ON THE FOCK REPRESENTATION OF THE CENTRAL EXTENSIONS OF THE HEISENBERG ALGEBRA

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ABSTRACT. We examine the possibility of a direct Fock representation of the recently obtained non-trivial central extensions $CE_{Heis}$ of the Heisenberg algebra, generated by elements $a, a^\dagger, h$ and $E$ satisfying the commutation relations $[a, a^\dagger]_{CE_{Heis}} = h$, $[h, a^\dagger]_{CE_{Heis}} = z E$ and $[a, h]_{CE_{Heis}} = \bar{z} E$, where $a$ and $a^\dagger$ are dual, $h$ is self-adjoint, $E$ is the non-zero self-adjoint central element and $z \in \mathbb{C} \setminus \{0\}$. We define the exponential vectors associated with the $CE_{Heis}$ Fock space, we compute their Leibniz function (inner product), we describe the action of $a, a^\dagger$ and $h$ on the exponential vectors and we compute the moment generating and characteristic functions of the classical random variable corresponding to the self-adjoint operator $X = a + a^\dagger + h$.

Key words and phrases: Heisenberg algebra, Central extension, Fock representation.

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1. The centrally extended Heisenberg \(\ast\)-Lie algebra.

The generators \(a, a^\dagger\) and \(h\) of the Heisenberg algebra \(Heis\) satisfy the Lie algebra commutation relations

\[
[a, a^\dagger]_{Heis} = h \quad ; \quad [a, h]_{Heis} = [h, a^\dagger]_{Heis} = 0
\]

and the duality relations (throughout this paper we use \(x^\ast\) to denote the dual of \(x\))

\[
(a)^\ast = a^\dagger \quad ; \quad h^\ast = h
\]

As shown in [1], the Heisenberg algebra can be centrally extended to the \(\ast\)-Lie algebra \(CEHeis\) generated by \(\{a, a^\dagger, h, E\}\) with (non-zero) commutation relations among generators

\[
[a, a^\dagger]_{CEHeis} = h + \lambda E \quad ; \quad [h, a^\dagger]_{CEHeis} = z E \quad ; \quad [a, h]_{CEHeis} = \bar{z} E
\]

where \(\lambda \in \mathbb{R}, z = \Re z + i \Im z \in \mathbb{C}\), and \(E \neq 0\) is the self-adjoint central element. The central extension \(CEHeis\) of \(Heis\) is trivial if and only if \(z = 0\). Duality relations (1.2) still hold. \(CEHeis\) is a nilpotent and thus solvable \(\ast\)-Lie algebra.

Renaming \(h + \lambda E\) to just \(h\) in (1.3) we obtain the equivalent commutation relations

\[
[a, a^\dagger]_{CEHeis} = h \quad ; \quad [h, a^\dagger]_{CEHeis} = z E \quad ; \quad [a, h]_{CEHeis} = \bar{z} E
\]

From now on we will use (1.4) and (1.2) as the defining commutation relations of \(CEHeis\).

2. Representations of \(CEHeis\)

As shown in [2], the generators \(a, a^\dagger, h\) and \(E\) of \(CEHeis\) can be expressed in terms of the generators of the Schrödinger \(\ast\)-Lie algebra generated by \(b, b^\dagger, b^2, b^\dagger 2, b^\dagger b\) and \(1\) where \(b^\dagger, b\) and \(1\) are the generators of a Boson Heisenberg algebra with

\[
[b, b^\dagger] = 1 \quad ; \quad (b^\dagger)^\ast = b
\]

and \(CEHeis\) can therefore be represented (as a proper sub–algebra of the Schrödinger algebra) on the usual Heisenberg Fock space defined as the Hilbert space completion of the linear span of the exponential vectors \(\{y(\lambda) = e^{\lambda b^\dagger} \Phi ; \lambda \in \mathbb{C}\}\) (where \(\Phi\) is the vacuum vector such that \(b \Phi = 0\) and \(||\Phi|| = 1\)) with respect to the inner product

\[
\langle y(\lambda), y(\mu) \rangle = e^{\bar{\lambda} \mu}
\]

by using the well-known representation for non-negative integers \(n\) and \(k\)

\[
b^{\dagger n} b^{k} y(\lambda) = \lambda^{k} \frac{\partial^{n}}{\partial \epsilon^{n}}|_{\epsilon=0} y(\lambda + \epsilon)
\]

In this section we examine the possibility of constructing a direct Fock representation of \(CEHeis\) in a manner similar to that used for the (non-extended) Heisenberg algebra.

**Definition 2.1.** A \(\ast\)-representation of \(CEHeis\) as linear, densely defined operators on a Hilbert space \(\mathcal{H}\) with a cyclic unit vector \(\Phi\) satisfying

\[
a \Phi = 0
\]
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and such that $\Phi$ is in the domain of all the operators of the form (2.12) below, where the exponentials are meant in the sense of series expansion, is called a Fock representation.

In what follows we replace the central element $E$ by the multiplication identity “1” and we simply write $[·, ·]$ instead of $[·, ·]_{CEHeis}$. Notice that if the central extension of the Heisenberg algebra is not trivial, i.e. if $z \neq 0$, then $\Phi$ cannot be an eigenvector of $h$ with eigenvalue $\lambda_h \in \mathbb{R}$ since then, denoting by $\langle ·, · \rangle$ the (linear in the first, conjugate linear in the second argument) Fock space inner product normalized to $\langle \Phi, \Phi \rangle = 1$, we have

\begin{equation}
0 \neq \bar{z} = \langle \Phi, [a, h] \Phi \rangle = \langle \Phi, a h \Phi \rangle = \langle \Phi, a \lambda_h \Phi \rangle = \lambda_h \langle \Phi, a \Phi \rangle = 0
\end{equation}

Therefore we cannot set $h \Phi = \lambda_h \Phi$ where $\lambda_h \in \mathbb{R}$. That, in particular, excludes the option of setting $h \Phi = 0$.

As shown in [2], for all $\lambda, \mu \in \mathbb{C}$ we have that

\begin{equation}
eq \mu a e^\lambda a = e^\mu a e^\lambda a \frac{\lambda h}{2} (\mu z - \lambda \bar{z})
\end{equation}

\begin{equation}a e^\mu a = e^\mu a (a + \mu h + \frac{\mu^2 z}{2})
\end{equation}

\begin{equation}e^\lambda a e^\mu h = e^\mu h e^\lambda a e^{\lambda \mu z}
\end{equation}

\begin{equation}e^\mu h e^\lambda a = e^\lambda a e^\mu h e^{\lambda \mu z}
\end{equation}

\begin{equation}a e^\mu h = e^\mu h (a + \mu \bar{z})
\end{equation}

and

\begin{equation}h e^\lambda a = e^\lambda a (h + \lambda z)
\end{equation}

In general, for $u, v, w, y \in \mathbb{C}$ the centrally extended Heisenberg group elements

\begin{equation}g(u, v, w, y) := e^u a e^v h e^w a e^y E
\end{equation}

obey (see [2] for a proof) the nonlinear group law

\begin{equation}g(\alpha, \beta, \gamma, \delta) g(A, B, C, D) =
\end{equation}

\begin{equation}g(\alpha + A, \beta + B + \gamma A, \gamma + C, \left(\frac{\gamma A^2}{2} + \beta A\right) z + \left(\frac{\gamma^2 A^2}{2} + \gamma B\right) \bar{z} + \delta + D)
\end{equation}

Definition 2.2. For $\alpha, \beta \in \mathbb{C}$ we define the exponential vector $\psi(\alpha, \beta)$ by

\begin{equation}\psi(\alpha, \beta) = e^{\alpha a} e^{\beta h} \Phi
\end{equation}

In the following proposition we compute the sesquilinear form (“Fock space inner product”) associated with two such exponential vectors. In analogy with [5] we refer to that as the “Leibniz function”.

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Proposition 2.1. (Leibniz function) For $w \in \mathbb{C}$ let $f_h(w) = \langle \Phi, e^{w \hbar} \Phi \rangle$. Then, for all $\alpha, \beta, A, B \in \mathbb{C}$

$$\langle \psi(\alpha, \beta), \psi(A, B) \rangle = e^{\left(\frac{\alpha^2 A + \alpha B}{2} + (\alpha + A \beta)\right) z} \langle \Phi, e^{(\alpha A + B + \beta) \hbar} \Phi \rangle$$

and

(2.15) \hspace{1cm} ||\psi(\alpha, \beta)||^2 = \langle \psi(\alpha, \beta), \psi(\alpha, \beta) \rangle = e^{\Re\left((\alpha A^2 + 2 \alpha \beta) z\right)} f_h(||\alpha||^2 + 2 \Re \beta)

Proof. Using (2.13) and the fact that $e^{\bar{\alpha} \cdot \Phi} = \Phi$ we have

$$\langle \psi(\alpha, \beta), \psi(A, B) \rangle = \langle e^{\alpha A} e^{\beta \hbar} \Phi, e^{A \alpha} e^{B \hbar} \Phi \rangle$$

which for $A = \alpha$ and $B = \beta$ yields

$$||\psi(\alpha, \beta)||^2 = \langle \psi(\alpha, \beta), \psi(\alpha, \beta) \rangle = e^{\Re\left((\alpha \alpha^2 + 2 \alpha \beta) z\right)} f_h(||\alpha||^2 + 2 \Re \beta)$$

Minimal requirements on $f_h : \mathbb{C} \rightarrow \mathbb{C}$ so that the Leibniz function $\langle \psi(\alpha, \beta), \psi(A, B) \rangle$ of Proposition 2.1 is positive semi-definite is that $f_h$ is an analytic function such that:

(i) $f_h(0) = 1$

(ii) $f_h(w) > 0$ for all $w \in \mathbb{R}$

(iii) $\overline{f_h(w)} = f_h(\overline{w})$ for all $w \in \mathbb{C}$ (so that the Leibniz function is Hermitian)

(iv) $f_1(w) = e^w$ for all $w \in \mathbb{C}$ (so that we recover the Heisenberg algebra Fock space)

(v) $\frac{\partial}{\partial w} |_{w=0} f(w) = \langle \Phi, h \Phi \rangle \geq 0$ for all $k \geq 0$ (so that $||a^n \hbar^m \Phi|| \geq 0$ for all $n, m \geq 0$, see Corollary 2.2 below)
Corollary 2.2. For all \( n, k \geq 0 \)

\[
(2.16) \quad ||a^\dagger n h^k \Phi||^2 = \sum_{\rho=0}^{k} \sum_{\sigma=0}^{n} k^{\rho \sigma} \delta_{\rho+\sigma,2\theta} \binom{k}{\rho} \binom{\rho}{\theta} \binom{\sigma}{\theta} n! \langle \Phi, h^{n+2k-3\theta} \Phi \rangle
\]

where \( x^{(y)} = x(x-1) \cdots (x-y+1) \) with \( x^{(0)} = 1 \). By condition (v) on \( f_h \).

\[
(2.17) \quad ||a^\dagger n h^k \Phi|| > 0
\]

**Proof.** By Proposition 2.1 for \( \alpha, \beta, A, B \in \mathbb{R} \)

\[
||a^\dagger n h^k \Phi||^2 = \frac{\partial^{n+k}}{\partial \alpha^n \partial \beta^k} \big|_{\alpha=\beta=0} \langle \psi(\alpha, \beta), \psi(A, B) \rangle
\]

\[
= \frac{\partial^{n+k}}{\partial \alpha^n \partial \beta^k} \big|_{\alpha=\beta=0} e^{\left(\frac{\alpha^2}{2} + A \beta\right) \bar{z} + (\frac{\alpha^2}{2} + A \beta) z} \langle \Phi, e^{(A+B+\beta) h} \Phi \rangle
\]

from which the result follows with the use of the Leibniz rule for derivatives.

Unlike the non-extended Heisenberg case, vectors of the form \( a^\dagger k \Phi \) are not orthogonal. For example, \( \langle a^\dagger 2 \Phi, a^\dagger \Phi \rangle = z \neq 0 \). Of course, in the Heisenberg algebra case \( z = 0 \). In general:

**Proposition 2.3.** For all \( n \geq k \geq 0 \)

\[
(2.18) \quad \langle a^\dagger n \Phi, a^\dagger k \Phi \rangle = \sum_{\rho=0}^{k} \sum_{\sigma=0}^{k-\rho} d_{n,2k-\rho-3\sigma} \binom{k}{\rho} \binom{k-\rho}{\sigma} \binom{\alpha(\sigma) n!}{2^{k-\rho-\sigma} z^2} \bar{z}^{k-\rho-\sigma} \langle \Phi, h^\rho \Phi \rangle
\]

where

\[
(2.19) \quad \alpha(\sigma) = \begin{cases} 
0 & \text{if } \sigma \text{ is odd} \\
1 & \text{if } \sigma = 0 \\
3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2\sigma - 1) & \text{if } \sigma \text{ is even}
\end{cases}
\]

**Proof.** By Proposition 2.1 for \( \lambda, \mu \in \mathbb{R} \)

\[
\langle a^\dagger n \Phi, a^\dagger k \Phi \rangle = \frac{\partial^{n+k}}{\partial \lambda^n \partial \mu^k} \big|_{\lambda=\mu=0} \langle e^{\lambda a^\dagger} \Phi, e^{\mu a^\dagger} \Phi \rangle
\]

\[
= \frac{\partial^{n+k}}{\partial \lambda^n \partial \mu^k} \big|_{\lambda=\mu=0} \langle \psi(\lambda, 0) \Phi, \psi(\mu, 0) \Phi \rangle
\]

\[
= \frac{\partial^{n+k}}{\partial \lambda^n \partial \mu^k} \big|_{\lambda=\mu=0} \langle \Phi, e^{\lambda \mu} h \Phi \rangle e^{\frac{\lambda^2}{2} z + \frac{\mu^2}{2} z}
\]

and the result follows by making repeated use of the Leibniz rule for derivatives and the fact that
The Leibniz function of Proposition 2.1 does not define an inner product for arbitrary \( z \) and \( h \). If it did, then we could apply the Cauchy-Schwartz inequality to \( \psi(\alpha, \beta) = e^{\alpha a^\dagger} e^{\beta h} \Phi \) and \( \psi(0, 0) = \Phi \), and we would have that

\[
\langle \psi(\alpha, \beta) , \Phi \rangle \leq \| \langle \psi(\alpha, \beta) \rangle \| \| \Phi \|
\]

which, by Proposition 2.1 and the fact that \( \| \Phi \| = 1 \), becomes

\[
|f_h(\bar{\beta})| \leq e^{\Re((|\alpha|^2 + 2 \beta) \alpha) z} f_h(|\alpha|^2 + 2 \Re \beta)
\]

and so, by condition (ii) on \( f_h \),

\[
e^{\Re((|\alpha|^2 + 2 \beta) \alpha) z} \geq \frac{|f_h(\bar{\beta})|}{f_h(|\alpha|^2 + 2 \Re \beta)}
\]

which, for \( \beta = 0 \) and \( \alpha = 1 \), implies that

\[
e^{\Re z} \geq \frac{1}{\langle \Phi, e^h \Phi \rangle}
\]

while, for \( \beta = 0 \) and \( \alpha = i \), it implies that

\[
e^{3z} \leq \langle \Phi, e^h \Phi \rangle
\]

Therefore, (2.23) and (2.24) are necessary conditions for the Leibniz function of Proposition 2.1 to define an inner product.

The problem of finding examples of \( f_h \) for which the Leibniz function of Proposition 2.1 defines an inner product is open. The natural choices \( f_h(w) = \cosh(w) \) and \( f_h(w) = e^{cw} \), where \( c > 0 \), do not work since in both cases we can find \( c_1, c_2, \alpha_1, \beta_1, \alpha_2, \beta_2 \) for which \( \| c_1 \psi(\alpha_1, \beta_1) + c_2 \psi(\alpha_1, \beta_1) \|^2 \) is either negative or has non-zero imaginary part. For example, for \( f_h(w) = e^w \) and \( z = 1 \) we find that \( \| - \psi(-2, -1) + 2 \psi(1, -2) \|^2 < 0 \). Similarly, for \( f_h(w) = \cosh(w) \) and \( z = 1 \) we find that \( \| \psi(-1, 1) - \psi(1, -1) - \psi(-1, -1) \|^2 < 0 \).

The action of \( a, a^\dagger \) and \( h \) on the exponential vectors \( \psi(\alpha, \beta) \) is described in the following:

Proposition 2.4. (The action of \( a, a^\dagger \) and \( h \) on the exponential vectors) For all \( \alpha, \beta \in \mathbb{C} \)
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\[ a^\dagger \psi(\alpha, \beta) = \frac{\partial}{\partial \epsilon} |_{\epsilon=0} \psi(\alpha + \epsilon, \beta) \]
\[ a \psi(\alpha, \beta) = \frac{\partial}{\partial \epsilon} |_{\epsilon=0} \psi(\alpha, \epsilon \alpha + \beta) + \left( \frac{\alpha^2 z}{2} + \beta \bar{z} \right) \psi(\alpha, \beta) \]
\[ h \psi(\alpha, \beta) = \frac{\partial}{\partial \epsilon} |_{\epsilon=0} \psi(\alpha, \beta + \epsilon) + \alpha z \psi(\alpha, \beta) \]

In the Heisenberg case, corresponding to \( h = 1, \beta = 0 \) and \( z = 0 \), letting \( y(\alpha) = \psi(\alpha, 0) \) we are reduced to the well known representation

\[ a^\dagger y(\alpha) = \frac{\partial}{\partial \epsilon} |_{\epsilon=0} y(\alpha + \epsilon) \]
\[ ay(\alpha) = \alpha y(\alpha) \]
\[ 1y(\alpha) = y(\alpha) \]

**Proof.** We have that

\[ a^\dagger \psi(\alpha, \beta) = a^\dagger e^{\alpha a^\dagger} e^{\beta h} \Phi = \frac{\partial}{\partial \epsilon} |_{\epsilon=0} e^{(\alpha + \epsilon) a^\dagger} e^{\beta h} \Phi = \frac{\partial}{\partial \epsilon} |_{\epsilon=0} \psi(\alpha + \epsilon, \beta) \]

Similarly, by (2.13) and the fact that \( a \Phi = 0 \),

\[ a \psi(\alpha, \beta) = ae^{\alpha a^\dagger} e^{\beta h} \Phi = e^{\alpha a^\dagger} \left( a + \alpha h + \frac{\alpha^2 z}{2} \right) e^{\beta h} \Phi = e^{\alpha a^\dagger} a e^{\beta h} \Phi + \alpha e^{\alpha a^\dagger} h e^{\beta h} \Phi + \frac{\alpha^2 z}{2} e^{\alpha a^\dagger} e^{\beta h} \Phi \]
\[ = e^{\alpha a^\dagger} e^{\beta h} (a + \beta \bar{z}) \Phi + \alpha e^{\alpha a^\dagger} h e^{\beta h} \Phi + \frac{\alpha^2 z}{2} e^{\alpha a^\dagger} e^{\beta h} \Phi \]
\[ = e^{\alpha a^\dagger} e^{\beta h} \Phi + \alpha e^{\alpha a^\dagger} h e^{\beta h} \Phi + \frac{\alpha^2 z}{2} e^{\alpha a^\dagger} e^{\beta h} \Phi \]
\[ = \frac{\partial}{\partial \epsilon} |_{\epsilon=0} e^{\alpha a^\dagger} e^{\epsilon(\alpha + \beta) h} \Phi + \left( \frac{\alpha^2 z}{2} + \beta \bar{z} \right) e^{\alpha a^\dagger} e^{\beta h} \Phi \]
\[ = \frac{\partial}{\partial \epsilon} |_{\epsilon=0} \psi(\alpha, \epsilon \alpha + \beta) + \left( \frac{\alpha^2 z}{2} + \beta \bar{z} \right) \psi(\alpha, \beta) \]

and also, again by (2.13),

\[ h \psi(\alpha, \beta) = h e^{\alpha a^\dagger} e^{\beta h} \Phi = e^{\alpha a^\dagger} (h + \alpha z) e^{\beta h} \Phi = e^{\alpha a^\dagger} h e^{\beta h} \Phi + e^{\alpha a^\dagger} \alpha z e^{\beta h} \Phi \]
\[ = \frac{\partial}{\partial \epsilon} |_{\epsilon=0} e^{\alpha a^\dagger} e^{(\epsilon + \beta) h} \Phi + \alpha z e^{\alpha a^\dagger} e^{\beta h} \Phi \]
\[ = \frac{\partial}{\partial \epsilon} |_{\epsilon=0} \psi(\alpha, \beta + \epsilon) + \alpha z \psi(\alpha, \beta) \]

***

**Proposition 2.5.** On the linear span of the exponential vectors of Definition 2.2 the operators \( a, a^\dagger \) and \( h \) defined in Proposition 2.4 satisfy \([a, a^\dagger] = h, [h, a^\dagger] = z, [a, h] = \bar{z}] (a^\dagger)^* = a \) and \( h^* = h \).
Proof. We have

\[
\langle \psi(\alpha, \beta), a^\dagger \psi(A, B) \rangle = \frac{\partial}{\partial \epsilon} \mid_{\epsilon=0} \langle \psi(\alpha, \beta), \psi(A + \epsilon, B) \rangle
\]

\[
= \frac{\partial}{\partial \epsilon} \mid_{\epsilon=0} \left( \left( \frac{\hat{\alpha}^2}{2}(A^+ + \hat{\alpha} B) \hat{z} + \left( \frac{\hat{\alpha}^2}{2} + A(\hat{\alpha} + \beta) \right) \hat{z} \right) \langle \Phi, e^{(\hat{\alpha}(A+\epsilon)+B+\beta)h} \Phi \rangle \\
+ \left( \frac{\hat{\alpha}^2}{2} + \beta \hat{z} \right) e^{\left( \frac{\hat{\alpha}^2}{2}(A^+ + \hat{\alpha} B) \hat{z} + \left( \frac{\hat{\alpha}^2}{2} + A(\hat{\alpha} + \beta) \right) \hat{z} \right)} \langle \Phi, e^{(\hat{\alpha}A+B+\beta)h} \Phi \rangle \right)
\]

\[
= \frac{\partial}{\partial \epsilon} \mid_{\epsilon=0} \langle \psi(\alpha, e \alpha + \beta), \psi(A, B) \rangle + \left( \frac{\hat{\alpha}^2}{2} + \beta \hat{z} \right) \langle \psi(\alpha, \beta), \psi(A, B) \rangle
\]

Similarly \(\langle \psi(\alpha, \beta), h \psi(A, B) \rangle = \langle h \psi(\alpha, \beta), \psi(A, B) \rangle\). Thus \((a^\dagger)^* = a\) and \(h^* = h\). To prove that the extended Heisenberg commutation relations (1.4) are satisfied on the exponential domain, we notice that using (2.13) to put expressions that involve \(a^\dagger, h\) and \(a\) in “normal order” i.e. \(a^\dagger\) is on the left, \(h\) is in the middle and \(a\) is on the right, we find that

\[
[a, a^\dagger] \psi(\alpha, \beta) = (a a^\dagger - a^\dagger a) \psi(\alpha, \beta) = (a a^\dagger - a^\dagger a) e^{\alpha a^\dagger} e^{\beta h} \Phi = e^{\alpha a^\dagger} h e^{\beta h} \Phi + \alpha z e^{\alpha a^\dagger} e^{\beta h} \Phi
\]

and also

\[
h \psi(\alpha, \beta) = h e^{\alpha a^\dagger} e^{\beta h} \Phi = e^{\alpha a^\dagger} h e^{\beta h} \Phi + \alpha z e^{\alpha a^\dagger} e^{\beta h} \Phi
\]

Therefore \([a, a^\dagger] \psi(\alpha, \beta) = h \psi(\alpha, \beta). Similarly, [h, a^\dagger] \psi(\alpha, \beta) = z \psi(\alpha, \beta)\) and \([a, h] \psi(\alpha, \beta) = \bar{z} \psi(\alpha, \beta)\).

3. Random Variables

If \(s \in \mathbb{R}\), \(\Phi\) is the Fock vacuum vector and \(X\) is a self-adjoint operator on a Fock space then \(\langle \Phi, e^{sX} \Phi \rangle\) and \(\langle \Phi, e^{isX} \Phi \rangle\) can be viewed, respectively, as the moment generating and characteristic functions of a classical random variable. In this section we compute the moment generating and characteristic functions of the self-adjoint operator \(X = a + a^\dagger + h\) with respect to the sesquilinear form of Proposition 2.1.

Lemma 3.1. For all \(X, Y \in \text{span}\{a, a^\dagger, h, E\}\)

\[
e^{X+Y} = e^X e^Y e^{-\frac{1}{2} [X,Y]} e^{\frac{1}{8} (2 [Y,[X,Y]]+[X,[X,Y]])}
\]

Proof. This is a special case of the general Zassenhaus formula (converse of the BCH formula). See [2] for a proof.

Lemma 3.2. For all \(s \in \mathbb{R}\)

\[
e^{s(a+a^\dagger+h)} = e^{s a^\dagger} e^{s a} e^{\left( \frac{s^2}{2} + s \right) h} e^{\frac{s^2}{2} (z-2z) + \frac{s^2}{4} (z-z)}
\]

Proof. By Lemma 3.1 with \(X = s (a + a^\dagger)\) and \(Y = s h\), we have
By Lemma 3.2 and the fact that

\[ e^{s(a+a^1+h)} = e^{s(a+a^1)} e^{s h} e^{-\frac{1}{2} [s(a+a^1), s h]} e^{\frac{1}{2} (2 [s h, [s(a+a^1), s h]] + [s(a+a^1), [s(a+a^1), s h]])} \]

\[ = e^{s(a+a^1)} e^{s h} e^{-\frac{s^2}{2} [a + a^1, h]} e^{\frac{s^2}{2} (2 [h, [a + a^1, h]] + [a + a^1, [a + a^1, h]])} \]

\[ = e^{s(a+a^1)} e^{s h} e^{-\frac{s^2}{2} (\bar{z} - z)} \]

Similarly,

\[ e^{s(a+a^1)} = e^{s(a^1+a)} = e^{s a^1} e^{\frac{s^2}{2} h} e^{\frac{1}{2} (-2 s^3 \bar{z} + s^3 z)} \]

Therefore

\[ e^{s(a+a^1+h)} = e^{s a^1} e^{s a} e^{\frac{s^2}{2} h} e^{\frac{1}{2} (-2 s^3 \bar{z} + s^3 z)} e^{s h} e^{-\frac{s^2}{2} (\bar{z} - z)} \]

\[ = e^{s a^1} e^{s a} e^{\left(\frac{s^2}{2} + s\right) h} e^{\frac{s^3}{2} (z - 2 \bar{z}) + \frac{s^2}{2} (z - \bar{z})} \]

\section*{Proposition 3.3. (Moment generating and Characteristic functions) (i) For all \( s \in \mathbb{R} \)}

\[ \langle \Phi, e^{s(a+a^1+h)} \Phi \rangle = e^{\left(\frac{s^3}{3} + s^2\right) \mathbb{R} z} f_h \left( \frac{s^2}{2} + s \right) \]

(ii) For all \( s \in \mathbb{R} \)

\[ \langle \Phi, e^{i s(a+a^1+h)} \Phi \rangle = e^{-\left(i \frac{s^3}{3} + s^2\right) \mathbb{R} z} f_h \left( -\frac{s^2}{2} - i s \right) \]

where \( f_h \) is as in Proposition 2.1

\textbf{Proof.} By Lemma 3.2 and the fact that \( e^{s a} \Phi = \Phi \) we have

\[ \langle \Phi, e^{s(a+a^1+h)} \Phi \rangle = \langle \Phi, e^{s a^1} e^{s a} e^{\left(\frac{s^2}{2} + s\right) h} e^{\frac{s^3}{2} (z - 2 \bar{z}) + \frac{s^2}{2} (z - \bar{z})} \Phi \rangle \]

\[ = e^{\frac{s^3}{2} (z - 2 \bar{z}) + \frac{s^2}{2} (z - \bar{z})} \langle \Phi, e^{s a} e^{\left(\frac{s^2}{2} + s\right) h} \Phi \rangle \]

\[ = e^{\frac{s^3}{2} (z - 2 \bar{z}) + \frac{s^2}{2} (z - \bar{z})} \langle \Phi, e^{\left(\frac{s^2}{2} + s\right) h} e^{s a} e^{\left(\frac{s^2}{2} + s\right) \mathbb{R} z} \Phi \rangle \]

\[ = e^{\left(\frac{s^3}{3} + s^2\right) \mathbb{R} z} \langle \Phi, e^{\left(\frac{s^2}{2} + s\right) h} \Phi \rangle \]

\[ = e^{\left(\frac{s^3}{3} + s^2\right) \mathbb{R} z} f_h \left( \frac{s^2}{2} + s \right) \]

The proof of (ii) is similar. It can also be directly obtained from (i) by replacing \( s \) by \( i s \).

\section*{References}


