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INCLUSION AND NEIGHBORHOOD PROPERTIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH CONVOLUTION STRUCTURE

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ABSTRACT. In this paper we introduce and investigate two new subclasses of multivalently analytic functions of complex order defined by using the familiar convolution structure of analytic functions. In this paper we obtain the coefficient estimates and the consequent inclusion relationships involving the neighborhoods of the p-valently analytic functions.

Key words and phrases: Analytic functions; p-valent functions; Hadamard product (or convolution); neighborhood.

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1. INTRODUCTION

Let $A_p(n)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_k z^k \quad (p < n; p, n \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. Also let us put $A_p(p+1) = A(p)$ and $A = A(1)$. If $f(z) \in A_p(n)$ is given by (1.1) and $g(z) \in A_p(n)$ is given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k \ . \tag{1.2}$$

Then the Hadamard product (or convolution) $(f * g)(z)$ of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=n}^{\infty} a_k b_k z^k. \tag{1.3}$$

For a function $f(z)$ in $A_p(n)$, we define

$$D_p^0 f(z) = f(z),$$

$$\begin{aligned} D_p^1 f(z) &= D(D_p^0 f(z)) = \frac{z}{p} f^{'}(z) \\ &= z^p + \sum_{k=n}^{\infty} (\frac{k}{p}) a_k z^k, \end{aligned}$$

and

$$D_p^{\sigma} f(z) = D(D_p^{\sigma-1} f(z)) \quad (\sigma \in N).$$

It is easy to see that

$$D_p^{\sigma} f(z) = z^p + \sum_{k=n}^{\infty} (\frac{k}{p})^{\sigma} a_k z^k \quad (\sigma \in N_0 = N \cup \{0\})\,. \tag{1.4}$$

When $p = 1$ and $n = 2$, the differential operator $D_1^{\sigma} = D^{\sigma}$ was introduced by Salagean [15].

For complex parameters $\alpha_1 ..., \alpha_r$ and $\beta_1, ..., \beta_s (\beta_j \in C \backslash \{0, -1, -2, ...\}, j = 1, .., s)$, we define the generalized hypergeometric function ${}_rF_s(\alpha_1 ..., \alpha_r; \beta_1, ..., \beta_s; z)$ by

$${}_rF_s(\alpha_1 ..., \alpha_r; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k ......(\alpha_r)_k}{(\beta_1)_k ....(\beta_s)_k} . \frac{z^k}{k!}$$

$$(r \leq s + 1; r, s \in N_0\,; \ z \in U), \tag{1.5}$$

where $(\theta)_k$ is the Pochhammer symbol defined, in terms of the Gamma function Γ by

$$(\theta)_k = \frac{\Gamma(\theta + k)}{\Gamma(\theta)} = \begin{cases} 1 & (k = 0) \\ \theta(\theta + 1)....(\theta + k - 1) & (k \in N). \end{cases} \tag{1.6}$$

Corresponding to a function $h_p(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; z)$ defined by

$$h_p(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; z) = z^p \, {}_rF_s(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; z),$$

we consider a linear operator $H_p(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s) : A(p) \to A(p)$, defined by the convolution

(1.7) $$H_p(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s)f(z) = h_p(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; z) * f(z).$$

We observe that, for a function $f(z)$ of the form (1.1), we have

(1.8) $$H_p(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s)f(z) = z^p + \sum_{k=n}^{\infty} \Gamma_k a_k z^k,$$

where

(1.9) $$\Gamma_k = \frac{(\alpha_1)_{k-p} ......(\alpha_r)_{k-p}}{(\beta_1)_{k-p} ......(\beta_s)_{k-p} \, (k-p)!} \, .$$

For convenience, we write

(1.10) $$H^p_{r,s}[\alpha_1] = H_p(\alpha_1, ..., \alpha_r; \beta_1, ...\beta_s).$$

The linear operator $H^p_{r,s}[\alpha_1]$ was introduced and studied by Dziok and Srivastava [7].

We denote by $T_p(n)$ the subclass of $A_p(n)$ consisting of functions of the form :

(1.11) $$f(z) = z^p - \sum_{k=n}^{\infty} a_k z^k \quad (p < n; a_k \geq 0 \ (k \geq n); p, n \in N).$$

For a given function $g(z) \in A_p(n)$ defined by

(1.12) $$g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k \quad (p < n; b_k \geq 0 \, (k \geq n); p, n \in N),$$

we introduce here a new subclass $S_g(n, p, q, \lambda, b, \beta)$ of the p-valently analytic function class $T_p(n)$ which consists of functions $f(z) \in T_p(n)$ satisfying the inequality :

$$\left| \frac{1}{b} \left\{ \frac{z(f*g)^{(1+q)}(z) + \lambda z^2 (f*g)^{(2+q)}(z)}{\lambda z (f*g)^{(1+q)}(z) + (1-\lambda)(f*g)^{(q)}(z)} - (p-q) \right\} \right| < \beta$$

(1.13) $$(z \in U; p, n \in N; q \in N_0; p > q; 0 \leq \lambda \leq 1; b \in C \backslash \{0\}; 0 < \beta \leq 1).$$

We note that :

(i) $S_g(n, p, q, 1, b, \beta) = C_g(n, p, q, b, \beta)$

$$= \{ f : f \in T_p(n) \text{ and } \left| \frac{1}{b} \left\{ 1 + \frac{z(f*g)^{(2+q)}(z)}{(f*g)^{(1+q)}(z)} - (p-q) \right\} \right| < \beta$$

(1.14) $$(z \in U; p, n \in N; q \in N_0; p > q; b \in C \backslash \{0\}; 0 < \beta \leq 1) \ \} ;$$

(ii) $S_g(n, p, q, 0, b, 1) = S_g(n, p, q, b)$ (Prajapat et al. [12]);

(iii) $S_g(n, p, 0, 0, p(1-\alpha), 1) = TS^*_g(n, p, \alpha)(p \in N; 0 \leq \alpha < 1)$ (Ali et al. [2]);

(iv) Replacing n by $n + p$ in (1.11) and (1.12) and taking the coefficients b_k in (1.12) as follows :

(1.15) $$b_k = \binom{\nu + k - 1}{k - p} \quad (v > -p),$$

then we have :

$S_g(n,p,q,0,b,1)=H^p_{n,q}(v,b)$ (this class involving the familiar Ruscheweyh derivative)(Raina and Srivastava [13]);

(vi) By taking the coefficients b_k in (1.12) as follows : $b_k=\Gamma_k\geq 0(k\geq n)$, where Γ_k is given by (1.9), then we have :

$S_g(n,p,q,0,b,1)=S^*(n,p,q,b)$ (Prajapat et al. [12]).

Also we note that by choosing different values for the coefficients b_k defined in (1.12), we obtain the following new classes :

(i) By taking the coefficients b_k in (1.12) as follows : $b_k=\Gamma_k\geq 0(k\geq n)$, where Γ_k is given by (1.9), then we have the new class :

$S^*_{[\alpha_1]}(n,p,q,\lambda,b,\beta)=\Big\{f: f(z)\in T_p(n)$ and

$$\left|\frac{1}{b}\left\{\frac{z(H^p_{r,s}[\alpha_1]f)^{(1+q)}(z)+\lambda z^2(H^p_{r,s}[\alpha_1]f)^{(2+q)}(z)}{\lambda z(H^p_{r,s}[\alpha_1]f)^{(1+q)}(z)+(1-\lambda)(H^p_{r,s}[\alpha_1]f)^{(q)}(z)}-(p-q)\right\}\right|<\beta$$

$$(z\in U;r\leq s+1;p,n\in N;p>q;0\leq\lambda\leq 1;b\in C\backslash\{0\};0<\beta\leq 1); \tag{1.16}$$

(ii) By taking the coefficients b_k in (1.12) as follows : $b_k=(\dfrac{k}{p})^\sigma(k\geq n;p,n\in N;\sigma\in N_0)$, then we have the new class :

$TS^*_\sigma(n,p,q,\lambda,b,\beta)=\Big\{f: f\in T_p(n)$ and

$$\left|\frac{1}{b}\left\{\frac{z(D^\sigma_p f(z))^{(1+q)}+\lambda z^2(D^\sigma_p f(z))^{(2+q)}}{\lambda z(D^\sigma_p f(z))^{(1+q)}+(1-\lambda)(D^\sigma_p f(z))^{(q)}}-(p-q)\right\}\right|<\beta$$

$$(z\in U;p,n\in N;q,\sigma\in N_0;0\leq\lambda\leq 1;b\in C\backslash\{0\};0<\beta\leq 1); \tag{1.17}$$

(iii) By taking the coefficients b_k in (1.12) as given by (1.15), then we have the new class :

$H^p_{n,q}(\nu,\lambda,b,\beta)=\Big\{f: f\in T_p(n)$ and

$$\left|\frac{1}{b}\left\{\frac{z(D^{\nu,p}f(z))^{(1+q)}+\lambda z^2(D^{\nu,p}f(z))^{(2+q)}}{\lambda z(D^{\nu,p}_p f(z))^{(1+q)}+(1-\lambda)(D^{\nu,p}f(z))^{(q)}}-(p-q)\right\}\right|<\beta$$

(1.18)

$$(z\in U;p,n\in N;q\in N_0;\nu\in R;p>\max\{q,-\nu\};0\leq\lambda\leq 1;b\in C\backslash\{0\};0<\beta\leq 1);$$

where the symbol $D^{\nu,1}f(z)=D^\nu f(z)$ for $\nu=n\in N_0$ was named the n- th order Ruscheweyh derivative of $f(z)\in A$ by Al-Amiri [1].

Now, following the earlier investigation by Goodman [8], Ruscheweyh [14], and others including Altintas and Owa [3], Altintas et al. ([4] and [5]), Murgusundaramoorthy and Srivastava [9], Raina and Sirvastava [13], Aouf [6], Prajapat et al. [12] and Srivastava and Orhan [16] (see also [10], [11] and [17]), we define the $(n,\delta)-$ neighborhood of a function $f(z)\in T_p(n)$ by (see, for example, [5], p. 1668)

$$N_{n,\delta}(f)=\left\{g: g\in T_p(n), g(z)=z^p-\sum_{k=n}^{\infty}b_k z^k \text{ and } \sum_{k=n}^{\infty}k\left|a_k-b_k\right|\leq\delta\right\}. \tag{1.19}$$

In particular, if

$$h(z)=z^p\ (p\in N), \tag{1.20}$$

we immediately have

$$N_{n,\delta}(h) = \left\{ g : g \in T_p(n), g(z) = z^p - \sum_{k=n}^{\infty} b_k z^k \text{ and } \sum_{k=n}^{\infty} k\,|b_k| \le \delta \right\}. \tag{1.21}$$

Also, let $P_g(n, p, q, \lambda, b, \beta)$ denote the subclass of $T_p(n)$ consisting of functions $f(z)$ of the form (1.11) which satisfy the inequality :

$$\left| \frac{1}{b} \left\{ \left[(1-\lambda) \frac{(f*g)^{(q)}(z)}{z^{p-q}} + \lambda \frac{(f*g)^{(1+q)}(z)}{(p-q)z^{p-q-1}} \right] - \theta(p,q) \right\} \right| < \beta$$

$$(z \in U; p, n \in N; q \in N_0; p > q; \lambda \ge 0; b \in C\backslash\{0\}; 0 < \beta \le 1), \tag{1.22}$$

where

$$\theta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} 1 & (q = 0)\,, \\ p(p-1)....(p-q+1) & (q \ne 0). \end{cases} \tag{1.23}$$

We note that :

(i) $P_g(n, p, q, 0, b, \beta) = P_g(n, p, q, b, \beta) = \{\ f : f \in T_p(n)$ and

$$\left| \frac{1}{b} \left[\frac{(f*g)^{(q)}(z)}{z^{p-q}} - \theta(p,q) \right] \right|$$

$$< \quad \beta\,(z \in U; p, n \in N; q \in N_0; p > q; b \in C\backslash\{0\}; 0 < \beta \le 1)\ \}\,; \tag{1.24}$$

(ii) $P_g(n, p, q, 1, b, \beta) = L_g(n, p, q, b, \beta)$

$$= \left\{ f : f \in T_p(n) \text{ and } \left| \frac{1}{b} \left[\frac{(f*g)^{(1+q)}(z)}{(p-q)z^{p-q-1}} - \theta(p,q) \right] \right| < \beta \right.$$

$$(z \in U; p, n \in N; q \in N_0; p > q; b \in C\backslash\{0\}; 0 < \beta \le 1)\ \}\,. \tag{1.25}$$

Remark 1.1. Throughout our present paper, we assume that $\theta(p,q)$ is defined by (1.23).

2. NEIGHBORHOODS FOR THE CLASSES $S_g(n, p, q, \lambda, b, \beta)$ AND $P_g(n, p, q, \lambda, b, \beta)$

In our investigation of the inclusion relations involving $N_{n,\delta}(h)$, we shall require Lemmas 2.1 and 2.2 below.

Lemma 2.1. *Let the function* $f(z) \in T_p(n)$ *be defined by* (1.11). *Then* $f(z)$ *is in the class* $S_g(n, p, q, \lambda, b, \beta)$ *if and only if*

$$\sum_{k=n}^{\infty} (k + \beta\,|b| - p)[1 + \lambda(k-q-1)]\theta(k,q) b_k a_k \le \beta\,|b|\,[1 + \lambda(p-q-1)]\theta(p,q), \tag{2.1}$$

Proof. Let a function $f(z)$ of the form (1.11) belong to the class $S_g(n, p, q, \lambda, b, \beta)$.
Then, in view of (1.11),(1.12) and (1.13), we obtain the following inequality

$$\mathrm{Re} \left\{ \frac{z(f*g)^{(1+q)}(z) + \lambda z^2((f*g)^{(2+q)}(z)}{\lambda z(f*g)^{(1+q)}(z) + (1-\lambda)(f*g)^{(q)}(z)} - (p-q) \right\} > -\beta\,|b| \ \ (z \in U), \tag{2.2}$$

or, equivalently,

$$Re \left\{ \frac{-\sum\limits_{k=n}^{\infty} (k-p)[1 + \lambda(k-q-1)]\theta(k,q) a_k b_k z^{k-p}}{[1 + \lambda(p-q-1)]\theta(p,q) - \sum\limits_{k=n}^{\infty} [1 + \lambda(k-q-1)]\theta(k,q) a_k b_k z^{k-p}} \right\} > -\beta\,|b|$$

(2.3) $(z \in U).$

Setting $z = r\,(0 \leq r < 1)$ in (2.3), we observe that the expression in the denominator of the left-hand side of (2.3) is positive for $r = 0$ and also for all $(0 < r < 1)$. Thus, by letting $r \to 1^-$ through real values, (2.3) leads us to the desired assertion of Lemma 2.1.

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we find from (1.13) that

$$\left|\frac{z(f*g)^{(1+q)}(z) + \lambda z^2((f*g)^{(2+q)}(z)}{\lambda z(f*g)^{(1+q)}(z) + (1-\lambda)(f*g)^{(q)}(z)} - (p-q)\right|$$

$$= \left|\frac{\sum\limits_{k=n}^{\infty}(k-p)[1+\lambda(k-q-1)]\theta(k,q)a_k b_k z^{k-p}}{[1+\lambda(p-q-1)]\theta(p,q) - \sum\limits_{k=n}^{\infty}[1+\lambda(k-q-1)]\theta(k,q)a_k b_k z^{k-p}}\right|$$

$$\leq \frac{\sum\limits_{k=n}^{\infty}(k-p)[1+\lambda(k-q-1)]\theta(k,q)a_k b_k}{[1+\lambda(p-q-1)]\theta(p,q) - \sum\limits_{k=n}^{\infty}[1+\lambda(k-q-1)]\theta(k,q)a_k b_k}$$

$$\leq \frac{\beta\,|b|\left\{[1+\lambda(p-q-1)]\theta(p,q) - \sum\limits_{k=n}^{\infty}[1+\lambda(k-q-1)]\theta(k,q)a_k b_k\right\}}{[1+\lambda(p-q-1)]\theta(p,q) - \sum\limits_{k=n}^{\infty}[1+\lambda(k-q-1)]\theta(k,q)a_k b_k} = \beta\,|b|\,.$$

Hence, by the maximum modulus theorem, we have $f(z) \in S_g(n,p,q,\lambda,b,\beta)$, which evidenlty completes the proof of Lemma 2.1.

Similarly, we can prove the following lemma. ∎

Lemma 2.2. *Let the function $f(z) \in T_p(n)$ be given by (1.11). Then $f(z) \in P_g(n,p,q,\lambda,b,\beta)$ if and only if*

$$\sum_{k=n}^{\infty}[p-q+\lambda(k-p)]\theta(k,q)b_k a_k \leq \beta\,|b|\,(p-q). \tag{2.4}$$

Our first inclusion relation $N_{n,\delta}(h)$ is given in the following theorem.

Theorem 2.3. *If*

$$b_k \geq b_n \ (k \geq n) \text{ and } \delta = \frac{n\beta\,|b|\,[1+\lambda(p-q-1)]\theta(p,q)}{(n+\beta\,|b|-p)[1+\lambda(n-q-1)]\theta(n,q)b_n} \quad (p > |b|), \tag{2.5}$$

then

$$S_g(n,p,q,\lambda,b,\beta) \subset N_{n,\delta}(h). \tag{2.6}$$

Proof. Let $f(z) \in S_g(n,p,q,\lambda,b,\beta)$.Then, in view of the assertion (2.1) of Lemma 2.1, and the given condition that

$$b_k \geq b_n \quad (k \geq n),$$

we have

$$(n+\beta\,|b|-p)[1+\lambda(n-q-1)]\theta(n,q)b_n\sum_{k=n}^{\infty}a_k$$

$$\leq \sum_{k=n}^{\infty}(k+\beta\,|b|-p)[1+\lambda(k-q-1)]\theta(k,q)b_k a_k$$

$$\leq \beta\,|b|\,[1+\lambda(p-q-1)]\,\theta(p,q), \tag{2.7}$$

which readily yields

$$\sum_{k=n}^{\infty} a_k \leq \frac{\beta\,|b|\,[1+\lambda(p-q-1)]\theta(p,q)}{(n+\beta\,|b|-p)[1+\lambda(n-q-1)]\theta(n,q)b_n}. \tag{2.8}$$

Making use of (2.1) again, in conjunction with (2.8), we get

$$[1+\lambda(n-q-1)]\theta(n,q)b_n \sum_{k=n}^{\infty} k a_k$$

$$\leq \beta\,|b|\,[1+\lambda(p-q-1)]\theta(p,q) + (p-\beta\,|b|)[1+\lambda(n-q-1)]\theta(n,q)b_n \sum_{k=n}^{\infty} a_k$$

$$\leq \beta\,|b|\,[1+\lambda(p-q-1)]\theta(p,q) + \ (p-\beta\,|b|)\frac{\beta\,|b|\,[1+\lambda(p-q-1)]\theta(p,q)}{(n+\beta\,|b|-p)}$$

$$= \frac{n\beta\,|b|\,[1+\lambda(p-q-1)]\theta(p,q)}{(n+\beta\,|b|-p)}.$$

Hence

$$\sum_{k=n}^{\infty} k a_k \leq \frac{n\beta\,|b|\,[1+\lambda(p-q-1)]\theta(p,q)}{(n+\beta\,|b|-p)[1+\lambda(n-q-1)]\theta(n,q)b_n} = \delta \quad (p>|b|), \tag{2.9}$$

which, by means of the definition (1.21), establishes the inclusion relation (2.6) asserted by Theorem 2.3. ∎

Remark 2.1. (i) Putting $\lambda = 0$ and $\beta = 1$ in Theorem 2.3, we obtain the result obtained by Prajapat et al. [[12], Theorem 3];

(ii) Putting $\lambda = 0$ and $\beta = 1$, replacing n by $n+p$ and choosing $b_n = \binom{\nu+n+p-1}{n} (\nu > -p)$, in Theorem 2.3, we obtain the result obtained by Raina and Srivastava [[13], Theorem 3].

In a similar manner, by applying the assertion (2.5) of Lemma 2.2 instead of the assertion (2.1) of Lemma 2.1 to functions in the class $P_g(n,p,q,\lambda,b,\beta)$, we can prove the following inclusion relationship.

Theorem 2.4. *If*

$$b_k \geq b_n \ (k \geq n) \textit{ and } \delta = \frac{n(p-q)\beta\,|b|}{[(p-q)+\lambda(n-p)]\,\theta(n,q)b_n} \quad (\lambda > 1), \tag{2.10}$$

then

$$P_g(n,p,q,\lambda,b,\beta) \subset N_{n,\delta}(h). \tag{2.11}$$

Remark 2.2. (i) We note that the result obtained by Prajapat et al. [[12], Theorem 4] is not correct. The correct result is given by (2.10) with $\beta = 1$;

(ii) We note that the result obtained by Raina and Srivastava [[13], Theorem 4] is not correct. The correct result is given by (2.10) by taking $\lambda = 0$, $\beta = 1$, replacing n by $n+p$ and choosing $b_n = \binom{\nu+n+p-1}{n} (\nu > -p)$.

3. NEIGHBORHOODS FOR THE CLASSES $\mathbf{S}_g^{(\alpha)}(\mathbf{n}, \mathbf{p}, \mathbf{q}, \boldsymbol{\lambda}, \mathbf{b}, \boldsymbol{\beta})$ AND $\mathbf{P}_g^{(\alpha)}(\mathbf{n}, \mathbf{p}, \mathbf{q}, \boldsymbol{\lambda}, \mathbf{b}, \boldsymbol{\beta})$

In this section, we determine the neighborhood for each of the classes

$$\mathbf{S}_g^{(\alpha)}(n,p,q,\lambda,b,\beta) \text{ and } P_g^{(\alpha)}(n,p,q,\lambda,b,\beta),$$

which we define as follows. A function $f(z) \in T_p(n)$ is said to be in the class $\mathbf{S}_g^{(\alpha)}(n,p,q,\lambda,b,\beta)$ if there exists a function $k(z) \in S_g(n,p,q,\lambda,b,\beta)$ such that

$$\left|\frac{f(z)}{k(z)} - 1\right| < p - \alpha \quad (z \in U;\ 0 \le \alpha < p - q). \tag{3.1}$$

Analogously, a function $f(z) \in T_p(n)$ is said to be in the class $P_g^{(\alpha)}(n,p,q,\lambda,b,\beta)$ if there exists a function $k(z) \in P_g(n,p,q,\lambda,b,\beta)$ such that the inequality (3.1) holds true.

Theorem 3.1. *If* $k(z) \in S_g(n,p,q,\lambda,b,\beta)$ *and*

$$\alpha = p - \frac{\delta(n+\beta|b|-p)[1+\lambda(n-q-1)]\theta(n,q)b_n}{n\left\{(n+\beta|b|-p)[1+\lambda(n-q-1)]\theta(n,q)b_n - \beta|b|[1+\lambda(p-q-1)]\theta(p,q)\right\}}, \tag{3.2}$$

then

$$N_{n,\delta}(k) \subset S_g^{(\alpha)}(n,p,q,\lambda,b,\beta), \tag{3.3}$$

where

$$\delta \le pn\Big[1 - \beta|b|[1+\lambda(p-q-1)]\theta(p,q)\ .$$

$$.\ \{(n+\beta|b|-p)[1+\lambda(n-q-1)]\theta(n,q)b_n\}^{-1}\Big]\ . \tag{3.4}$$

Proof. Suppose that $f(z) \in N_{n,\delta}(k)$. We find from (1.19) that

$$\sum_{k=n}^{\infty} k|a_k - b_k| \le \delta, \tag{3.5}$$

which readily implies that

$$\sum_{k=n}^{\infty} |a_k - b_k| \le \frac{\delta}{n} \quad (n \in N). \tag{3.6}$$

Next, since $k(z) \in S_g(n,p,q,\lambda,b,\beta)$, we have [cf. equation (2.8)] that

$$\sum_{k=n}^{\infty} b_k \le \frac{\beta|b|[1+\lambda(p-q-1)]\theta(p,q)}{(n+\beta|b|-p)[1+\lambda(n-q-1)]\theta(n,q)b_n}, \tag{3.7}$$

so that

$$\left|\frac{f(z)}{k(z)} - 1\right| \le \frac{\sum_{k=n}^{\infty}|a_k - b_k|}{1 - \sum_{k=n}^{\infty} b_k}$$

$$\le \frac{\delta(n+\beta|b|-p)[1+\lambda(n-q-1)]\theta(n,q)b_n}{n\left\{(n+\beta|b|-p)[1+\lambda(n-q-1)]\theta(n,q)b_n - \beta|b|[1+\lambda(p-q-1)]\theta(p,q)\right\}}$$

$$= p - \alpha, \tag{3.8}$$

provided that α is given by (3.2). Thus, by the above definition, $f(z) \in S_g^{(\alpha)}(n,p,q,\lambda,b,\beta)$ for α given by (3.2). This evidently proves Theorem 3.1. ∎

Remark 3.1. (i) Putting $\lambda = 0$ in Theorem 3.1, we obtain the result obtained by Prajapat et al. [[12], Theorem 5 with $q = 0$];

(ii) Putting $\lambda = 0$, $\beta = 1$, replacing n by $n + p$ and choosing $b_n = \begin{pmatrix} \nu + n + p - 1 \\ n \end{pmatrix} (\nu > -p),$ in Theorem 3.1, we obtain the result obtained by Raina and Srivastava [[13], Theorem 5].

The proof of Theorem 3.2 below is similar to that of Theorem 3.1 above, therefore, we omit the details involved.

Theorem 3.2. *If* $k(z) \in P_g(n, p, q, \lambda, b, \beta)$ *and*

$$\alpha = p - \frac{\delta[(p-q) + \lambda(n-p)]\theta(n,q)b_n}{n\{(p-q) + \lambda(n-p)]\theta(n,q)b_n - (p-q)\beta\,|b|\}} \quad , \tag{3.9}$$

then

$$N_{n,\delta}(k) \subset P_g^{(\alpha)}(n, p, q, \lambda, b, \beta), \tag{3.10}$$

where

$$\delta \leq pn\left[1 - (p-q)\beta\,|b| \;\; .\left\{[(p-q) + \lambda(n-p)]\,\theta(n,q)b_n\right\}^{-1}\right]. \tag{3.11}$$

Remark 3.2. (i) We note that the result obtained by Prajapat et al. [[12], Theorem 6] is not correct. The correct result is given by (3.9) with $\beta = 1$;

(ii) We note that the result obtained by Raina and Srivastava [[13], Theorem 6] is not correct. The correct result is given by (3.9) by taking $\lambda = 0$ and $\beta = 1$, replacing n by $n + p$ and choosing $b_n = (\begin{smallmatrix} \nu + n + p - 1 \\ n \end{smallmatrix}) (\nu > -p)$.

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