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REAL INTERPOLATION METHODS AND QUASILOGARITHMIC OPERATORS

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ABSTRACT. The purpose of this paper is to deal with nonlinear quasilogarithmic operators, which possesse the uniformly bounded commutator property on various interpolation spaces in the sense of Brudnyi-Krugljak associated with the quasi-power parameter spaces. The duality, and the domain and range spaces of these operators are under consideration. Some known inequalities for the Lebesgue integration spaces and the trace classes are carried over to the non-commutative symmetric spaces of measurable operators affliated with a semi-finite von Neumann algebra.

Key words and phrases: Commutator, Domain, Non-commutative symmetric space, Orbital method, Quasilogarithmic operator, Quasi-power parameter space, Range, Real interpolation, Reiteration, Twisted sum.

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1. INTRODUCTION AND PRELIMINARIES

The main topic of interpolation theory is concerned with the boundedness of linear operators acting between several Banach spaces. Recently, the interpolation results for certain nonlinear operators were obtained, for which cancellation plays an important roll. Many important inequalities for the Lebesgue integration spaces and the trace classes can be formulated and studied in terms of commutator estimates between a bounded linear operator and certain quasilogarithmic operator which is neither bounded nor linear with the interpolation scale indexed by a numerical parameter between 0 and 1. See [5] and references therein.

In [9], the author formulated the commutator theorem for a wide family of the real interpolation methods, the K- and J-methods due to Brudnyi and Krugljak associated with the function space parameters of quasi-power type, which replace the numerical parameters in the classical interpolation methods. The purpose of the present paper is to study the nonlinear quasilogarithmic operators arising from the above mentioned interpolation methods and the corresponding commutator estimates by combining the results obtained in [9] with the reiteration of interpolation methods, which will be formulated in the sequel, and the well-known Aronszajn-Gagliardo construction [1]. Some open problems posted in [5] are also handled.

This paper is organized as follows. In the the rest of this introductory section, we collect some useful definitions and notations from interpolation theory. Section 2 is devoted to a reiteration result concerning the K- and J-methods of interpolation with quasi-power parameters. In Section 3, we study the quasilogarithmic operators related to K- and J-functionals, which are determined by a much wide choice of Lipschitz functions and possesse the uniformly bounded commutator property on the real interpolation spaces. The main result in this section is an inequality concerning the duality of the quasilogarithmic operators, and the relationship between the corresponding domain and range spaces. In Section 4, we investigate the quasilogarithmic operators related to E_α-functionals, and establish the equivalence of the various domain and range spaces of different operators. In the final section, we apply the results from Sections 2 & 3 on the non-commutative symmetric spaces of measurable operators affilated with a semi-finite von Neumann algebra. The non-commutative version of some known inequalities is obtained.

Throughout this paper, the notations $\subseteq$ and $=$ between two Hausdorff topological vector spaces stand for continuous inclusion and isomorphic equivalence respectively. Let $\overline{X} = (X_0, X_1)$ be a Banach couple with $\Delta\overline{X} = X_0 \cap X_1$ and $\Sigma\overline{X} = X_0 + X_1$, and let X be a quasi-normed intermediate space for $\overline{X}$, we denote by X^0 the regularization of X for $\overline{X}$, by X' the Banach space dual of X^0 when X is a Banach space, and we write $\overline{X}' = (X_0', X_1')$ as the dual couple of $\overline{X}$. According to (4.1.1) of [4], an operator (not necessarily linear) T acting from a Banach space X to a Banach space Y is said to be bounded if

$$\|T\|_{X,Y} = \sup\left\{ \|Tx\|_Y / \|x\|_X \,\middle|\, x \in X,\ x \neq 0 \right\} < \infty.$$

The notation $\mathcal{B}(X, Y)$ (resp., $\mathcal{B}(\overline{X}, \overline{Y})$) stands for the space of all bounded linear operators from quasi-Banach space X to quasi-Banach space Y (resp., from Banach couple $\overline{X}$ to Banach couple $\overline{Y}$). We simply write $\mathcal{B}(X) = \mathcal{B}(X, X)$ and $\mathcal{B}(\overline{X}) = \mathcal{B}(\overline{X}, \overline{X})$. For $T \in \mathcal{B}(X_j, Y_j)$ $(j = 0, 1)$, we denote

$$\|T\|_j = \|T\|_{X_j, Y_j} \qquad (j = 0, 1).$$

For $T \in \mathcal{B}(\overline{X}, \overline{Y})$, we denote $\|T\|_{\overline{X},\overline{Y}} = \|T\|_0 \vee \|T\|_1$. Further information about interpolation theory can be found in [3] and [4].

Let X and Y be Banach spaces. Recall that an operator $\Omega\colon X \longrightarrow Y$ is said to be quasi-additive if $\Omega(-x) = \Omega(x)$ and $\lim_{\lambda\to 0}\Omega(\lambda x) = 0$ in Y for all x in X, and if there exists a positive constant c such that

$$\Big\|\Omega\big(x_1+x_2\big)-\Omega\big(x_1\big)-\Omega\big(x_2\big)\Big\|_Y \le c\left(\big\|x_1\big\|_X+\big\|x_2\big\|_X\right) \tag{1.1}$$

for all $x_1,\ x_2$ in X. A homogeneous quasi-additive operator is said to be quasi-linear. Now let $\overline{X}$ be a Banach couple, and let X be an interpolation space for $\overline{X}$. If $\Omega\colon X \to X$ is a quasi-linear operator, then one can define the twisted (direct) sum of X associated with Ω, denoted by $X \oplus_\Omega X$, by the set of all pairs of elements $(x, y) \in \Sigma\overline{X} \oplus \Sigma\overline{X}$ such that

$$\big\|(x,y)\big\|_{X\oplus_\Omega X} = \big\|x\big\|_X + \big\|\Omega x - y\big\|_X < \infty.$$

Observe that $X \oplus_\Omega X$ is a quasi-Banach space. The domain space of Ω is defined by

$$\mathrm{Dom}(\Omega) = \mathrm{Dom}_X(\Omega) = \Big\{\, x\in X \,\Big|\, \Omega x \in X \,\Big\}$$

with the quasi-norm $\|x\|_{\mathrm{Dom}(\Omega)} = \|x\|_X + \|\Omega x\|_X$. If we identify x with $(x, 0)$ and consider $\mathrm{Dom}(\Omega)$ as a subspace of $X \oplus_\Omega X$, then the range space of Ω is defined by

$$\mathrm{Ran}(\Omega) = \mathrm{Ran}_X(\Omega) = \big(X\oplus_\Omega X\big)\big/\mathrm{Dom}_X(\Omega)$$

with the quotient quasi-norm. If we only assume that $\Omega\colon X \to X$ is quasi-additive satisfying the condition

$$\lim_{\lambda\to\lambda_0}\big\|\Omega(\lambda x)-\Omega(\lambda_0 x)\big\|_X = 0 \tag{1.2}$$

for all $x \in X$, then the spaces $X \oplus_\Omega X$, $\mathrm{Dom}_X(\Omega)$ and $\mathrm{Ran}_X(\Omega)$ can be defined in a similar way. These spaces are Hausdorff topological vector spaces in this case. In fact, if $(x_1, y_1), (x_2, y_2) \in X \oplus_\Omega X$, then $(x_1 + x_2, y_1 + y_2) \in X \oplus_\Omega X$ by (1.1). If $(x, y) \in X \oplus_\Omega X$ and $\lambda \in \mathbf{R}$, then $\lambda(x, y) \in X \oplus_\Omega X$ by (1.1) repeatedly when λ is a rational number and by (1.2) when λ is an irrational number. For $T \in \mathcal{B}(\overline{X}, \overline{Y})$, we denote the commutator

$$\big[T,\Omega\big] = T\Omega - \Omega T.$$

Let F be an interpolation functor, and let $\Omega\colon F(\overline{X}) \to F(\overline{X})$ be a quasi-additive operator satisfying (1.2) for each Banach couple $\overline{X}$. The operator Ω is said to possese the uniformly bounded commutator property (UBCP in short) if there exists a positive constant C depending on Ω and F such that

$$\Big\|\big[T,\Omega\big]x\Big\|_Y \le C\|T\|_{\overline{X},\overline{Y}}\|x\|_X \tag{1.3}$$

for all Banach couples $\overline{X}$ and $\overline{Y}$ with $X = F(\overline{X})$ and $Y = F(\overline{Y})$, and for all $T \in \mathcal{B}(\overline{X}, \overline{Y})$.

Let $\overline{X}$ be a Banach couple. For $t > 0$, the J- and K-functionals defined on $\Delta\overline{X}$ and $\Sigma\overline{X}$, respectively, are given by

$$J(t,x) = J\big(t,x;\overline{X}\big) = \big\|x\big\|_0 \vee \Big(t\big\|x\big\|_1\Big)$$

for $x \in \Delta\overline{X}$, and

$$K(t,x) = K\big(t,x;\overline{X}\big) = \inf\Big\{\, \big\|x_0\big\|_0 + t\big\|x_1\big\|_1 \,\Big|\, x = x_0 + x_1,\ x_j \in X_j\ (j=0,1) \,\Big\}$$

for $x \in \Sigma\overline{X}$. Let Φ be a Banach function space over $\big(\mathbf{R}_+, dt/t\big)$ such that $1 \wedge t \in \Phi$ and

$$\int_0^\infty 1\wedge(1/t)\,|x(t)|\,\frac{dt}{t} < \infty \qquad \text{for all } x\in\Phi.$$

We define

$$K_\Phi\big(\overline{X}\big) = \Big\{\, x \in \Sigma\overline{X} \;\Big|\; \big\|x\big\|_{K_\Phi} = \big\|K(t,x)\big\|_\Phi < \infty \,\Big\}$$

by (3.3.1) of [4], and define $J_\Phi\big(\overline{X}\big)$ as the space of all $x \in \Sigma\overline{X}$, which permits a canonical representation $x = \int_0^\infty u(t)dt/t$ for a strongly measurable function $u\colon \mathbf{R}_+ \to \Delta\overline{X}$, with the norm

$$\big\|x\big\|_{J_\Phi} = \inf_u \big\|J\big(t, u(t)\big)\big\|_\Phi < \infty$$

by (3.4.3) of [4]. According to Corollary 4.1.9 of [4], K_Φ and J_Φ are exact interpolation functors for all Banach couples under the action of bounded (not necessarily linear) operators. More precisely, if $\overline{X}$ and $\overline{Y}$ are Banach couples and if $T\colon \overline{X} \to \overline{Y}$ are bounded (not necessarily linear) operators in the sense of Definition 4.1.1 of [4] that T acting from $\Sigma\overline{X}$ to $\Sigma\overline{Y}$ such that, there exists $\lambda > 0$, for any $x_j \in X_j$ $(j = 0, 1)$ and for any $\epsilon > 0$,

$$T(x_0 + x_1) = y_0 + y_1$$

for some $y_j \in Y_j$ with $\big\|y_j\big\|_{Y_j} \le \lambda\big\|x_j\big\|_{X_j} + \epsilon$ $(j = 0, 1)$. We put $\big\|T\big\|_{\overline{X},\overline{Y}} = \inf \lambda$. Thus,

$$\big\|Tx\big\|_{K_\Phi(\overline{Y})} \le \big\|T\big\|_{\overline{X},\overline{Y}}\big\|x\big\|_{K_\Phi(\overline{X})} \tag{1.4}$$

for all $x \in K_\Phi\big(\overline{X}\big)$, and

$$\big\|Tx\big\|_{J_\Phi(\overline{Y})} \le \big\|T\big\|_{\overline{X},\overline{Y}}\big\|x\big\|_{J_\Phi(\overline{X})} \tag{1.5}$$

for all $x \in J_\Phi\big(\overline{X}\big)$.

The function space Φ is said to be a quasi-power parameter space for real interpolation if the Calderón operator S is bounded on Φ, where S is defined by

$$\big(Sf\big)(t) = \int_0^\infty 1 \wedge \big(t/s\big) f(s)\frac{ds}{s}.$$

In this case, the equivalence $J_\Phi\big(\overline{X}\big) = K_\Phi\big(\overline{X}\big)$ holds with the isomorphism constant depending on Φ. By a slight abuse of notation, we may use the quasi-power parameter Φ to represent the corresponding methods J_Φ and K_Φ, and may simply write

$$\overline{X}_\Phi = J_\Phi\big(\overline{X}\big) = K_\Phi\big(\overline{X}\big).$$

Let Φ' be the Köthe dual of Φ. Then Φ' is also a quasi-power parameter space for real interpolation, and the duality

$$\big(\overline{X}_\Phi\big)' = \big(\overline{X}'\big)_{\Phi'} \tag{1.6}$$

holds with the isomorphism constant depending on Φ. For $1 \le p \le \infty$ and $0 \le \theta \le 1$, let

$$\begin{aligned} L^p_\theta &= \left\{ f \in L^0(\mathbf{R}_+, dt/t) \;\middle|\; \big\|f\big\|_{L^p_\theta} = \left(\int_0^\infty \left(\frac{f(t)}{t^\theta}\right)^p \frac{dt}{t}\right)^{1/p} \right\}, \\ l^p_\theta &= \left\{ a = (a_\nu)_{\nu=-\infty}^\infty \;\middle|\; \big\|a\big\|_{l^p_\theta} = \left(\sum_\nu \big(2^{-\nu\theta}|a_\nu|\big)^p\right)^{1/p} \right\}, \end{aligned}$$

and let $\overline{L}^p = \big(L^p_0, L^p_1\big)$ and $\overline{l}^p = \big(l^p_0, l^p_1\big)$. In particular, we denote

$$\overline{X}_{\theta,p} = \overline{X}_{L^p_\theta} = \overline{X}_{l^p_\theta}$$

for $0 < \theta < 1$ and $1 \le p < \infty$.

2. REITERATION FOR INTERPOLATION SPACES

In this section, we formulate a reiteration result concerning the interpolation space $\overline{X}_\Phi$, where Φ is a quasi-power parameter space for real interpolation. This result is a natural extension of Theorem 3.5.3 in [3], which is interesting in its own right and plays an important role in the sequel. For $1 \le p \le \infty$, let $L^p = L^p(\mathbf{R}_+, dt)$. For rearrangement invariant (r. i. in short) function spaces, we mean those Banach function spaces of measurable real valued functions on $(\mathbf{R}_+, dt)$ which are exact interpolation spaces for the Banach couple $\left(L^1, L^\infty\right)$. We refer to [2] for the background of r. i. function spaces.

It is known that each r. i. function space can be obtained by the real interpolation method [6]. For convenience of references, we include a lemma concerning the connection between the r. i. function spaces with the non-trivial Boyd indices and the quasi-power parameter spaces for real interpolation.

Lemma 2.1. *Assume that $\mathcal{E}$ is an r. i. function space over $(\mathbf{R}^+, dt)$ with the Boyd indices $\underline{\alpha}_\mathcal{E}$ and $\overline{\alpha}_\mathcal{E}$. Then*

$$0 < \underline{\alpha}_\mathcal{E} \le \overline{\alpha}_\mathcal{E} < 1$$

iff there exists a quasi-power parameter space Φ for real interpolation such that $\mathcal{E}$ is equivalent to $\left(L^1, L^\infty\right)_\Phi$.

Proof. If $0 < \underline{\alpha}_\mathcal{E} \le \overline{\alpha}_\mathcal{E} < 1$, then we define

$$\Phi = \left\{ f \mid f(t)/t \in \mathcal{E} \right\}$$

with the norm $\left\|f\right\|_\Phi = \left\|f(t)/t\right\|_\mathcal{E}$. We choose now p_0 and p_1 with

$$1 < p_0 < 1/\overline{\alpha}_\mathcal{E} \le 1/\underline{\alpha}_\mathcal{E} < p_1 < \infty,$$

According to Theorem 3.5.16 in [2], $\mathcal{E}$ is an interpolation space for the couple $\left(L^{p_0}, L^{p_1}\right)$. As a consequence, the Calderón operator S is bounded on Φ. In addition, $1 \wedge \left(1/t\right) \in L^{p_0} \cap L^{p_1} \subseteq \mathcal{E}$. This implies that $1 \wedge t \in \Phi$ and

$$\int_0^\infty 1 \wedge \left(1/t\right) |f(t)| \,\frac{dt}{t} \le \left\|1 \wedge \left(1/t\right)\right\|_{L^{p_0'} \cap L^{p_1'}} \cdot \left\|f(t)/t\right\|_{L^{p_0} + L^{p_1}} < \infty$$

for all $f \in \Phi$. Therefore, Φ is a quasi-power parameter space for real interpolation. Let $\hat{\Phi} = K_\Phi\left(L^1, L^\infty\right)$. For $f \in \hat{\Phi}$, by Proposition 3.1.18 in [4], we have

$$K\left(t, f; L^1, L^\infty\right) = \int_0^t f^*(s)ds,$$

where f^* is the nonincreasing rearrangement of f, and hence

$$\left\|f\right\|_{\hat{\Phi}} = \left\|\int_0^t f^*(s)ds\right\|_\Phi. \tag{2.1}$$

If $f \in \mathcal{E}$, then $\left\|f\right\|_\mathcal{E} = \left\|f^*\right\|_\mathcal{E} = \left\|tf^*(t)\right\|_\Phi$. We claim that this norm is equivalent to $\left\|f\right\|_{\hat{\Phi}}$. In fact, the inequality $tf^*(t) \le \int_0^t f^*(s)ds$ implies

$$\left\|tf^*(t)\right\|_\Phi \le \left\|\int_0^t f^*(s)ds\right\|_\Phi. \tag{2.2}$$

On the other hand,

$$\left\|\int_0^t f^*(s)ds\right\|_\Phi = \left\|\int_0^t sf^*(s)ds/s\right\|_\Phi \le \left\|S\right\|_\Phi \left\|tf^*(t)\right\|_\Phi. \tag{2.3}$$

Conversely, if Φ is a quasi-power parameter space for real interpolation, then $\hat{\Phi} = K_\Phi\big(L^1, L^\infty\big)$ is an r.i. space over $\big(\mathbf{R}^+, dt\big)$. If $f \in \hat{\Phi}$, then by Proposition 3.5.2 in [2] and (2.1)–(2.3), we have

$$\left\|\int_0^\infty \left(\frac{1}{t} \wedge \frac{1}{s}\right) f(s)ds\right\|_{\hat{\Phi}} \le \left\|\int_0^\infty \left(1 \wedge \frac{t}{s}\right) f^*(s)ds\right\|_{\Phi}$$
$$\le \big\|S\big\|_\Phi \big\|tf^*(t)\big\|_\Phi \le \big\|S\big\|_\Phi \big\|f\big\|_{\hat{\Phi}}.$$

By Theorem 3.5.15 of [2], $0 < \underline{\alpha}_{\hat{\Phi}} \le \overline{\alpha}_{\hat{\Phi}} < 1$. Therefore, if $\mathcal{E}$ is equivalent to $\big(L^1, L^\infty\big)_\Phi$, then $0 < \underline{\alpha}_{\mathcal{E}} \le \overline{\alpha}_{\mathcal{E}} < 1$. ∎

Theorem 2.2. *Let Φ be a quasi-power parameter space for real interpolation, and let $\hat{\Phi} = K_\Phi\big(L^1, L^\infty\big)$. If θ_0, θ_1, p_0, p_1 are real numbers satisfying*

$$1 < \frac{1}{1-\theta_0} < p_0 < 1/\overline{\alpha}_{\hat{\Phi}} \le 1/\underline{\alpha}_{\hat{\Phi}} < \frac{1}{1-\theta_1} < p_1 < \infty,$$

then there exists another quasi-power parameter space Ψ for real interpolation such that

$$\overline{X}_\Phi = \big(\overline{X}_{\theta_0,p_0}, \overline{X}_{\theta_1,p_1}\big)_\Psi$$

with the isomorphism constant depending on θ_0, θ_1, p_0, p_1 and Φ.

Proof. By Theorem 3.5.16 of [2], $\hat{\Phi}$ is an interpolation space for the couple $\big(L^{p_0}, L^{p_1}\big)$ over $(\mathbf{R}_+, dt)$. We define now a function space Ψ over $\big(\mathbf{R}_+, dt/t\big)$ by

$$\Psi = \left\{\, f \;\middle|\; g(t) = t^{\theta_0-1} f(t^{\theta_1-\theta_0}) \in \hat{\Phi} \,\right\}$$

with the norm $\big\|f\big\|_\Psi = \big\|g\big\|_{\hat{\Phi}}$, and show that Ψ is a quasi-power parameter space for real interpolation. Observe first that $t^{\theta_0-1} \wedge t^{\theta_1-1} \in L^{p_0} \cap L^{p_1} \subseteq \hat{\Phi}$, which implies that

$$1 \wedge t = t^{\frac{1-\theta_0}{\theta_1-\theta_0}} \left(t^{\frac{\theta_0-1}{\theta_1-\theta_0}} \wedge t^{\frac{\theta_1-1}{\theta_1-\theta_0}}\right) \in \Psi.$$

If $f \in \Psi$ with $g(t) = t^{\theta_0-1} f(t^{\theta_1-\theta_0}) \in \hat{\Phi}$, then

$$f(t) = t^{\frac{1-\theta_0}{\theta_1-\theta_0}} g\left(t^{\frac{1}{\theta_1-\theta_0}}\right).$$

This, together with integration-by-parts and Hölder's inequality, gives that

$$\int_0^\infty \left(1 \wedge \frac{1}{t}\right)|f(t)|\frac{dt}{t} = \int_0^\infty \left(1 \wedge \frac{1}{t}\right) t^{\frac{1-\theta_0}{\theta_1-\theta_0}} \left|g\left(t^{\frac{1}{\theta_1-\theta_0}}\right)\right| \frac{dt}{t}$$
$$= (\theta_1 - \theta_0)\left(\int_0^1 t^{-\theta_0}|g(t)|dt + \int_1^\infty t^{-\theta_1}|g(t)|dt\right)$$
$$\le \left(\frac{\theta_1-\theta_0}{(1-\theta_0 p_0')^{1/p_0'}}\left(\int_0^1 |g(t)|^{p_0}dt\right)^{1/p_0} + \right.$$
$$\left. + \frac{\theta_1-\theta_0}{(\theta_1 p_1' - 1)^{1/p_1'}}\left(\int_1^\infty |g(t)|^{p_1}dt\right)^{1/p_1}\right)$$
$$< \infty.$$

Let us now consider the Calderón operator S on the function space Ψ. For $g \in \hat{\Phi}$, let

$$\big(\hat{S}g\big)(t) = \int_0^\infty \left(\frac{s}{t}\right)^{1-\theta_0} \wedge \left(\frac{s}{t}\right)^{1-\theta_1} g(s)\frac{ds}{s}.$$

By Theorem 3.5.15 of [2], $\hat{S}$ is a bounded operator on $\hat{\Phi}$ with the norm $\|\hat{S}\|_{\hat{\Phi}} < \infty$. Consequently, if $f \in \Psi$ with $g(t) = t^{\theta_0 - 1} f(t^{\theta_1 - \theta_0}) \in \hat{\Phi}$, then

$$\begin{aligned}(Sf)(t) &= \int_0^t f(s)\frac{ds}{s} + t\int_t^\infty \frac{f(s)}{s}\frac{ds}{s} \\ &= \int_0^t s^{\frac{1-\theta_0}{\theta_1-\theta_0}} g\Big(s^{\frac{1}{\theta_1-\theta_0}}\Big)\frac{ds}{s} + t\int_t^\infty s^{\frac{1-\theta_1}{\theta_1-\theta_0}} g\Big(s^{\frac{1}{\theta_1-\theta_0}}\Big)\frac{ds}{s} \\ &= (\theta_1-\theta_0)\bigg(\int_0^{t^{\frac{1}{\theta_1-\theta_0}}} s^{1-\theta_0} g(s)\frac{ds}{s} + t\int_{t^{\frac{1}{\theta_1-\theta_0}}}^\infty s^{1-\theta_1} g(s)\frac{ds}{s}\bigg),\end{aligned}$$

and hence

$$\begin{aligned} t^{\theta_0-1}(Sf)\big(t^{\theta_1-\theta_0}\big) &= (\theta_1-\theta_0)\int_0^\infty \Big(\frac{s}{t}\Big)^{1-\theta_0} \wedge \Big(\frac{s}{t}\Big)^{1-\theta_1} g(s)\frac{ds}{s} \\ &= (\theta_1-\theta_0)(\hat{S}g)(t).\end{aligned}$$

This implies that

$$\|Sf\|_\Psi = (\theta_1-\theta_0)\|\hat{S}g\|_{\hat{\Phi}} \le (\theta_1-\theta_0)\|\hat{S}\|_{\hat{\Phi}}\|g\|_{\hat{\Phi}} = (\theta_1-\theta_0)\|\hat{S}\|_{\hat{\Phi}}\|f\|_\Psi,$$

which means that S is bounded on Ψ.

Let $\overline{Y} = \big(\overline{X}_{\theta_0,p_0}, \overline{X}_{\theta_1,p_1}\big)$. Then $K_\Psi\big(\overline{Y}\big) = J_\Psi\big(\overline{Y}\big) = \overline{Y}_\Psi$. We are ready to show the equivalence

$$\overline{X}_\Phi = \overline{Y}_\Psi. \tag{2.4}$$

For $x \in \overline{Y}_\Psi$, let

$$K_{\overline{X}}(t) = K\big(t, x; \overline{X}\big) \quad \text{and} \quad K_{\overline{Y}}(t) = K\big(t, x; \overline{Y}\big).$$

As in the proof of Theorem 3.5.3 of [3], we have

$$K\big(t, x; \overline{X}\big) \le C_1 t^{\theta_0} K\big(t^{\theta_1-\theta_0}, x; \overline{Y}\big), \tag{2.5}$$

where C_1 is a positive constant depending on $\theta_0, \theta_1, p_0, p_1$. Recall that $K_{\overline{X}}(t)$ is increasing while $K_{\overline{X}}(t)/t$ is decreasing. Thus $K_{\overline{X}}(t)/t = \big(K_{\overline{X}}(t)/t\big)^* \in \hat{\Phi}$. This, together with (2.2) and (2.5), implies that

$$\begin{aligned}\|x\|_{K_\Phi(\overline{X})} &= \|K_{\overline{X}}(x)\|_\Phi = \Big\|t\big(K_{\overline{X}}(t)/t\big)^*\Big\|_\Phi \le \|K_{\overline{X}}(t)/t\|_{\hat{\Phi}} \\ &\le C_1\Big\|t^{\theta_0-1}K_{\overline{Y}}\big(t^{\theta_1-\theta_0}\big)\Big\|_{\hat{\Phi}} = C_1\Big\|K\big(t, x; \overline{Y}\big)\Big\|_\Psi = C_1\|x\|_{K_\Psi(\overline{Y})}.\end{aligned}$$

On the other hand, for $x \in \overline{X}_\Phi$, we see that

$$\int_0^\infty \Big(1 \wedge \frac{1}{t}\Big) K\big(t, x; \overline{X}\big)\frac{dt}{t} < \infty,$$

which implies that $\big(1 \wedge (1/t)\big)K\big(t, x; \overline{X}\big) \to 0$ as $t \to 0$ or ∞. According to the continuous version of the fundamental lemma of real interpolation [11], there exists a decomposition $x = \int_0^\infty u(t)dt/t$ such that

$$J\big(t, u(t); \overline{X}\big) \le 2K\big(t, x; \overline{X}\big).$$

By combining this with another estimate in the proof of Theorem 3.5.3 of [3], we have

$$t^{\theta_0}K\big(t^{\theta_1-\theta_0}, x; \overline{Y}\big) \le C_2\int_0^\infty \Big(\frac{t}{s}\Big)^{\theta_0} \wedge \Big(\frac{t}{s}\Big)^{\theta_1} K_{\overline{X}}(s)\frac{ds}{s} = C_2 t\hat{S}\big(K_{\overline{X}}(t)/t\big)$$

for a positive constant C_2 depending on θ_0, θ_1, p_0, p_1. Consequently,

$$\begin{aligned}\|x\|_{K_\Psi(\overline{Y})} &= \Big\|K\big(t,x;\overline{Y}\big)\Big\|_\Psi = \Big\|t^{\theta_0-1}K_{\overline{Y}}\big(t^{\theta_1-\theta_0}\big)\Big\|_{\hat{\Phi}} \\ &\le C_2\Big\|\hat{S}\big(K_{\overline{X}}(t)/t\big)\Big\|_{\hat{\Phi}} \le C_2\|\hat{S}\|_{\hat{\Phi}}\|K_{\overline{X}}(t)/t\|_{\hat{\Phi}} \le C_2\|\hat{S}\|_{\hat{\Phi}}\|S\|_\Phi\|x\|_{K_\Phi(\overline{X})}.\end{aligned}$$

The last inequality follows from (2.1) and (2.3). Therefore,

$$\overline{X}_\Phi = \big(\overline{X}_{\theta_0,p_0},\overline{X}_{\theta_1,p_1}\big)_\Psi$$

with the isomorphism constant depending on θ_0, θ_1, p_0, p_1 and Φ. ∎

3. ON OPERATOR Ω_ψ AND DUALITY

Here and throughout, we always assume that $\psi\colon \mathbf{R}\longrightarrow\mathbf{R}$ is a Lipschitz function satisfying

$$\big|\psi(t)-\psi(s)\big| \le \gamma_\psi\big|t-s\big|, \tag{3.1}$$

where γ_ψ is a positive constant depending on ψ. We introduce now the quasilogarithmic operators Ω^J_ψ and Ω^K_ψ as below. Let $c>1$ be a constant which is fixed during the discussion. Given $x\in J_\Phi\big(\overline{X}\big)$, we may choose the decomposition $x=\int_0^\infty u(t)\,dt/t$ in $\Sigma\overline{X}$ with $u\colon\mathbf{R}^+\to\Delta\overline{X}$ and $\Big\|J\big(t,u(t)\big)\Big\|_\Phi \le c\,\|x\|_{J_\Phi}$. Let

$$\Omega^J_\psi(x)=\Omega^J_{\psi,\overline{X}}(x)=\int_0^\infty \psi\big(\log t\big)u(t)\frac{dt}{t}.$$

For $x\in\Sigma\overline{X}$, we choose the decomposition $x=x_0(t)+x_1(t)$, $t>0$, satisfying

$$K(t,x)\le \big\|x_0(t)\big\|_0 + t\big\|x_1(t)\big\|_1 \le c\,K(t,x), \qquad t>0.$$

It is always possible to choose $x_j(t)$ $(j=0,1)$ to be continuous. Now we define

$$\Omega^K_\psi(x)=\Omega^K_{\psi,\overline{X}}(x) = -\int_0^1 x_0(t)d\psi\big(\log t\big)+\int_1^\infty x_1(t)d\psi\big(\log t\big)+\psi(0)x. \tag{3.2}$$

Observe that both Ω^J_ψ and Ω^K_ψ are quasi-linear operators. According to Theorem 3.3 of [9], if Φ is a quasi-power parameter space for real interpolation, then $\Omega^K_\psi=\Omega^J_\psi$ modulo bounded operators on $\overline{X}_\Phi$; and thus we may denote $\Omega_\psi=\Omega^K_\psi=\Omega^J_\psi$. Observe that, by Theorem 4.3 of [9], the operator

$$\Omega_\psi\colon \overline{X}_\Phi\to\overline{X}_\Phi$$

possesses the UBCP (1.3). Consequently, the operator Ω_ψ is well-defined up to a bounded error, and hence the domain and range spaces of Ω_ψ are independent of the choice of the decompositions of $x\in\overline{X}_\Phi$. Moreover, let Ψ and $\overline{Y}=\big(\overline{X}_{\theta_0,p_0},\overline{X}_{\theta_1,p_1}\big)$ as given in Theorem 2.2, and let $\eta=\theta_1-\theta_0$ and $\psi_\eta(t)=\psi(\eta t)$. Then

$$\Omega_{\psi,\overline{X}}=\Omega_{\psi_\eta,\overline{Y}} \tag{3.3}$$

modulo bounded linear operators on $\overline{X}_\Phi$. This can be easily induced from (2.5) of [10] and Theorem 2.2.

Let $\overline{A}$ be a Banach couple, and let A be an interpolation space for $\overline{A}$. The orbital interpolation space $\mathrm{Orb}_A\big(\overline{A},\overline{X}\big)$ consists of all $x\in\Sigma\overline{X}$ such that $x=\sum_{\nu=1}^\infty T_\nu a_\nu$ for some $T_\nu\in\mathcal{B}\big(\overline{A},\overline{X}\big)$ and $a_\nu\in A$ for which

$$\sum_{\nu=1}^\infty \|T_\nu\|_{\overline{A},\overline{X}}\|a_\nu\|_A<\infty$$

with the orbit norm $\|x\|_{\mathrm{Orb}} = \inf \sum_{\nu=1}^{\infty} \|T_\nu\|_{\overline{A},\overline{X}} \|a_\nu\|_A$. It is known that $\mathrm{Orb}_A(\overline{A}, -)$ is minimal among all interpolation functors F for which $F(\overline{A}) = A$ [1]. For a quasi-power parameter space Φ for real interpolation, we set $L_\Phi = (\overline{L}^1)_\Phi$ and $l_\Phi = (\overline{l}^1)_\Phi$. In view of Theorem 3.4.3 and Proposition 3.4.15 of [4], we have

$$\overline{X}_\Phi = \mathrm{Orb}_{L_\Phi}(\overline{L}^1, \overline{X}) = \mathrm{Orb}_{l_\Phi}(\overline{l}^1, \overline{X})$$

with the isomorphism constant depending on Φ. Moreover, we may present the discrete version of Lemma 3.4.5 of [4].

Lemma 3.1. *Let Φ be a quasi-power parameter space for real interpolation, and let $x \in \overline{X}_\Phi$. If $\epsilon > 0$, then there exist $T \in \mathcal{B}(\overline{l}^1, \overline{X})$ and $a \in l_\Phi$ equipped with the J_Φ-norm such that $x = Ta$ and*

$$\|T\|_{\bar{l}^1,\overline{X}} \|a\|_{J_\Phi(\bar{l}^1)} \le \frac{(1+\epsilon)}{\log 2} \|x\|_{J_\Phi}.$$

Proof. For $\epsilon > 0$, we may choose the decomposition $x = \int_0^\infty u(t)dt/t$, where $u\colon \mathbf{R}_+ \to \Delta\overline{X}$ and

$$\Big\| J\big(t, u(t)\big) \Big\|_\Phi \le (1+\epsilon)\|x\|_{J_\Phi}.$$

For $\nu \in \mathbf{Z}$, let $u_\nu = \int_{2^\nu}^{2^{\nu+1}} u(t)dt/t$, $a_\nu = J(2^\nu, u_\nu)$ and $a = (a_\nu)_\nu \in l_\Phi$. Then

$$J\big(2^\nu, u_\nu\big) \le \int_{2^\nu}^{2^{\nu+1}} J\big(t, u(t)\big)dt/t,$$

and hence

$$\|a\|_{J_\Phi(\bar{l}^1)} \le \frac{(1+\epsilon)}{\log 2} \|x\|_{J_\Phi}.$$

Assume that $u_\nu / a_\nu = 0$ if $a_\nu = 0$ and hence $u_\nu = 0$. Now we define $T \in \mathcal{B}(\overline{l}^1, \overline{X})$ by

$$T\lambda = \sum_\nu \lambda_\nu u_\nu / a_\nu, \qquad \text{for} \quad \lambda = (\lambda_\nu)_\nu \in \Sigma\overline{l}^1.$$

This implies that $\|T\|_{\bar{l}^1,\overline{X}} \le 1$, $x = Ta$, and hence

$$\|T\|_{\bar{l}^1,\overline{X}} \|a\|_{J_\Phi(\bar{l}^1)} \le \frac{(1+\epsilon)}{\log 2} \|x\|_{J_\Phi},$$

which completes the proof. ∎

The main result of this section is formulated as follows:

Theorem 3.2. *Let Φ be a quasi-power parameter space for real interpolation with the Köthe dual Φ', let ψ be a Lipschitz function satisfying (3.1), and let*

$$\psi^\times(t) = -\psi(-t).$$

If $\overline{X}_\Phi$ is equipped with the J_Φ-norm, then the operator $\Omega = \Omega_{\psi,\overline{X}}$ on $\overline{X}_\Phi$ and the operator $\Omega_\times = \Omega_{\psi^\times,\overline{X}'}$ on $(\overline{X}')_{\Phi'}$ are related by

$$\Big| \langle \Omega x, y \rangle + \langle x, \Omega_\times y \rangle \Big| \le C \|x\|_{\overline{X}_\Phi} \|y\|_{(\overline{X}')_{\Phi'}}$$

for $x \in \overline{X}_\Phi$ and $y \in (\overline{X}')_{\Phi'}$. Consequently, $\langle \Omega x, y \rangle = -\langle x, \Omega_\times y \rangle$ modulo bounded operators. Here C is a positive constant depending on ψ and Φ.

Proof. We start with the couple $\bar{l}^1$ and the dual couple $\big(\bar{l}^1\big)' = \bar{l}^\infty$. By applying Theorem 2.2 on Φ', we have

$$\begin{aligned}(l_\Phi)' &= \Big(\big(\bar{l}^1\big)_\Phi\Big)' = \big(\bar{l}^\infty\big)_{\Phi'} = \Big(\big(\bar{l}^\infty\big)_{\theta_0,p_0}, \big(\bar{l}^\infty\big)_{\theta_1,p_1}\Big)_\Psi = \\ &= \big(l^{p_0}_{\theta_0}, l^{p_1}_{\theta_1}\big)_\Psi = \Big(\big(\bar{l}^1\big)_{\theta_0,p_0}, \big(\bar{l}^1\big)_{\theta_1,p_1}\Big)_\Psi = l_{\Phi'}\end{aligned}$$

for some $1 < p_0 < p_1 < \infty$, $0 < \theta_0 < \theta_1 < 1$, and for a quasi-power parameter space Ψ for real interpolation. Let $\overline{B} = \big(l^{p_0}_{\theta_0}, l^{p_1}_{\theta_1}\big)$. Then

$$\Omega_{\psi^\times,\bar{l}^\infty} = \Omega_{(\psi^\times)_\eta,\overline{B}} = \Omega_{\psi^\times,\bar{l}^1} \tag{3.4}$$

modulo bounded operators on $l_{\Phi'}$ by (3.3). Let $a = \big(a_\nu\big)_\nu \in l_\Phi$ and $b = \big(b_\nu\big)_\nu \in l_{\Phi'}$. For $t > 0$, we set

$$a_0(t) = \big(a_\nu \chi_{\{2^\nu \le t\}}\big)_\nu \quad \text{and} \quad a_1(t) = \big(a_\nu \chi_{\{2^\nu > t\}}\big)_\nu.$$

This implies that

$$K(t) = K\big(t, x; \bar{l}^1\big) = \sum_\nu \Big(1 \wedge \big(t/2^\nu\big)\Big)|a_\nu| = \big\|a_0(t)\big\|_{l^1_0} + t\big\|a_1(t)\big\|_{l^1_1},$$

and hence

$$\Omega(a) = \big(a_\nu \psi(\nu \log 2)\big)_\nu. \tag{3.5}$$

We have, by (3.4) and (3.5),

$$\big\langle \Omega a, b\big\rangle = \sum_{\nu=-\infty}^{\infty} a_\nu b_{-\nu} \psi\big(\nu \log 2\big) \quad \text{and} \quad \big\langle a, \Omega_\times b\big\rangle = -\sum_{\nu=-\infty}^{\infty} a_\nu b_{-\nu} \psi\big(\nu \log 2\big).$$

It gives that

$$\big\langle \Omega a, b\big\rangle + \big\langle a, \Omega_\times b\big\rangle = 0. \tag{3.6}$$

Let now $\overline{X}$ be an arbitrary Banach couple, and let $x \in \overline{X}_\Phi$ and $y \in \big(\overline{X}'\big)_{\Phi'}$. For $\epsilon > 0$, we choose $a \in l_\Phi$ and $T \in \mathcal{B}\big(\bar{l}^1, \overline{X}\big)$ by Lemma 3.1 such that $x = Ta$ and

$$\big\|T\big\|_{\bar{l}^1,\overline{X}} \big\|a\big\|_{l_\Phi} \le \frac{(1+\epsilon)}{\log 2} \big\|x\big\|_{\overline{X}_\Phi}.$$

Let $u = [T, \Omega]a$. Then $u \in \overline{X}_\Phi$ such that $\Omega x = \Omega T a = T\Omega a - u$, and

$$\big\|u\big\|_{\overline{X}_\Phi} \le C\big\|T\big\|_{\bar{l}^1,\overline{X}}\big\|a\big\|_{l_\Phi}$$

by the UBCP of Ω. If we denote by T' the dual operator of T, then

$$\big\|T'\big\|_{\overline{X}',\bar{l}^\infty} = \big\|T\big\|_{\bar{l}^1,\overline{X}}$$

Thus, $T'y \in \big(\bar{l}^\infty\big)_{\Phi'} = l_{\Phi'}$, and $T'\Omega_\times y = \Omega_\times T'y + \big[T', \Omega_\times\big]y$ with

$$\Big\|\big[T', \Omega_\times\big]y\Big\|_{(\bar{l}^\infty)_{\Phi'}} \le C\big\|T\big\|_{\bar{l}^1,\overline{X}}\big\|y\big\|_{(\overline{X}')_{\Phi'}}.$$

This, together with (3.6), implies that

$$\begin{aligned}\big\langle \Omega x, y\big\rangle + \big\langle x, \Omega_\times y\big\rangle &= \big\langle \Omega T a, y\big\rangle + \big\langle Ta, \Omega_\times y\big\rangle \\ &= \big\langle \Omega a, T'y\big\rangle + \big\langle a, \Omega_\times T'y\big\rangle - \langle u, y\rangle + \Big\langle a, \big[T', \Omega_\times\big]y\Big\rangle \\ &= \Big\langle a, \big[T', \Omega_\times\big]y\Big\rangle - \langle u, y\rangle.\end{aligned}$$

Therefore,

$$\Big|\langle \Omega x, y\rangle + \langle x, \Omega_\times y\rangle\Big| \le 2C\|T\|_{\bar{l}^1,\overline{X}}\|a\|_{l_\Phi}\|y\|_{(\overline{X}')_{\Phi'}} \le 2C\frac{(1+\epsilon)}{\log 2}\|x\|_{\overline{X}_\Phi}\|y\|_{(\overline{X}')_{\Phi'}}.$$

The desired inequality is obtained by letting $\epsilon \to 0$ an by rewriting the constant. ∎

Remarks:

(i) For $(x_1, x_2) \in \overline{X}_\Phi \oplus_\Omega \overline{X}_\Phi$ and $(y_1, y_2) \in (\overline{X}')_{\Phi'} \oplus_{\Omega_\times} (\overline{X}')_{\Phi'}$, we have

$$\Big|\big\langle (x_1, x_2), (y_1, y_2)\big\rangle\Big| \le C\Big\|(x_1, x_2)\Big\|_{\overline{X}_\Phi \oplus_\Omega \overline{X}_\Phi}\Big\|(y_1, y_2)\Big\|_{(\overline{X}')_{\Phi'}\oplus_{\Omega_\times}(\overline{X}')_{\Phi'}}. \tag{3.7}$$

This estimate can be obtained by following the proof of Theorem 3.15 of [5]. For $\psi(t) = t$, we may solve Question 5 of [5] by proving Proposition 3.7 of [5] without using function theory.

(ii) By (1.6) and (3.7), we can put equivalent norms on the twisted sums $\overline{X}_\Phi \oplus_\Omega \overline{X}_\Phi$ and $(\overline{X}')_{\Phi'} \oplus_{\Omega_\times} (\overline{X}')_{\Phi'}$ as in Section 3 of [5], and obtain the duality relation

$$\big(\overline{X}_\Phi \oplus_\Omega \overline{X}_\Phi\big)' = (\overline{X}')_{\Phi'} \oplus_{\Omega_\times} (\overline{X}')_{\Phi'}.$$

Consequently,

$$\mathrm{Dom}_{\overline{X}_\Phi}(\Omega)' = \mathrm{Ran}_{(\overline{X}')_{\Phi'}}(\Omega_\times) \quad \text{and} \quad \mathrm{Ran}_{\overline{X}_\Phi}(\Omega)' = \mathrm{Dom}_{(\overline{X}')_{\Phi'}}(\Omega_\times). \tag{3.8}$$

Before proceeding, we show now that the range space $\mathrm{Ran}_X(\Omega)$ naturally appears in the context of Aronszajn-Gagliardo construction. The essential tool used here is similar to that in the proof of Theorem 3.2.

Theorem 3.3. *Let $X = \mathrm{Orb}_A(\overline{A}, \overline{X})$. Assume that $\Omega\colon X \to X$ is a quasi-linear operator possessing the UBCP (1.3) with $R = \mathrm{Ran}_A(\Omega)$. If, for each $x \in X$ and for $\epsilon > 0$, there exist $T \in \mathcal{B}(\overline{A}, \overline{X})$ and $a \in A$ such that $x = Ta$ and $\|T\|_{\overline{A},\overline{X}}\|a\|_A \le (1+\epsilon)\|x\|_X$, and if R is equivalent to a Banach space, then*

$$\mathrm{Ran}_X(\Omega) = \mathrm{Orb}_R(\overline{A}, \overline{X}).$$

Proof. Since $R = \mathrm{Ran}_A(\Omega) = \mathrm{Orb}_R(\overline{A}, \overline{A})$, it implies that

$$\mathrm{Orb}_R(\overline{A}, \overline{X}) \subseteq \mathrm{Ran}_X(\Omega)$$

by the minimality of the orbital functor. It is enough to show the converse inclusion

$$\mathrm{Ran}_X(\Omega) \subseteq \mathrm{Orb}_R(\overline{A}, \overline{X}).$$

For $\epsilon > 0$, by the definition of $\mathrm{Ran}_X(\Omega)$, we may choose $(x, y) \in X \oplus_\Omega X$ such that

$$\|x\|_X + \|\Omega x - y\|_X \le (1+\epsilon)\|(x, y)\|_{\mathrm{Ran}_X(\Omega)}.$$

Observe that $x = T_1 a_1$ and $y - \Omega x = T_2 a_2$ for some $T_1, T_2 \in \mathcal{B}(\overline{A}, \overline{X})$ and $a_1, a_2 \in A$ satisfying

$$\|T_1\|_{\overline{A},\overline{X}}\|a_1\|_A \le (1+\epsilon)\|x\|_X \quad \text{and} \quad \|T_2\|_{\overline{A},\overline{X}}\|a_2\|_A \le (1+\epsilon)\|y - \Omega x\|_X.$$

Consequently, we have

$$(x, y) = \big(T_1 a_1, \Omega T_1 a_1 + T_2 a_2\big) = \big(T_1 a_1, T_1 \Omega a_1 + T_2 a_2 - [T_1, \Omega] a_1\big).$$

Let $u = [T_1, \Omega] a_1$. Then, by (1.3), $u \in X$ with

$$\|u\|_X \le C\|T_1\|_{\overline{A},\overline{X}}\|a_1\|_A \le C(1+\epsilon)\|x\|_X,$$

and hence $u = T_3 a_3$ for some $T_3 \in \mathcal{B}(\overline{A}, \overline{X})$ and $a_3 \in A$ with

$$\|T_3\|_{\overline{A},\overline{X}} \|a_3\|_A \le (1+\epsilon)\|u\|_X \le C(1+\epsilon)^2 \|x\|_X.$$

Now we have

$$(x,y) = T_1(a_1, \Omega a_1) + T_2(0, a_2) - T_3(0, a_3)$$

satisfying

$$\begin{aligned}
&\|(x,y)\|_{\mathrm{Orb}_R} \\
&\quad \le \|T_1\|_{\overline{A},\overline{X}} \|(a_1, \Omega a_1)\|_R + \|T_2\|_{\overline{A},\overline{X}} \|(0, a_2)\|_R + \|T_3\|_{\overline{A},\overline{X}} \|(0, a_3)\|_R \\
&\quad = \|T_1\|_{\overline{A},\overline{X}} \|a_1\|_A + \|T_2\|_{\overline{A},\overline{X}} \|a_2\|_A + \|T_3\|_{\overline{A},\overline{X}} \|a_3\|_A \\
&\quad \le (C+2)(1+\epsilon)^2 \|(x,y)\|_{\mathrm{Ran}_X(\Omega)}.
\end{aligned}$$

Therefore, $\|(x,y)\|_{\mathrm{Orb}_R} \le (C+2)\|(x,y)\|_{\mathrm{Ran}_X(\Omega)}$ by letting $\epsilon \to 0$. ∎

4. ON OPERATOR Ω_ψ^α WITH DOMAIN AND RANGE SPACES

Assume that $\alpha \ge 1$. Let us now consider the E_α-functional on $\Sigma\overline{X}$, which is given by

$$\begin{aligned}
E_\alpha(r,x) &= E_\alpha(r,x;\overline{X}) \\
&= \inf\left\{ \max_{j=0,1} \left(\|x_j\|_j \big/ r \right)^{1/(\alpha-j)} \;\middle|\; x = x_0 + x_1,\ x_j \in X_j\ (j=0,1) \right\}
\end{aligned}$$

for $\alpha > 1$, and

$$\begin{aligned}
E_1(r,x) &= E_1(r,x;\overline{X}) \\
&= \inf\left\{ \|x_0\|_0 \big/ r \;\middle|\; x = x_0 + x_1,\ x_j \in X_j\ (j=0,1),\ \|x_1\|_1 \le r \right\},
\end{aligned}$$

where $r > 0$ and $x \in \Sigma\overline{X}$. For $c > 1$ fixed and for $x \in \Sigma\overline{X}$, we have decomposition $x = x_0(r) + x_1(r)$, $r > 0$, for which

$$E_\alpha(r,x) \le \left(\|x_0(r)\|_0 \big/ r \right)^{1/\alpha} \vee \left(\|x_1(r)\|_1 \big/ r \right)^{1/(\alpha-1)} \le E_\alpha(r/c, x)$$

for $\alpha > 1$, and

$$E_1(r,x) \le \|x_0(r)\|_0 \big/ r \le E_1(r/c, x) \quad \text{with } \|x_1(r)\|_1 \le r.$$

The corresponding quasilogarithmic operator Ω_ψ^α can be defined by

$$\Omega_\psi^\alpha(x) = -\int_0^1 x_0(r)\, d\psi(\log r) + \int_1^\infty x_1(r)\, d\psi(\log r) \tag{4.1}$$

for $x \in \Sigma\overline{X}$. It is known that $\Omega_\psi^\alpha \colon \overline{X}_{\theta,p} \to \overline{X}_{\theta,p}$, for $0 < \theta < 1$ and $1 \le p < \infty$, possesses the UBCP (1.3). If Φ is a quasi-power parameter space for real interpolation, then by using (1.4), (1.5) and Theorem 2.2, $\Omega_\psi^\alpha \colon \overline{X}_\Phi \to \overline{X}_\Phi$ also possesses the UBCP. Similar to (3.3), we have

$$\Omega_{\psi,\overline{X}}^\alpha = \Omega_{\psi_\eta,\overline{Y}}^\beta \tag{4.2}$$

modulo bounded operators on $\overline{X}_\Phi$ by (2.7) of [10] and Theorem 2.2 again, where $\overline{Y}$, η, ψ_η are given as before, and $\beta = (\alpha - \theta_0)\big/(\theta_1 - \theta_0) > 1$.

Let now $K(t) = K(t,x)$ and $E_\alpha(r) = E_\alpha(r,x)$. If we use the change of variable $r = K(t)/t^\alpha$, then $E_\alpha(r) = t$ for each $x \in \Sigma\overline{X}$ by (2.7) of [10]. Moreover, if $x \neq 0$ then

$$1 \wedge t \leq K(t)/\|x\|_{\Sigma\overline{X}} \leq 1 \vee t,$$

and hence

$$\alpha - 1 \leq \left| \frac{\log\Big(K(t)\Big/\big(t^\alpha \|x\|_{\Sigma\overline{X}}\big)\Big)}{\log t} \right| \leq \alpha. \tag{4.3}$$

If $u\colon \mathbf{R}^+ \longrightarrow \Delta\overline{X}$, for which $\int_0^\infty u(t)\,dt/t = 0$ and $\Big\|J\big(t,u(t)\big)\Big\|_\Phi < \infty$, then by using Lemma 3.4 of [9], we obtain

$$\psi\left(\log \frac{K(t)}{t^\alpha \|x\|_{\Sigma\overline{X}}}\right) \int_0^t u(s)\,\frac{ds}{s} \longrightarrow 0 \qquad \text{as } t \to 0 \text{ or } \infty.$$

This, together with the estimate

$$\left|\psi\left(\log \frac{K(t)}{t^\alpha}\right)\right| \leq \left|\psi\left(\log \frac{K(t)}{t^\alpha \|x\|_{\Sigma\overline{X}}}\right)\right| + \gamma_\psi \Big|\log\|x\|_{\Sigma\overline{X}}\Big|,$$

implies that

$$\psi\left(\log \frac{K(t)}{t^\alpha}\right) \int_0^t u(s)\,\frac{ds}{s} \longrightarrow 0 \qquad \text{as } t \to 0 \text{ or } \infty.$$

If $x = \int_0^\infty u(t)dt/t$ with $\Big\|J\big(t,u(t)\big)\Big\|_\Phi \leq c\|x\|_X$, then as in the proof of Theorem 3.5 of [9], we obtain

$$\Omega_\psi^\alpha(x) = \int_0^\infty \psi\left(\log \frac{K(t)}{t^\alpha}\right) u(t)\frac{dt}{t} - \psi\Big(\log\|x\|_{\Sigma\overline{X}}\Big)x \tag{4.4}$$

modulo bounded operators on $\overline{X}_\Phi$.

Observe that Ω_ψ^α is quasi-additive satisfying (1.2) but not always homogenuous. We need the following result for the further study.

Proposition 4.1. *Assume that Φ is a quasi-power parameter space for real interpolation. If we define $\tilde\Omega_\psi^\alpha$ by*

$$\tilde\Omega_\psi^\alpha(x) = \|x\|_X \Omega_\psi^\alpha\Big(x\Big/\|x\|_X\Big)$$

on $X = \overline{X}_\Phi$ equipped with the J_Φ-norm, then

(i) *$\tilde\Omega_\psi^\alpha$ is quasi-linear,*

(ii) *$X \oplus_{\tilde\Omega_\psi^\alpha} X = X \oplus_{\Omega_\psi^\alpha} X$, and hence*

$$\mathrm{Dom}_X\big(\tilde\Omega_\psi^\alpha\big) = \mathrm{Dom}_X\big(\Omega_\psi^\alpha\big) \quad \textit{and} \quad \mathrm{Ran}_X\big(\tilde\Omega_\psi^\alpha\big) = \mathrm{Ran}_X\big(\Omega_\psi^\alpha\big).$$

Proof. For part (i), let $x = \int_0^\infty u(t)dt/t$ with $\Big\|J\big(t,u(t)\big)\Big\|_\Phi \leq c\|x\|_X$. Then by (4.4), we may choose

$$\Omega_\psi^\alpha(x) = \int_0^\infty \left(\psi\left(\log \frac{K(t,x)}{t^\alpha}\right) - \psi\Big(\log\|x\|_{\Sigma\overline{X}}\Big)\right)u(t)\frac{dt}{t},$$

and hence

$$\tilde\Omega_\psi^\alpha(x) = \int_0^\infty \left(\psi\left(\log \frac{K(t,x)}{t^\alpha\|x\|_X}\right) - \psi\left(\log \frac{\|x\|_{\Sigma\overline{X}}}{\|x\|_X}\right)\right)u(t)\frac{dt}{t}.$$

This implies that

(4.5) $$\left\|\tilde{\Omega}_\psi^\alpha(x)-\Omega_\psi^\alpha(x)\right\|_X \le 2\gamma_\psi\left|\log\|x\|_X\right|\left\|J\big(t,u(t)\big)\right\|_\Phi \le 2c\gamma_\psi\|x\|_X\left|\log\|x\|_X\right|.$$

Moreover, if $\|x\|_X \le 1$, then

(4.6) $$\|x\|_X\left|\log\|x\|_X\right| \le 1.$$

Observe that

$$\begin{aligned}&\left\|\tilde{\Omega}_\psi^\alpha(x+y)-\tilde{\Omega}_\psi^\alpha(x)-\tilde{\Omega}_\psi^\alpha(y)\right\|_X\\ &\quad\le \left\|\Omega_\psi^\alpha(x+y)-\Omega_\psi^\alpha(x)-\Omega_\psi^\alpha(y)\right\|_X+\left\|\tilde{\Omega}_\psi^\alpha(x+y)-\Omega_\psi^\alpha(x+y)\right\|_X+\\ &\qquad+\left\|\tilde{\Omega}_\psi^\alpha(x)-\Omega_\psi^\alpha(x)\right\|_X+\left\|\tilde{\Omega}_\psi^\alpha(y)-\Omega_\psi^\alpha(y)\right\|_X.\end{aligned}$$

It follows that $\tilde{\Omega}_\psi^\alpha$ is quasi-additive by (4.5), (4.6) and the quasi-additivity of Ω_ψ^α.

Now we turn to part (ii). If $\|x_\nu\|_X \to 0$, then $\|x_\nu\|_X \log\|x_\nu\|_X \to 0$, and hence

$$\left\|\tilde{\Omega}_\psi^\alpha(x_\nu)-\Omega_\psi^\alpha(x_\nu)\right\|_X \to 0$$

by (4.5). Consequently,

$$\left\|\big(x_\nu,y_\nu\big)\right\|_{X\oplus_{\Omega_\psi^\alpha}X} \to 0 \qquad \text{iff} \qquad \left\|\big(x_\nu,y_\nu\big)\right\|_{X\oplus_{\tilde{\Omega}_\psi^\alpha}X} \to 0.$$

This implies that $X\oplus_{\tilde{\Omega}_\psi^\alpha}X = X\oplus_{\Omega_\psi^\alpha}X$. Therefore,

$$\text{Dom}\big(\tilde{\Omega}_\psi^\alpha\big)=\text{Dom}\big(\Omega_\psi^\alpha\big) \quad \text{and} \quad \text{Ran}\big(\tilde{\Omega}_\psi^\alpha\big)=\text{Ran}\big(\Omega_\psi^\alpha\big),$$

which completes the proof. ∎

For $a=\big(a_\nu\big)_\nu \in l_\Phi$ and $K(t)=K\big(t,a;\bar{l}^1\big)$, we have

$$\widetilde{\Omega}^\alpha(a)=\Big(a_\nu\psi\big(\log K(2^\nu)-\alpha\nu\log 2\big)-a_\nu\psi\big(\log K(1)\big)\Big)_\nu.$$

This, together with (4.2) and a similar argument as in the proof of Theorem 3.2, implies

Proposition 4.2. *The estimates in Theorem 3.2 and (3.7) are also valid for the operator $\tilde{\Omega}_\psi^\alpha$ on the space $X=\overline{X}_\Phi$.*

In particular, this gives an affirmative answer to Question 7 of [5]. As a consequence of Proposition 4.2, we have

(4.7) $$\text{Dom}_X\big(\Omega_\psi^\alpha\big)'=\text{Ran}_{X'}\big(\Omega_{\psi^\times}^\alpha\big) \quad \text{and} \quad \text{Ran}_X\big(\Omega_\psi^\alpha\big)'=\text{Dom}_{X'}\big(\Omega_{\psi^\times}^\alpha\big).$$

For $\psi(t)=t$, let us now pay attention to the connection between operators $\Omega=\Omega_\psi$ and $\Omega^\alpha=\Omega_\psi^\alpha$. As mentioned in [5], it is not generally true that both operators differ from each other on $\overline{X}_{\theta,p}$ by a bounded error. However, the equivalence of the domain spaces

$$\text{Dom}_X\big(\Omega^\alpha\big)=\text{Dom}_X\big(\Omega\big)$$

is given by Theorem 4.2 of [5] with a rather complicated computational proof. Cwikel et al asked if it was possible to give a conceptual proof for this equivalence and if there was an analogous equivalence for the range spaces. We show that, as an application of Theorem 3.3 and Proposition 4.1, this is the case even for $\overline{X}_\Phi$ and our proof is very simple.

Proposition 4.3. *Let Φ be a quasi-power parameter space for real interpolation, and let $X = \overline{X}_\Phi$ equipped with the J_Φ-norm. Then*

$$\mathrm{Dom}_X\big(\Omega^\alpha\big) = \mathrm{Dom}_X\big(\Omega\big) \quad \textit{and} \quad \mathrm{Ran}_X\big(\Omega^\alpha\big) = \mathrm{Ran}_X\big(\Omega\big).$$

Proof. By Proposition 4.1, we may replace Ω^α by $\tilde{\Omega}^\alpha$, and thus assume without loss of generality that $\|x\|_X = 1$. Let $x = \int_0^\infty u(t)dt/t$ with $\Big\|J\big(t, u(t)\big)\Big\|_\Phi \le c\|x\|_X$. Observe that

$$\begin{aligned}\Omega^\alpha(x) &= \int_0^\infty \log\Big(K(t,x)\Big/\big(t^\alpha\|x\|_{\Sigma\overline{X}}\big)\Big)u(t)\frac{dt}{t}\\ &= \int_0^\infty \log t \cdot \frac{\log\Big(K(t,x)\Big/\big(t^\alpha\|x\|_{\Sigma\overline{X}}\big)\Big)}{\log t}u(t)\frac{dt}{t},\end{aligned}$$

and

$$\begin{aligned}\Omega(x) &= \int_0^\infty \log t \cdot u(t)\frac{dt}{t}\\ &= \int_0^\infty \log\Big(K(t,x)\Big/\big(t^\alpha\|x\|_{\Sigma\overline{X}}\big)\Big) \cdot \frac{\log t}{\log\Big(K(t,x)\Big/\big(t^\alpha\|x\|_{\Sigma\overline{X}}\big)\Big)}u(t)\frac{dt}{t}.\end{aligned}$$

It turns out by (4.3) that

$$\frac{\alpha-1}{c}\big\|\Omega(x)\big\|_X \le \big\|\Omega^\alpha(x)\big\|_X \le \alpha c\,\big\|\Omega(x)\big\|_X.$$

For $\alpha > 1$, the identity

$$\mathrm{Dom}_X\big(\Omega^\alpha\big) = \mathrm{Dom}_X\big(\Omega\big) \tag{4.8}$$

is a direct consequence of this estimate. By the reiteration in (4.2), we obtain that the identity (4.8) holds true also for $\alpha = 1$.

For the identity $\mathrm{Ran}_X\big(\Omega^\alpha\big) = \mathrm{Ran}_X\big(\Omega\big)$, observe that, by (3.8), (4.7) and (4.8), we have

$$\mathrm{Ran}_{X'}\big(\Omega^\alpha\big) = \mathrm{Dom}_X\big(\Omega^\alpha_\times\big)' = \mathrm{Dom}_X\big(\Omega_\times\big)' = \mathrm{Ran}_{X'}\big(\Omega\big)$$

for all dual couples $\overline{X}'$ and for $X' = \big(\overline{X}'\big)_{\Phi'}$. Especially,

$$R = \mathrm{Ran}_{l_\Phi}\big(\Omega^\alpha\big) = \mathrm{Ran}_{l_\Phi}\big(\Omega\big)$$

for the couple $\overline{l}^1$. By Lemma 3.1 and Theorem 3.3, we obtain

$$\mathrm{Ran}_X\big(\Omega^\alpha\big) = \mathrm{Orb}_R\big(\overline{l}^1, \overline{X}\big) = \mathrm{Ran}_X\big(\Omega\big)$$

for all Banach couples $\overline{X}$. ∎

5. ON NON-COMMUTATIVE SYMMETRIC SPACES OF MEASURABLE OPERATORS

Let $\mathcal{M}$ be a semifinite von Neumann algebra acting on a Hilbert space H with the given normal faithful semifinite trace τ and the identity 1. The densely-defined closed linear operator x on H is said to be affiliated with $\mathcal{M}$ if $xu = ux$ for all unitary operators u commuting with $\mathcal{M}$. The operator x is said to be τ-measurable if, for each $\epsilon > 0$, there is a project e in $\mathcal{M}$ for which $e(H)$ is included in the domain of x and $\tau(1-e) < \epsilon$. Now we denote by $\widetilde{\mathcal{M}}$ the space of all τ-measurable operators affliated with $\mathcal{M}$. For $x \in \widetilde{\mathcal{M}}$ and $t > 0$, the corresponding singular number is defined by

$$\mu_t(x) = \inf\Big\{ \, \|xe\|_{\mathcal{M}} \, \Big| \, e \text{ is a projection in } \mathcal{M} \text{ with } \tau(1-e) \le t \, \Big\}.$$

The function $\mu(x)\colon t \mapsto \mu_t(x)$ is said to be the generalized singular value function or the nonincreasing rearrangement of x. The space $\widetilde{\mathcal{M}}$ is equipped with the measure topology in the sense that a basis of neighbourhoods at zero is given by the sets

$$M_{\epsilon,\delta} = \big\{ \, x \in \widetilde{\mathcal{M}} \, \big| \, \mu_\delta(x) < \epsilon \, \big\} \qquad \text{for } \epsilon, \, \delta > 0.$$

Observe that $\mu(x) = \mu\big(|x|\big)$, where $|x|$ is the absolute value of x. See [13] for details. Furthermore, let us assume that $|x|$ has the spectral representation

$$|x| = \int_0^\infty s \, de_s^x,$$

and that $e_{(s_0,s_1)}\big(|x|\big)$ is the spectral projection of $|x|$ with respect to the interval (s_0, s_1) for $0 \le s_0 < s_1 \le \infty$. According to Proposition 2.2 of [8], we have

$$\mu_t(x) = \inf\Big\{ \, r > 0 \, \Big| \, \tau\big(e_{(r,\infty)}\big(|x|\big)\big) \le t \, \Big\} \tag{5.1}$$

for $t > 0$. Now let $\mathcal{E}$ be an r. i. function space over $\mathbf{R}^+$ satisfying $0 < \underline{\alpha}_{\mathcal{E}} \le \overline{\alpha}_{\mathcal{E}} < 1$. We define the symmetric space $\mathcal{E}(\mathcal{M})$ of measurable operators associated with $\mathcal{E}$ and $\mathcal{M}$ by

$$\mathcal{E}(\mathcal{M}) = \big\{ \, x \in \widetilde{\mathcal{M}} \, \big| \, \mu(x) \in \mathcal{E} \, \big\}$$

equipped with the norm $\|x\|_{\mathcal{E}(\mathcal{M})} = \|\mu(x)\|_{\mathcal{E}}$. If we denote by $\mathcal{E}(\mathcal{M})'$ the Köthe dual of $\mathcal{E}(\mathcal{M})$ in the sense of Definition 5.1 of [7], then the following duality relation

$$\mathcal{E}(\mathcal{M})' = \mathcal{E}'(\mathcal{M})$$

holds by Theorem 5.6 of [7]. In particular, $L^\infty(\mathcal{M}) = \mathcal{M}$ equipped with the usual operator norm. We refer to [7] and references therein for the further information.

We turn now our attention to the interpolation of r. i. spaces and the corresponding symmetric operator spaces. By Lemma 2.1, $\mathcal{E} = K_\Phi\big(L^1, L^\infty\big)$ for a quasi-power parameter space Φ for real interpolation. Recall that $\big(L^1(\mathcal{M}), \mathcal{M}\big)$ is a Banach couple with the K-functional

$$K(t,x) = K\big(t, x; L^1(\mathcal{M}), \mathcal{M}\big) = \int_0^t \mu_s(x) \, ds \tag{5.2}$$

by Proposition 2.5 of [7].

Proposition 5.1. $\mathcal{E}(\mathcal{M}) = K_\Phi\big(L^1(\mathcal{M}), \mathcal{M}\big)$.

Proof. For $x \in L^1(\mathcal{M}) \cap \mathcal{M}$, we have

$$\begin{aligned} \|x\|_{\mathcal{E}(\mathcal{M})} &= \|\mu_t(x)\|_{\mathcal{E}} \simeq \Big\| K\big(t, \mu_t(x); (L^1, L^\infty)\big) \Big\|_\Phi \\ &= \Big\| \int_0^t \mu_s(x) ds \Big\|_\Phi = \Big\| K\Big(t, x; \big(L^1(\mathcal{M}), \mathcal{M}\big)\Big) \Big\|_\Phi, \end{aligned}$$

by Proposition 3.1.18 of [4] and (5.2). Since $L^1 \cap L^\infty$ is dense in $\mathcal{E}$, this implies that $L^1(\mathcal{M}) \cap \mathcal{M}$ is dense in both $\mathcal{E}(\mathcal{M})$ and $K_\Phi\big(L^1(\mathcal{M}), \mathcal{M}\big)$ by Proposition 2.8 of [7] and Corollary 3.6.3 of [4] respectively. Therefore, the identity

$$\mathcal{E}(\mathcal{M}) = K_\Phi\big(L^1(\mathcal{M}), \mathcal{M}\big)$$

holds true. ∎

We are ready to calculate the operators Ω_ψ and Ω^1_ψ for the Banach couple $\Big(L^1(\mathcal{M}),\mathcal{M}\Big)$. Without loss of generality, we may assume that $\psi(0)=0$.

Proposition 5.2. *For $x\in\widetilde{\mathcal{M}}$, let $x=u|x|$, where u is a partial isometry and $|x|=\int_0^\infty s\,de^x_s$. Then*

$$\Omega_\psi(x)=u\int_0^\infty s\psi\big(\log\lambda_s(x)\big)\,de^x_s, \tag{5.3}$$

and

$$\Omega^1_\psi(x)=u\int_0^\infty s\psi\big(\log s\big)\,de^x_s. \tag{5.4}$$

Proof. If we set $t=\lambda_r(x)=\tau\big(e_{(r,\infty)}(|x|)\big)$, $K(t)=K(t,x)$ and $\tilde r=K(t)/t$, then by (5.1) and (5.2),

$$r=r(t)=\mu_t(x)=K'(t)$$

and $\tilde r\ge r$. For the operator Ω_ψ, we set $x=x_0(t)+x_1(t)$, where

$$x_0(t)=u\int_{(r(t),\infty)} s\,de^x_s=u|x|e_{(r(t),\infty)},$$

$$x_1(t)=u\int_{(0,r(t))} s\,de^x_s=u|x|e_{(0,r(t))}.$$

Recall that

$$\mu_s\big(|x|e_{(r(t),\infty)}\big)=\chi_{(0,t)}(s)\mu_s(x)$$

by an argument in the proof of Proposition 2.7 of [7]. Combining this with (5.2), we obtain that

$$\begin{aligned}\big\|x_0(t)\big\|_{L^1(\mathcal{M})} &= \int_0^\infty \mu_s\big(x_0(t)\big)\,ds \;=\; \int_0^\infty \mu_s\big(|x|e_{(r(t),\infty)}\big)\,ds\\ &= \int_0^t \mu_s(x)\,ds \;=\; K(t,x),\end{aligned}$$

and

$$t\big\|x_1(t)\big\|_{\mathcal{M}}=t\Big\|\int_0^{\mu_t(x)} s\,de^x_s\Big\|_{\mathcal{M}}\le t\mu_t(x)\le\int_0^t\mu_s(x)\,ds=K(t,x).$$

Therefore,

$$\big\|x_0(t)\big\|_{L^1(\mathcal{M})}+t\big\|x_1(t)\big\|_{\mathcal{M}}\le 2\,K(t,x).$$

It turns out that

$$\begin{aligned}\Omega_\psi(x) &= -u\int_0^1 x_0(t)\,d\psi\big(\log t\big)+u\int_1^\infty x_1(t)\,d\psi\big(\log t\big)\\ &= -u\int_0^1\Big(\int_{r(t)}^\infty s\,de^x_s\Big)\,d\psi\big(\log t\big)+u\int_1^\infty\Big(\int_0^{r(t)} s\,de^x_s\Big)\,d\psi\big(\log t\big).\end{aligned}$$

We choose now p_0 and p_1 satisfying $0<1/p_1<\underline{\alpha}_{\mathcal{E}}\le\overline{\alpha}_{\mathcal{E}}<1/p_0<1$. Since $\mu_t\in\mathcal{E}\subseteq L^{p_0}+L^{p_1}$, this follows that

$$\int_0^1\mu_t(x)^{p_0}\,dt<\infty\quad\text{and}\quad\int_1^\infty\mu_t(x)^{p_1}\,dt<\infty.$$

Observe that, for $0 < t < 1$,

$$\psi\big(\log t\big)\big\|x_0(t)\big\|_{L^1(\mathcal{M})} \le \psi\big(\log t\big)\int_0^t \mu_s(x)\,ds$$
$$\le \psi\big(\log t\big)t^{1/p_0'}\bigg(\int_0^1 \mu_t(x)^{p_0}\,dt\bigg)^{1/p_0},$$

and, for $t > 1$,

$$\psi\big(\log t\big)\big\|x_1(t)\big\|_{\mathcal{M}} \le \frac{\psi\big(\log t\big)}{t}\int_0^t \mu_s(x)\,ds$$
$$\le \frac{\psi\big(\log t\big)}{t}\bigg(\int_0^1 \mu_t(x)^{p_0}\,dt\bigg)^{1/p_0} + \frac{\psi\big(\log t\big)}{t^{1/p_1}}\bigg(\int_1^\infty \mu_t(x)^{p_1}\,dt\bigg)^{1/p_1}.$$

Consequently,

$$\lim_{t\to 0}\psi\big(\log t\big)\big\|x_0(t)\big\|_{L^1(\mathcal{M})} = \lim_{t\to\infty}\psi\big(\log t\big)\big\|x_1(t)\big\|_{\mathcal{M}} = 0.$$

We obtain (5.3) in terms of interpolation-by-parts. For the operator Ω_ψ^1, we set $x = x_0(\tilde{r}) + x_1(\tilde{r})$, where

$$x_0(\tilde{r}) = u\int_{\tilde{r}}^\infty s\,de_s^x = u|x|e_{(\tilde{r},\infty)} \quad \text{and} \quad x_1(\tilde{r}) = u\int_0^{\tilde{r}} s\,de_s^x = u|x|e_{(0,\tilde{r})}.$$

By (2.7) of [10], $\tilde{r} = K(t)\big/t$ iff $E_1(\tilde{r}) = t$, and hence

$$E_1(r,x) \le \big\|x_0(r)\big\|_0\Big/r \le E_1\big(r/c,x\big), \qquad \big\|x_1(r)\big\|_1 \le r.$$

A similar calculation implies (5.4). ∎

By applying Theorem 4.3 of [9], Theorem 3 and Proposition 4.2 on the operators given in (5.3) and (5.4), we obtain immediately the following result:

Proposition 5.3. (i) *If $a, b \in \mathcal{M}$ with $\big\|a\big\|_{\mathcal{M}} \le 1$, $\big\|b\big\|_{\mathcal{M}} \le 1$, and if $x \in \mathcal{E}(\mathcal{M})$, then*

$$\begin{aligned}\big\|\Omega_\psi(axb) - a\Omega_\psi(x)b\big\|_{\mathcal{E}(\mathcal{M})} &\le C\big\|x\big\|_{\mathcal{E}(\mathcal{M})},\\ \big\|\Omega_\psi^1(axb) - a\Omega_\psi^1(x)b\big\|_{\mathcal{E}(\mathcal{M})} &\le C\big\|x\big\|_{\mathcal{E}(\mathcal{M})}.\end{aligned}$$

(ii) *If $x \in \mathcal{E}(\mathcal{M})$ and $y \in \mathcal{E}'(\mathcal{M})$, then*

$$\begin{aligned}\Big|\tau\big(\Omega_\psi(x)y - x\Omega_\psi(y)\big)\Big| &\le C\big\|x\big\|_{\mathcal{E}(\mathcal{M})}\big\|y\big\|_{\mathcal{E}'(\mathcal{M})},\\ \Big|\tau\big(\Omega_\psi^1(x)y - x\Omega_\psi^1(y)\big)\Big| &\le C\big\|x\big\|_{\mathcal{E}(\mathcal{M})}\big\|y\big\|_{\mathcal{E}'(\mathcal{M})}.\end{aligned}$$

Here C is a constant depending on $\mathcal{E}$ and ψ.

Observe that part (i) extends Theorems 4.2 and 4.3 of [12].

REFERENCES

[1] N. ARONSZJAN & E. GAGLIARDO, Interpolation spaces and interpolation methods, *Ann. Mat. Pura Appl.,* **68** (1965), pp. 51–118.

[2] C. BENNETT & R. SHARPLEY, *Interpolation of Operators*, Pure and applied mathmatics, **129**, Academic press, New York, 1988.

[3] J. BERGH & J. LÖFSTRÖM, *Interpolation Spaces*, Grundlehren Math. Wiss., **223**, Springer-Verlag, Berlin/Heidelberg/New York, 1976.

[4] YU. A. BRUDNYI & N. JA. KRUGLJAK, *Interpolation Functors and Interpolation Spaces*, North-Holland, Amsterdam, 1991.

[5] M. CWIKEL, B. JAWERTH, M. MILMAN & R. ROCHBERG, *Differential Estimates and Commutator in Interpolation Theory*, "Analysis at Urbana IIt't', London Math. Soc., Cambridge Univ. Press 1989, pp. 170–220.

[6] M. CWIKEL & E. PUSTYLNIK, Weak type interpolation near "endpoint" spaces, *J. Funct. Anal.*, **171** (2000), pp. 235–277.

[7] P. G. DODDS, T. K. DODDS & B. DE PAGTER, Non-commutative Köthe duality, *Trans. Amer. Math. Soc.*, **339** (1993), pp. 717–750.

[8] T. FACK & H. KOSAKI, Generalized s-numbers of τ-measurable operators, *Pacific J. Math.*, **123** (1986), pp. 269–300.

[9] M. FAN, Commutators in real interpolation with quasi-power parameters, *Abstr. Appl. Anal.*, **7** (2002), pp. 239–257. Erratum `[Online http://users.du.se/~fmi/pub/Comm1.pdf]`.

[10] B. JAWERTH, R. ROCHBERG & G. WEISS, Commutators and other second order estimates in real interpolation theory, *Ark. Mat.,* **24** (1986), pp. 191–219.

[11] S. KAIJSER, A new proof of the fundamental lemma of interpolation theory, *Math. Scand.*, **78** (1996), pp. 266–270.

[12] N. J. KALTON, Trace-class operators and commutators, *J. Funct. Anal.*, **86** (1989), pp. 41–74.

[13] E. NELSON, Notes on non-commutative integration, *J. Funct. Anal.*, **15** (1974), pp. 103–116.