SOLUTION OF ONE CONJECTURE ON INEQUALITIES WITH POWER-EXPONENTIAL FUNCTIONS

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Received 12 June, 2010; accepted 2 July, 2010; published 13 December, 2010.

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ABSTRACT. In this paper, we prove the open inequality $a^c + b^e \geq a^e + b^c$ for all positive real numbers $a$ and $b$.

Key words and phrases: Power-exponential function, Logarithmic mean, Convex function.

2000 Mathematics Subject Classification 26D15.

ISSN (electronic): 1449-5910
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1. Introduction

In the paper [1], V. Cîrtoaje conjectured the following inequality
\[(1.1) \quad a^e a + b^e b \geq a^e b + b^e a \]
for all positive real numbers \(a\) and \(b\). We will prove the conjecture.

V. Cîrtoaje proved the inequality \((1.1)\) for all cases, except the cases \(0 < b < \frac{1}{e} < a < 1\) and \(0 < a < \frac{1}{e} < b < 1\). So we will prove the inequality \((1.1)\) for these two cases.

2. Main Results and The Proofs

The logarithmic mean \(L(x, y)\) is defined for \(0 < y \leq x\) as
\[L(x, y) = \frac{x - y}{\ln x - \ln y} \quad \text{and} \quad L(x, x) = x.\]

**Theorem 2.1.** If \(0 < y < x\) and \(1 < x\), then
\[\frac{x^x - y^x}{x^y - y^y} > \frac{x}{y} L(x, y)^{x-y}.\]

**Lemma 2.2.** Define \(f(t) = \frac{A^{t} - 1}{t} (A > 0, t > 0)\). If \(A > 1\), then \(f(t)\) is a convex function. If \(0 < A < 1\), then \(f(t)\) is a concave function.

**Proof.** Put \(u = A^t\), and \(f''(t) = \frac{u}{A^t} a(u)\), where
\[a(u) = (\ln u)^2 - 2 \ln u + 2 \left(1 - \frac{1}{u}\right).\]

It is easy to see that \(a(u) \geq 0 \iff u \geq 1.\]

In the paper [2], J. Sándor showed the convexity of \(f(t)\) for \(A > 1\), in a more stronger form (i.e., log-convexity).

**Proof of Proposition 2.1** Put \(g(t) = \frac{A^t - B^t}{t} (0 < B < 1 < A, t > 0)\). Since \(g(t) = \frac{A^t - 1}{t} - \frac{B^t - 1}{t}\), \(g(t)\) is a convex function. Thus,
\[h(t) = \left(\frac{x}{L(x, y)}\right)^{t} - \left(\frac{y}{L(x, y)}\right)^{t}\]
is convex function. Since
\[\lim_{t \to 0} h(t) = h(1) (= \ln x - \ln y),\]
\(h(t)\) has a single minimum point \(c\) in \((0, 1)\) and \(h(t)\) is strictly increasing for \(t > c\). So \(h(y) < h(x)\). This inequality is equivalent to \((2.1)\).

**Theorem 2.3.** If \(0 < y < x\), then
\[\frac{x}{y} L(x, y)^{x-y} \geq e^{x-y}.\]

**Proof.** From the inequality \(\ln t \geq 1 - \frac{1}{t}\) for all positive real numbers \(t\),
\[1 + L(x, y) \ln L(x, y) \geq L(x, y).\]

Therefore,
\[\left(\frac{x}{y}\right)^{1+L(x,y)\ln L(x,y)} \geq \left(\frac{x}{y}\right)^{L(x,y)}.\]
The inequality \((2.3)\) becomes the desired result \((2.2)\).
3. Proof of the Conjecture

Proof. Without loss of generality, assume that $a \geq b$. As mentioned in the introduction, we will prove the inequality (1.1) for the case $0 < b < \frac{1}{e} < a < 1$. Let $x = ea$ and $y = eb$, where $0 < y < 1 < x < e$. The inequality (1.1) becomes

$$\frac{x^x - y^x}{x^y - y^y} > e^{x-y}.$$ 

This is obvious by Theorem 2.1 and Theorem 2.3.

References
