_n-b_{1n}(1-k)\|x_{n+1}-T_1^n x_{n+1}\|^2, \tag{2.7}$$

where $\{r_n\}$ and $\{s_n\}$ are sequences such that $\sum_{n=1}^{\infty} r_n<\infty, \sum_{n=1}^{\infty} s_n<\infty$. It follows from Lemma 1.8(i) that $\lim_{n\to\infty}\|x_n-p\|$ exists. Therefore, $\{x_n\}$ is bounded. This completes the proof of part(i).

Now, we prove the claim (ii). It follows from (2.7) that

$$b_{1n}(1-k)\|x_{n+1}-T_1^n x_{n+1}\|^2 \le (1+r_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2+s_n, \tag{2.8}$$

Then $\sum_{n=1}^{\infty} b_{1n}(1-k)\|x_{n+1}-T_1^n x_{n+1}\|^2<\infty$. Notice the condition $\sum_{n=1}^{\infty} b_{1n}=\infty$, therefore, $\liminf_{n\to\infty}\|x_{n+1}-T_1^n x_{n+1}\|=0$.

Observe that $\lim_{n\to\infty}\|x_{n+1}-y_{1n}\|=0$ and $\lim_{n\to\infty}\|y_{1n}-x_n\|=0$, we have

$$\begin{aligned}\|x_n-T_1^n x_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-T_1^n x_{n+1}\|+\|T_1^n x_{n+1}-T_1^n x_n\|\\ &\le (1+L)\|x_{n+1}-x_n\|+\|x_{n+1}-T_1^n x_{n+1}\|\\ &\le (1+L)(\|x_{n+1}-y_{1n}\|+\|y_{1n}-x_n\|)+\|x_{n+1}-T_1^n x_{n+1}\|, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \|x_n - T_1x_n\| &\le \|x_n - T_1^n x_{n-1}\| + \|T_1^n x_{n-1} - T_1 x_{n-1}\| + \|T_1 x_{n-1} - T_1 x_n\| \\ &\le \|x_n - T_1^n x_n\| + L\|x_n - x_{n-1}\| + L\|T_1^{n-1}x_{n-1} - x_{n-1}\|, \end{aligned} \tag{2.10}$$

Therefore, $\liminf_{n\to\infty} \|x_n - T_1x_n\| = 0$.
∎

Theorem 2.2. *Let X be a real Banach space and C a nonempty closed convex subset of X. Let $T_i : C \to C, i = 1, 2, \cdots, r$ be a finite family uniformly L_i-lipschitzian asymptotically demicontractive mappings with a sequence $\{k_{in}\} \subseteq [1, \infty)$ and $\sum_{n=1}^{\infty}(k_{in}^2 - 1) < \infty$. If $F := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Suppose the sequence $\{x_n\}$ is defined by (1.7) and satisfying the conditions in Lemma 2.1. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{r}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. The necessity is obvious. We just need to prove the sufficiency. It follows from (2.7) that

$$d(x_{n+1}, p)^2 \le (1 + r_n)d(x_n, p)^2 + s_n, \quad \forall p \in F.$$

By Lemma 1.8, we know that $\lim_{n\to\infty} d(x_n, F)$ exists. With the help of inequality $1 + x \le e^x, x \ge 0$. For any integer $m \ge 1$, we have

$$\begin{aligned} \|x_{n+m} - p\|^2 &\le (1 + r_{n+m-1})\|x_{n+m-1} - p\|^2 + s_{n+m-1} \\ &\le e^{r_{n+m-1}}\|x_{n+m-1} - p\|^2 + s_{n+m-1} \\ &\le e^{r_{n+m-1}}e^{r_{n+m-2}}\|x_{n+m-2} - p\|^2 + e^{r_{n+m-1}}s_{n+m-2} + s_{n+m-1} \\ &\quad \cdots \\ &\le e^{\sum_{k=n}^{n+m-1} r_k}\|x_n - p\|^2 + e^{\sum_{k=n}^{n+m-1} r_k}\sum_{k=n}^{n+m-1} s_k \\ &\le e^{\sum_{n=1}^{\infty} r_n}\|x_n - p\|^2 + e^{\sum_{n=1}^{\infty} r_n}\sum_{k=n}^{n+m-1} s_k \\ &= M^{'}\|x_n - p\|^2 + M^{'}\sum_{k=n}^{n+m-1} s_k, \end{aligned} \tag{2.11}$$

where $M' = e^{\sum_{n=1}^{\infty} r_n}$.

Since $\liminf_{n\to\infty} d(x_n, F) = 0$, without loss of generality, we may assume that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_{n_k}\} \subset F$ such that $\|x_{n_k} - p_{n_k}\| \to 0$ as $k \to \infty$. Then for any $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that

$$\|x_{n_k} - p_{n_k}\| < \frac{\varepsilon}{2\sqrt{2M'}}, \text{ and } \sum_{k=n_{k_\varepsilon}}^{\infty} s_k < \frac{\varepsilon^2}{8M'},$$

for all $k \ge k_\varepsilon$.

For any $m \geq 1$ and for all $n > n_{k_\varepsilon}$, by (2.11), we have

$$\begin{aligned}\|x_{n+m} - x_n\|^2 &\leq (\|x_{n+m} - p_{n_{k_\varepsilon}}\| + \|x_n - p_{n_{k_\varepsilon}}\|)^2 \\ &\leq 2(\|x_{n+m} - p_{n_{k_\varepsilon}}\|^2 + \|x_n - p_{n_{k_\varepsilon}}\|^2) \\ &\leq 2M' \|x_{n_{k_\varepsilon}} - p_{n_{k_\varepsilon}}\|^2 + 2M' \sum_{k=n_{k_\varepsilon}}^{\infty} s_k + 2M' \|x_{n_{k_\varepsilon}} - p_{n_{k_\varepsilon}}\|^2 \\ &\quad + 2M' \sum_{k=n_{k_\varepsilon}}^{\infty} s_k \\ &\leq 4M' \frac{\varepsilon^2}{8M'} + 4M' \frac{\varepsilon^2}{8M'} = \varepsilon^2.\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since C is a nonempty closed convex subset of Banach space X, so there exists a $q \in C$ such that $x_n \to q$ as $n \to \infty$. Finally, we prove that $q \in F$. In fact, notice that $d(q, F) = 0$. Therefore, for any $\varepsilon_1 > 0$, there exists a $p_2 \in F$ such that $\|p_2 - p\| < \varepsilon_1$. Then, we have

$$\begin{aligned}\|T_i q - q\| &\leq \|T_i q - p_2\| + \|p_2 - q\| \\ &\leq (L+1)\|p_2 - q\| < (L+1)\varepsilon_1.\end{aligned}$$

By the arbitrary of ε_1, we know that $T_i q = q$, for all $i = 1, 2, \cdots, r$, i.e., $q \in F$.

∎

Remark 2.1. If $T_1 = T_2 = \cdots = T_r$ and T_1 satisfies the Condition(A), then $\{x_n\}$ converges strongly to a fixed point of T_1. In fact, with the help of Lemma 2.1(ii), we know that $\liminf_{n\to\infty} f(d(x_n, F)) = 0$. Hence, $\liminf_{n\to\infty} d(x_n, F) = 0$. Thus, Theorem 1.4 of Cho et al. [7] becomes a corollary of Theorem 2.2.

Theorem 2.3. *Let X be a real normed space and C be a nonempty closed convex subset of X. Let $T_i : C \to C, i = 1, 2, \cdots, r$ be a finite family of asymptotically demicontractive mappings with a sequence $\{k_{in}\} \subseteq [1, \infty)$ and $\sum_{n=1}^{\infty}(k_{in}^2 - 1) < \infty$, and $F := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Suppose the sequence $\{x_n\}$ is defined by (1.7) and satisfying the conditions in Lemma 2.1. Let the family of $\{T_i\}_{i=1}^r$ satisfy*

$$\|T_i^n x - T_j^n y\| \leq L\|x - y\|,$$

for all $x, y \in C, n \geq 1$ and all pairs (i, j), $L > 0$. Suppose in addition that T_1 is demicompact, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. It follows from Lemma 2.1(ii) that

$$\liminf_{n\to\infty} \|x_n - T_1 x_n\| = 0.$$

So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} \|x_{n_j} - T_1 x_{n_j}\| = 0$.

Since T_1 is demicompact and $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ such that $x_{n_{j_k}} \to q \in C$ as $k \to \infty$. Therefore,

$$\begin{aligned}\|q - T_1 q\| &\leq \|q - x_{n_{j_k}}\| + \|x_{n_{j_k}} - T_1 x_{n_{j_k}}\| + \|T_1 x_{n_{j_k}} - T_1 q\| \\ &\leq (L+1)\|q - x_{n_{j_k}}\| + \|x_{n_{j_k}} - T_1 x_{n_{j_k}}\| \to 0 \text{ as } k \to \infty.\end{aligned}$$

Hence, $q \in F(T_1)$. Now, we prove q is a common fixed point of $\{T_i\}_{i=2}^r$,

$$\begin{aligned}\|T_j q - q\| &\leq \|T_j q - T_j x_{n_{j_k}}\| + \|T_j x_{n_{j_k}} - T_1 q\| \\ &\leq L\|x_{n_{j_k}} - q\| + L\|x_{n_{j_k}} - q\| \\ &\leq 2L\|x_{n_{j_k}} - q\| \to 0 \text{ as } k \to \infty.\end{aligned}$$

Hence, $\|T_j q - q\| = 0$, for all $i = 2, 3, \cdots, r$. Therefore, q is a common fixed point of $\{T_i\}_{i=1}^r$, i.e., $q \in F$. By Lemma 1.8(ii) and (2.7), we know that $\lim_{n\to\infty} \|x_n - q\| = 0$.

∎

Remark 2.2. Theorem 2.2 and Theorem 2.3 not only extend the corresponding results of Igbokwe [6], Cho et al. [7], Moore and Nnoli [8] and Hu [9] from one mapping to a finite family of mappings, but also to a more general iteration methods.

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