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UNIQUENESS OF MEROMORPHIC FUNCTIONS AND WEIGHTED SHARING

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ABSTRACT. With the help of the notion of weighted sharing of values, we prove a result on uniqueness of meromorphic functions and as a consequence we improve a result of P. Li.

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1. INTRODUCTION, DEFINITIONS AND RESULTS

Two non-constant meromorphic functions f and g defined in the open complex plane $\mathbb{C}$ are said to share the value a CM (counting multiplicities), for some $a \in \mathbb{C} \cup \{\infty\}$, if the locations and multiplicities of the a-points of f and g coincide. If the a-points coincide in locations only then the functions f and g are said to share the value a IM (ignoring multiplicities). For the standard notations and definitions of the value distribution theory we refer to [2]. However, we explain some notations that will be needed in the sequel.

Definition 1.1. [7] *For any positive integer k, we denote by $E(a, k; f)$ the set of all a-points of f with multiplicities less than or equal to k, where each a -point is counted according to its multiplicity.*

Definition 1.2. *For any positive integer s, we denote by $\overline{N}(r, a; f \mid\geq s)$ the counting function of those a-points of f whose multiplicities are greater than or equal to s, where each a-point is counted only once.*

The behaviour of two non-constant meromorphic functions sharing three values is being talked about largely and continuous effort is being put in to relax the hypotheses of the results. In [7] E. Mues proved the following theorem.

Theorem A. *{Theorem 10 [7]} Let f and g be non-constant meromorphic functions sharing $0,\ 1,\ \infty$ CM. Suppose additionally that there exists a complex number $a(\neq 0, 1, \infty)$ such that $E(a, 1; f) = E(a, 1; g)$. If f is not a bilinear transformation of g then there exists a bilinear transformation L permuting $\{0, 1, \infty\}$ such that Lof and Log have the form*

$$\frac{e^{3\gamma} - 1}{e^{\gamma} - 1} \quad \textit{and} \quad \frac{e^{-3\gamma} - 1}{e^{-\gamma} - 1}$$

with $L(a) = \frac{3}{4}$ or Lof and Log have the form

$$\frac{e^{\gamma} - 1}{-e^{2\gamma} - 1} \quad \textit{and} \quad \frac{e^{-\gamma} - 1}{-e^{-2\gamma} - 1}$$

with $L(a)$ as a solution of $\frac{1}{4a^2} = 1 - \frac{1}{a}$, where γ is a non-constant entire function.

In this direction P. Li [5] proved the following result.

Theorem B. *{Theorem 1 [5]} Let f and g be non-constant meromorphic functions sharing $0, 1, \infty$ CM. Suppose additionally that f is not a bilinear transformation of g and that there exists a complex number $a(\neq 0, 1, \infty)$ such that*

$$T(r, f) \leq c\overline{N}(r, a; f \mid\geq 2) + S(r, f),$$

where $c > 0$ is a constant, then there exists a non-constant entire function γ, a non-zero constant λ, and two integers $s, t (t > 0)$ which are mutually prime such that

$$f = \frac{e^{t\gamma} - 1}{\lambda e^{-s\gamma} - 1} \quad \textit{and} \quad g = \frac{e^{-t\gamma} - 1}{\frac{1}{\lambda} e^{s\gamma} - 1},$$

$$\textit{where} \quad \frac{(1-a)^{s+t}}{a^t} = \frac{\lambda^t (1-\theta)^{s+t}}{\theta^t} \quad \textit{with} \quad \theta = -\frac{t}{s} \neq 1, a.$$

In 2001 the first author [3] introduced the notion of a gradation of value sharing by non-constant meromorphic functions and called it weighted sharing, which measures how close a shared value is to being shared CM or to being shared IM.

Definition 1.3. [3] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_o is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_o is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In this paper we use this notion and relax the mode of sharing of values by the functions in Theorem B. The main result of the paper is stated as follows.

Theorem 1.1. *Let f and g be non-constant meromorphic functions sharing $(0, 1)$, $(1, \infty)$, (∞, ∞). If there exists a complex number $a(\neq 0, 1, \infty)$ such that*

$$T(r, f) \leq c\overline{N}(r, a; f \mid\geq 2) + S(r, f),$$

where $c(> 0)$ is a constant, then f and g share $(0, \infty)$, $(1, \infty)$, (∞, ∞).

Combining Theorem 1.1 and Theorem B we obtain the following corollary.

Corollary 1.1. *Let f and g be non-constant meromorphic functions sharing $(0, 1)$, $(1, \infty)$, (∞, ∞). If there exists a complex number $a(\neq 0, 1, \infty)$ such that*

$$T(r, f) \leq c\overline{N}(r, a; f \mid\geq 2) + S(r, f),$$

where $c(> 0)$ is a constant, then the conclusion of Theorem B holds.

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *Let f and g be non-constant meromorphic functions sharing $(0, 0)$, $(1, 0)$, $(\infty, 0)$. If f is a bilinear transformation of g, then f and g share $(0, \infty)$, $(1, \infty)$, (∞, ∞).*

Proof. If f and g share $(0, 0)$, $(1, 0)$, $(\infty, 0)$ and f is a bilinear transformation of g then

$$f = \frac{ag + b}{cg + d}, \quad \text{where} \quad ad - bc \neq 0$$

and the following cases come up for consideration :

CASE 1 Let f and g have zeros, 1-points and poles. Then at a common zero of f and g we have $b = 0$. Therefore $f = \frac{ag}{cg+d}$. At a common 1-point of f and g, we have $a = c + d$ so that we can write

$$\frac{1}{f} = \frac{c + \frac{d}{g}}{c + d}.$$

Hence for any common pole of f and g we have $c = 0$. Therefore $a = d$ and consequently $f \equiv g$, from which we can easily conclude that f and g share $(0, \infty)$, $(1, \infty)$, (∞, ∞).

CASE 2 Let f and g have no zero while they have at least one common pole and at least one common 1-point. Then a set of similar calculations as Case 1 at the common 1-points and poles of f and g show that $a+b=c+d$ and $c=0$. Therefore,

$$f=\frac{ag+d}{a+b}$$

and so we conclude that f and g share (∞,∞). Also

$$f-1=\frac{a}{a+b}(g-1),$$

which shows that f and g share $(1,\infty)$. Hence f and g share $(0,\infty)$, $(1,\infty)$, (∞,∞).

CASE 3 Let f and g have no 1-point while they have at least one common pole and at least one common zero. Then by a set of similar calculations as Case 1 at the common zeros and poles of f and g we obtain $b=0$ and $c=0$. Therefore $df=ag$ and so f and g share $(0,\infty)$, $(1,\infty)$, (∞,∞).

CASE 4 Let f and g have no pole while they have at least one common zero and at least one common 1-point. Arguments as Case 1 at the common zeros and 1-points of f and g show that $b=0$ and $a=c+d(\neq 0)$. Therefore

$$f=\frac{(c+d)g}{cg+d} \quad \text{and} \quad f-1=\frac{d(g-1)}{cg+d},$$

which shows that f and g share $(0,\infty)$, $(1,\infty)$. Since f and g have no pole, it is clear that f and g share (∞,∞).

CASE 5 Let f and g have no zero and 1-point. Then f and g have at least one common pole and so we obtain $c=0$. Therefore $df=ag+c$ and so f and g share $(0,\infty)$, $(1,\infty)$, (∞,∞).

CASE 6 Let f and g have no zero and pole. Then f and g have at least one common 1-point and so we have $a+b=c+d$. Then

$$f-1=\frac{(a-c)(g-1)}{cg+d}$$

and so f, g share $(0,\infty)$, $(1,\infty)$, (∞,∞).

CASE 7 Let f and g have no 1-point and pole. Then f and g have at least one common zero so that $b=0$ and

$$f=\frac{ag}{cg+d}$$

so that f and g share $(0,\infty)$. Hence f and g share $(0,\infty)$, $(1,\infty)$, (∞,∞). This proves the lemma. ∎

Lemma 2.2. *{Lemma 4* [4]*} If f and g share $(0,1)$, $(1,\infty)$, (∞,∞) and $f\not\equiv g$ then*

$$\frac{f-1}{g-1}=e^{\alpha} \quad \textit{and} \quad \frac{g}{f}=h,$$

where α is an entire function and h is a meromorphic function with $\overline{N}(r,0;h)=S(r,f)$ and $\overline{N}(r,\infty;h)=S(r,f)$.

Lemma 2.3. *{Theorem 3* [1]*} If f and g share $(0,0)$, $(1,0)$, $(\infty,0)$ then $T(r,f)\leq 3T(r,g)+S(r,f)$ and $T(r,g)\leq 3T(r,f)+S(r,g)$.*

Clearly then $S(r,f)=S(r,g)$. Henceforth we shall denote either of them by $S(r)$.

Lemma 2.4. *{Lemma 7* [6]*} Let* f_1 *and* f_2 *be two non-constant meromorphic functions satisfying* $\overline{N}(r,0;f_i)+\overline{N}(r,\infty;f_i)=S(r;f_1,f_2)$ *for* $i=1,2$*. If* $f_1^s f_2^t-1$ *is not identically zero for arbitrary integers* s *and* t $(|s|+|t|>0)$*, then for any positive* ε*, we have*

$$N_0(r,1;f_1,f_2)\leq \varepsilon T(r)+S(r;f_1,f_2),$$

where $N_0(r,1;f_1,f_2)$ *denotes the reduced counting function related to the common* 1 *-points of* f_1 *and* f_2 *and* $T(r)=T(r,f_1)+T(r,f_2)$*,* $S(r;f_1,f_2)=o(T(r))$ *as* $r\to\infty$ *possibly outside a set of finite linear measure.*

3. PROOF OF THEOREM 1.1

Proof. If f is a bilinear transformation of g then the result is proved by Lemma 2.1. Therefore let us suppose that f is not a bilinear transformation of g. Then by Lemma 2.2 we get

$$\frac{f-1}{g-1}=e^{\alpha} \quad \text{and} \quad \frac{g}{f}=h, \tag{3.1}$$

where α is an entire function and h is a meromorphic function with $\overline{N}(r,0;h)=S(r,f)$ and $\overline{N}(r,\infty;h)=S(r,f)$.

Then from (3.1) we get

$$f=\frac{e^{\alpha}-1}{he^{\alpha}-1} \quad \text{and} \quad g=\frac{h(e^{\alpha}-1)}{he^{\alpha}-1}. \tag{3.2}$$

From (3.1) using Lemma 2.3 we get

$$\begin{aligned} T(r,e^{\alpha}) &\leq T(r,f)+T(r,g)+O(1)\\ &\leq T(r,f)+3T(r,f)+S(r)\\ &= 4T(r,f)+S(r) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} T(r,h) &\leq T(r,f)+T(r,g)+O(1)\\ &\leq 4T(r,f)+S(r). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4) we obtain $S(r,e^{\alpha})\leq S(r)$ and $S(r,h)\leq S(r)$.

Let z_0 be a multiple a-point of f but not a zero of α' or h. Since

$$f-a=\frac{e^{\alpha}-ahe^{\alpha}+(a-1)}{he^{\alpha}-1}, \tag{3.5}$$

we have

$$e^{\alpha(z_0)}-ah(z_0)e^{\alpha(z_0)}+(a-1)=0 \tag{3.6}$$

and

$$\alpha'(z_0)e^{\alpha(z_0)}-ah'(z_0)e^{\alpha(z_0)}-ah(z_0)\alpha'(z_0)e^{\alpha(z_0)}=0. \tag{3.7}$$

Putting $h'(z)=\gamma(z)h(z)$ we get from (3.7)

$$ah(z_0)=\frac{\alpha'(z_0)}{\alpha'(z_0)+\gamma(z_0)}.$$

Therefore from(3.6) we obtain

$$e^{\alpha(z_0)}=\frac{(1-a)\{\alpha'(z_0)+\gamma(z_0)\}}{\gamma(z_0)} \quad \text{and} \quad h(z_0)e^{\alpha(z_0)}=\frac{(1-a)\alpha'(z_0)}{a\gamma(z_0)}.$$

Let

$$F_1 = \frac{e^\alpha \gamma}{(1-a)(\alpha' + \gamma)} \quad \text{and} \quad F_2 = \frac{a h e^\alpha \gamma}{(1-a)\alpha'}.$$

Therefore

$$\begin{aligned}
T(r, F_1) &\leq T(r, e^\alpha) + 2T(r, \gamma) + T(r, \alpha') + S(r, e^\alpha) \\
&= T(r, e^\alpha) + 2T(r, \frac{h'}{h}) + S(r, e^\alpha) \\
&\leq T(r, e^\alpha) + 2N(r, \alpha; \frac{h'}{h}) + S(r) \\
&\leq T(r, e^\alpha) + 2\overline{N}(r, 0; h) + 2\overline{N}(r, \infty; h) + S(r) \\
&\leq T(r, e^\alpha) + 4T(r, h) + S(r) \\
&\leq 20T(r, f) + S(r)
\end{aligned}$$

and

$$\begin{aligned}
T(r, F_2) &\leq T(r, h) + T(r, e^\alpha) + T(r, \gamma) + T(r, \alpha') + S(r, e^\alpha) \\
&\leq T(r, h) + T(r, e^\alpha) + T(r, \frac{h'}{h}) + S(r) \\
&\leq T(r, h) + T(r, e^\alpha) + \overline{N}(r, 0; h) + \overline{N}(r, \infty; h) + S(r) \\
&\leq T(r, e^\alpha) + 3T(r, h) + S(r) \\
&\leq 16T(r, f) + S(r).
\end{aligned}$$

From above we obtain $S(r; F_1, F_2) \leq S(r)$. Since $F_1(z_0) = 1$ and $F_2(z_0) = 1$, we have $\overline{N}(r, a; f \mid\geq 2) \leq N_0(r, 1; F_1, F_2) + S(r)$. Therefore

$$\begin{aligned}
T(r, F_1) + T(r, F_2) &\leq 36T(r, f) + S(r) \\
&\leq 36c\overline{N}(r, a; f \mid\geq 2) + S(r) \\
&\leq 36cN_0(r, 1; F_1, F_2) + S(r).
\end{aligned}$$

Since $\overline{N}(r, 0; F_i) + \overline{N}(r, \infty; F_i) = S(r; F_1, F_2)$ for $i = 1, 2$, by Lemma 2.4 there exist two mutually prime integers s and t ($|s| + |t| > 0$) such that $F_1^s F_2^t \equiv 1$. This gives

$$e^{(s+t)\alpha} = \frac{(1-a)^{s+t}}{a^t} \times \frac{(1 + \frac{\gamma}{\alpha'})^s}{h^t \left(\frac{\gamma}{\alpha'}\right)^{s+t}}.$$

Now logarithmic differentiation gives

$$(s+t)\alpha' + t\gamma = \left(\frac{\gamma}{\alpha'}\right)' \left[\frac{s}{1 + \frac{\gamma}{\alpha'}} - \frac{s+t}{\frac{\gamma}{\alpha'}}\right].$$

If $(s+t)\alpha' + t\gamma \not\equiv 0$, then from above we get

$$\frac{h'}{h} = -\frac{\left(\frac{\gamma}{\alpha'}\right)'}{1 + \frac{\gamma}{\alpha'}}$$

which gives on integration

$$h = \frac{1}{c_1 \left(1 + \frac{\gamma}{\alpha'}\right)}, \tag{3.8}$$

where c_1 is a non-zero constant. This shows that

$$T(r, h) \leq S(r). \tag{3.9}$$

Again from (3.8) we get

$$\alpha' = \frac{c_1 h'}{1 - c_1 h}$$

which gives on integration

$$e^{\alpha} = \frac{1}{c_2(1 - c_1 h)}, \tag{3.10}$$

for some non-zero constant c_2. Thus in view of (3.9) we obtain

$$T(r, e^{\alpha}) \leq S(r). \tag{3.11}$$

So from (3.2), (3.9) and (3.11) we see that $T(r, f) \leq S(r)$, which is a contradiction. So

$$(s + t)\alpha' + t\gamma \equiv 0. \tag{3.12}$$

If $t = 0$, we see from (3.12) that α is a constant. If f and g have any zero then $\frac{f-1}{g-1} = e^{\alpha}$ implies that $e^{\alpha} = 1$ and so $f \equiv g$. Hence f and g share $(0, \infty)$, $(1, \infty)$, (∞, ∞). Now let $t \neq 0$. Since α is an entire function, from (3.12) we see that $\frac{h'}{h}$ is also an entire function. Hence $h = \frac{g}{f}$ has no zero and no pole. Therefore f and g share $(0, \infty)$, $(1, \infty)$, (∞, ∞). This proves the theorem. ∎

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