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SOLUTION OF ONE CONJECTURE ON INEQUALITIES WITH POWER-EXPONENTIAL FUNCTIONS

SEIICHI MANYAMA

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GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, JAPAN
manchanr4@gmail.com

ABSTRACT. In this paper, we prove the open inequality $a^{ea} + b^{eb} \geq a^{eb} + b^{ea}$ for all positive real numbers a and b.

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1. INTRODUCTION

In the paper [1], V. Cîrtoaje conjectured the following inequality

$$a^{ea} + b^{eb} \geq a^{eb} + b^{ea} \tag{1.1}$$

for all positive real numbers a and b. We will prove the conjecture.

V. Cîrtoaje proved the inequality (1.1) for all cases, except the cases $0 < b < \frac{1}{e} < a < 1$ and $0 < a < \frac{1}{e} < b < 1$. So we will prove the inequality (1.1) for these two cases.

2. MAIN RESULTS AND THE PROOFS

The logarithmic mean $L(x, y)$ is defined for $0 < y \leq x$ as

$$L(x, y) = \frac{x - y}{\ln x - \ln y} \quad \text{and} \quad L(x, x) = x.$$

Theorem 2.1. *If $0 < y < x$ and $1 < x$, then*

$$\frac{x^x - y^x}{x^y - y^y} > \frac{x}{y} L(x, y)^{x-y}. \tag{2.1}$$

Lemma 2.2. *Define $f(t) = \frac{A^t - 1}{t}$ ($A > 0$, $t > 0$). If $A > 1$, then $f(t)$ is a convex function. If $0 < A < 1$, then $f(t)$ is a concave function.*

Proof. Put $u = A^t$, and $f''(t) = \frac{u}{t^3} a(u)$, where

$$a(u) = (\ln u)^2 - 2\ln u + 2\left(1 - \frac{1}{u}\right).$$

It is easy to see that $a(u) \gtreqless 0 \Leftrightarrow u \gtreqless 1$. ∎

In the paper [2], J. Sándor showed the convexity of $f(t)$ for $A > 1$, in a more stronger form (i.e., log-convexity).

Proof of Proposition 2.1. Put $g(t) = \frac{A^t - B^t}{t}$ ($0 < B < 1 < A$, $t > 0$). Since $g(t) = \frac{A^t - 1}{t} - \frac{B^t - 1}{t}$, $g(t)$ is a convex function. Thus,

$$h(t) = \frac{\left(\frac{x}{L(x,y)}\right)^t - \left(\frac{y}{L(x,y)}\right)^t}{t}$$

is convex function. Since

$$\lim_{t \to 0} h(t) = h(1) (= \ln x - \ln y),$$

$h(t)$ has a single minimum point c in $(0, 1)$ and $h(t)$ is strictly increasing for $t > c$. So $h(y) < h(x)$. This inequality is equivalent to (2.1). ∎

Theorem 2.3. *If $0 < y < x$, then*

$$\frac{x}{y} L(x, y)^{x-y} \geq e^{x-y}. \tag{2.2}$$

Proof. From the inequality $\ln t \geq 1 - \frac{1}{t}$ for all positive real numbers t,

$$1 + L(x, y) \ln L(x, y) \geq L(x, y).$$

Therefore ,

$$\left(\frac{x}{y}\right)^{1 + L(x,y)\ln L(x,y)} \geq \left(\frac{x}{y}\right)^{L(x,y)}. \tag{2.3}$$

The inequality (2.3) becomes the desired result (2.2). ∎

3. PROOF OF THE CONJECTURE

Proof. Without loss of generality, assume that $a \geq b$. As mentioned in the introduction, we will prove the inequality (1.1) for the case $0 < b < \frac{1}{e} < a < 1$. Let $x = ea$ and $y = eb$, where $0 < y < 1 < x < e$. The inequality (1.1) becomes

$$\frac{x^x - y^x}{x^y - y^y} > e^{x-y}.$$

This is obvious by Theorem 2.1 and Theorem 2.3. ∎

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