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NONTRIVIAL SOLUTIONS OF SINGULAR SUPERLINEAR THREE-POINT BOUNDARY VALUE PROBLEMS AT RESONANCE

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ABSTRACT. The singular superlinear second order three-point boundary value problems at resonance

$$\begin{cases} x''(t) = f(t, x(t)), & 0 < t < 1, \\ x'(0) = 0, \ \ x(\eta) = x(1), \end{cases}$$

are considered under some conditions concerning the first eigenvalues corresponding to the relevant linear operators, where $\eta \in (0, 1)$ is a constant, f is allowed to be singular at both $t = 0$ and $t = 1$. The existence results of nontrivial solutions are given by means of the topological degree theory.

Key words and phrases: Singular, Nontrivial solutions, Boundary value problems, Topology degree, Resonance.

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1. INTRODUCTION

This paper is concerned with the existence of at least one nontrivial solution for the nonlinear singular superlinear second order three-point boundary value problems at resonance

$$\begin{cases} x''(t) = f(t, x(t)), & 0 < t < 1, \\ x'(0) = 0, \ \ x(\eta) = x(1), \end{cases} \tag{1}$$

where $\eta \in (0, 1)$ is a constant, f is allowed to be singular at both $t = 0$ and $t = 1$. The problem (1) happens to be at resonance in the sense that the associated linear homogeneous boundary value problems

$$\begin{cases} x''(t) = 0, & 0 < t < 1, \\ x'(0) = 0, \ \ x(\eta) = x(1), \end{cases}$$

has $x(t) = c, c \in \mathbf{R}$ as a nontrivial solution.

Multi-point boundary value problems of second order differential equations have been studied intensively and the resonance cases have received more attention [1]-[4]. Recently, Ma [2] developed the method of upper and lower solutions for nonlinear three-point boundary value problem at resonance and established some multiplicity results. Bai [3] generalized the existence results to the four-point boundary value problems at resonance by using coincidence degree theory due to Mawhin [5]. Han [4] studied the existence of positive solutions of the BVP (1) by rewriting the BVP as an equivalent one, so that Guo-Kranoselskii fixed point theorem can be applied. Motivated by [4], we establish the existence results of nontrivial solutions for the singular boundary value problem (1) at resonance by means of the topological degree theory under some conditions concerning the first eigenvalue corresponding to the relevant linear operator. The eigenvalue criteria of this sort for nonlinear two-point boundary value problems is established in [6]. For the concepts and properties about the cone theory and the topological degree we refer to [7]–[9].

2. PRELIMINARIES

In this section, we shall give some preliminaries. In the Banach space $C[0, 1]$ in which the norm is defined by $\|x\| = \max\limits_{0 \le t \le 1} |x(t)|$. We set

$$P = \{x \in C[0, 1] \mid x(t) \ge 0, \quad t \in [0, 1]\}. \tag{2}$$

P is a positive cone in $C[0, 1]$. Throughout this section, the partial ordering is always given by P. We denote by $B_r = \{x \in C[0, 1] \mid \|x\| < r\} (r > 0)$ the open ball of radius r.

Define $g(t, x) = f(t, x) + \beta^2 x$. For convenience, we make the following assumptions:

(H_1) $\beta \in (0, \frac{\pi}{2})$ is a constant.

(H_2) $f : (0, 1) \times (-\infty, +\infty) \to (-\infty, +\infty)$ is continuous and

$$h_1(t)g_1(x) \le g(t, x) \le h_2(t)g_2(x), \ \ (t, x) \in (0, 1) \times (-\infty, +\infty),$$

where $h_1(t), h_2(t) \in C(0, 1), \ g_1(x), g_2(x) \in C(-\infty, +\infty)$, and $h_1(t) \not\equiv 0, \ h_2(t) \not\equiv 0, t \in (0, 1)$.

(H_3) $\displaystyle\int_0^1 h_2(t)dt < +\infty.$

It is known (see [4]) that BVP (1) is equivalent to the problem

$$\begin{cases} x''(t) + \beta^2 x(t) = g(t, x(t)), & 0 < t < 1, \\ x'(0) = 0, \ \ x(\eta) = x(1). \end{cases}$$

As is well known, the singular nonlinear boundary value problems (1) can be converted into the equivalent Hammerstein nonlinear integral equation

$$x(t) = \int_0^1 G(t,s)g(s,x(s))ds, \;\; t \in [0,1], \tag{3}$$

where

$$\begin{aligned} G(t,s) \;\; &= \;\; \begin{cases} \frac{1}{\beta}\sin\beta(t-s), & 0 \le s \le t \le 1, \\ 0, & 0 \le t \le s \le 1, \end{cases} \\ &+ \;\; \frac{\cos\beta t}{\beta\sin\frac{\beta(\eta+1)}{2}} \begin{cases} \cos\frac{\beta(2s-\eta-1)}{2}, & 0 \le s \le \eta < 1, \\ \frac{\sin\beta(1-s)}{2\sin\frac{\beta(1-\eta)}{2}}, & 0 < \eta \le s \le 1. \end{cases} \end{aligned}$$

By Lemma 2.3 in [4], we know that

$$c\phi(s) \le G(t,s) \le \phi(s), \;\; \forall\, t,s \in [0,1],$$

where $\phi : [0,1] \to [0,\infty)$ is a continuous function and $c \in (0,1]$.

Define

$$P_1 = \{x \in C[0,1] | x(t) \ge c\|x\|, \; t \in [0,1]\}$$

then P_1 is a cone in C[0,1] and $P_1 \subset P$.

Let

$$(Ax)(t) = \int_0^1 G(t,s)g(s,x(s))ds, \;\; t \in [0,1]. \tag{4}$$

$$(T_1x)(t) = \int_0^1 G(t,s)h_1(s)x(s)ds, \;\; t \in [0,1]. \tag{5}$$

$$(T_2x)(t) = \int_0^1 G(t,s)h_2(s)x(s)ds, \;\; t \in [0,1]. \tag{6}$$

It is not difficult to verify that the nonzero fixed points of the operator A are the solutions of singular boundary value problems (1).

By the method similar to that in [6], we have

Lemma 2.1 ([6])**.** *Suppose that $(H_1) - (H_3)$ are satisfied, then $A : C[0,1] \to C[0,1]$ is a completely continuous operator, $T_1, T_2 : C[0,1] \to C[0,1]$ are completely continuous linear operators and $T_1, T_2 : P \to P_1$.*

Lemma 2.2 ([6])**.** *Suppose that the conditions $(H_1), (H_2)$ are satisfied, then for the operators T_1, T_2 defined by $(5), (6)$, the spectral radius $r(T_1), r(T_2) \neq 0$ and T_1, T_2 have the positive eigenfunctions corresponding to their first eigenvalue $\lambda_1 = (r(T_1))^{-1}, \widetilde{\lambda_1} = (r(T_2))^{-1}$ respectively.*

We also need the following lemmas in [7].

Lemma 2.3 ([7])**.** *Let E be a Banach space and Ω be a bounded open set of E, and $A : \overline{\Omega} \to E$ a completely continuous operator. Assume that there exists $u_0 \neq \theta$ such that*

$$x - Ax \neq \mu u_0,$$

for all $x \in \partial\Omega$ and $\mu \ge 0$, then the topological degree

$$\deg(I - A, \Omega, \theta) = 0.$$

Lemma 2.4 ([7]). *Let E be a Banach space and Ω be a bounded open set of E with $\theta \in \Omega$, and $A : \overline{\Omega} \to E$ a completely continuous operator. If*

$$Ax \neq \mu x,$$

for all $x \in \partial\Omega$ and $\mu \geq 1$, then the topological degree

$$\deg(I - A, \Omega, \theta) = 1.$$

3. MAIN RESULTS

Theorem 3.1. *Suppose that the conditions $(H_1) - (H_3)$ are satisfied. If there exists a constant $b \geq 0$ such that*

$$g_1(x) \geq -b, \ \ \forall x \in \mathbf{R}, \tag{7}$$

$$\liminf_{x \to +\infty} \frac{g_1(x)}{x} > \lambda_1, \tag{8}$$

$$\limsup_{x \to 0} \left| \frac{g_2(x)}{x} \right| < \widetilde{\lambda_1}, \tag{9}$$

where $\lambda_1, \widetilde{\lambda_1}$ are the first eigenvalue of T_1 and T_2 respectively. Then the singular boundary value problems (1) have at least one nontrivial solution.

Proof. It follows from (8) that there exists $\varepsilon > 0$ such that $g_1(x) \geq (\lambda_1 + \varepsilon)x$ when x is sufficiently large. We know from (7) that there exists $b_1 \geq 0$ such that

$$g_1(x) \geq (\lambda_1 + \varepsilon)x - b_1, \ \ \forall\, x \in (-\infty, +\infty). \tag{10}$$

Take

$$R_1 > \max\left\{1, \frac{bc - b_1 - b\lambda_1 \int_0^1 \phi(s)h_1(s)ds}{c^2\varepsilon}\right\},$$

Let x^* be the positive eigenfunction of T corresponding to λ_1, thus $x^* = \lambda_1 T x^*$, and $x^* \in P_1$ by Lemma 1.

We may suppose that A has no fixed point on ∂B_{R_1} (otherwise, the proof completes). Now we show that

$$x - Ax \neq \tau x^*, \ \ \forall\, x \in \partial B_{R_1} \cap P, \ \tau \geq 0. \tag{11}$$

If otherwise, there exist $x_1 \in \partial B_{R_1}$ and $\tau_0 \geq 0$ such that $x_1 - Ax_1 = \tau_0 x^*$. Hence $\tau_0 > 0$ and

$$x_1 = Ax_1 + \tau_0 x^*.$$

Let

$$\widetilde{x}(t) = b\int_0^1 G(t,s)h_1(s)ds,$$

then we have

$$\begin{aligned} x_1(t) + \widetilde{x}(t) &= Ax_1(t) + \widetilde{x}(t) + \tau_0 x^*(t) \\ &= \int_0^1 G(t,s)g(s, x_1(s))ds + b\int_0^1 G(t,s)h_1(s)ds + \tau_0 x^*(t), \end{aligned} \tag{12}$$

which implies

$$x_1 + \widetilde{x} \in P_1 \tag{13}$$

since (7), $T(P) \subset P_1$ and noticing

$$\int_0^1 G(t,s)g(s, x_1(s))ds + b\int_0^1 G(t,s)h_1(s)ds \geq \int_0^1 G(t,s)h_1(s)(g_1(x_1) + b)ds.$$

By condition (H_2), (10) and (13), we get

$$\begin{aligned}
& Ax_1 + \widetilde{x}(t)dtds \\
= & \int_0^1 G(t,s)g(s,x_1(s))ds + b\int_0^1 G(t,s)h_1(s)ds \\
\geq & \int_0^1 G(t,s)h_1(s)g_1(x_1)ds + b\int_0^1 G(t,s)h_1(s)ds \\
\geq & (\lambda_1+\varepsilon)\int_0^1 G(t,s)h_1(s)x_1(s)ds - b_1\int_0^1 G(t,s)h_1(s)ds + b\int_0^1 G(t,s)h_1(s)ds \\
= & \lambda_1\int_0^1 G(t,s)h_1(s)\Big(x_1(s)+b\int_0^1 G(s,t)h_1(t)dt\Big)ds + \varepsilon\int_0^1 G(t,s)h_1(s)x_1(s)ds \\
& -b_1\int_0^1 G(t,s)h_1(s)ds + b\int_0^1 G(t,s)h_1(s)ds \\
& -b\lambda_1\int_0^1 G(t,s)h_1(s)\int_0^1 G(s,t)h_1(t)dtds \\
\geq & \lambda_1 T_1(x_1+\widetilde{x}) + c^2\varepsilon\int_0^1 \phi(s)h_1(s)ds\|x_1\| - b_1\int_0^1 \phi(s)h_1(s)ds \\
& +bc\int_0^1 \phi(s)h_1(s)ds - b\lambda_1(\int_0^1 \phi(s)h_1(s)ds)^2 \\
\geq & \lambda_1 T_1(x_1+\widetilde{x}).
\end{aligned}$$

Therefore by (12) and (13), we have

$$x_1+\widetilde{x} \geq \lambda_1 T_1(x_1+\widetilde{x}) + \tau_0 x^* \geq \tau_0 x^*. \tag{14}$$

Put

$$\tau^* = \sup\{\tau | x_1+\widetilde{x} \geq \tau x^*\}.$$

It is easy to see that $\tau^* \geq \tau_0 > 0$ and $x_1+\widetilde{x} \geq \tau^* x^*$. We have from $T_1(P) \subset P$ that

$$\lambda_1 T_1(x_1+\widetilde{x}) \geq \tau^*\lambda_1 T_1 x^* = \tau^* x^*.$$

Therefore by (14),

$$x_1+\widetilde{x} \geq \lambda_1 T_1(x_1+\widetilde{x}) + \tau_0 x^* \geq \tau^* x^* + \tau_0 x^*,$$

which contradicts the definition of τ^*. Hence (11) is true and we have from Lemma 2.3 that

$$\deg(I-A, B_{R_1}, \theta) = 0. \tag{15}$$

It follows from (9) that there exists $0 < r < 1$ such that

$$|g_2(x)| \leq \widetilde{\lambda_1}|x|, \ \ \forall\, |x| \leq r. \tag{16}$$

Define $\widetilde{T_2}x = \widetilde{\lambda_1}T_2x,\ x \in C[0,1]$. Hence $\widetilde{T_2} : C[0,1] \to C[0,1]$ is a linear completely continuous operator and

$$\widetilde{T_2}(P) \subset P,\ r(\widetilde{T_2}) = 1.$$

Now we show that

$$Ax \neq \tau x,\ \ \forall\, x \in B_r,\ \tau \geq 1. \tag{17}$$

If otherwise, there exist $x_2 \in \partial B_r$ and $\tau_2 \geq 1$ such that $Ax_2 = \tau_2 x_2$. We may suppose that $\tau_2 > 1$(otherwise we are done). Let $\overline{x} = |x_2|$, then $\overline{x} \in P \cap \partial B_r$. From (16), it follows that

$$\tau_2\overline{x} = |Ax_2| \leq \widetilde{T_2}\overline{x}.$$

By induction, we have $\tau_2^n \overline{x} \leq \widetilde{T}_2^{\,n} \overline{x} (n = 1, 2, \ldots)$. Hence

$$\|\widetilde{T}_2\| \geq \frac{\|\widetilde{T}_2^{\,n} \overline{x}\|}{\|\overline{x}\|} \geq \tau_2^n, \ \ (n = 1, 2, \ldots).$$

In view of Gelfand's formula, we have

$$r(\widetilde{T}_2) = \lim_{n\to\infty} \sqrt[n]{\|\widetilde{T}_2^{\,n}\|} \geq \lim_{n\to\infty} \sqrt[n]{\tau_2^n} = \tau_2 > 1,$$

which is a contradiction. Hence (17) is true and by Lemma 2.4 we have

$$\deg(I - A, B_r, \theta) = 1. \tag{18}$$

By (15) and (18) we have that

$$\deg(I - A, B_{R_1} \backslash \overline{B}_r, \theta) = \deg(I - A, B_{R_1}, \theta) - \deg(I - A, B_r, \theta) = -1,$$

which implies that A has at least one fixed point on $B_{R_1} \backslash \overline{B}_r$. This means that the singular nonlinear boundary value problems (1) have at least one nontrivial solution. ∎

Corollary 3.2. *Suppose that the conditions $(H_1) - (H_3)$ are satisfied. If there exists a constant $b^* \geq 0$ such that*

$$f(t, x) \geq -\beta^2 x - \frac{b^*}{\widetilde{M}}, \ \ \forall\, t \in [0, 1], \ x \geq -b^*,$$

where $\widetilde{M} = \max_{t\in[0,1]} \int_0^1 G(t, s)ds$ and in addition, (8) and (9) hold, then the singular boundary value problems (1) have at least one nontrivial solution.

Proof. Denote

$$g_1(t, x) = \begin{cases} f(t, x) + \beta^2 x, & t \in [0, 1], \ x \geq -b^*, \\ f(t, -b^*) + \beta^2 x, & t \in [0, 1], \ x < -b^*. \end{cases} \tag{19}$$

Define

$$(A_1 x)(t) = \int_0^1 G(t, s) g_1(s, x(s))ds, \ \ t \in [0, 1].$$

By Theorem 1 we know that A_1 has at least one nontrivial fixed point $\widetilde{x}$. Then

$$\widetilde{x}(t) = \int_0^1 G(t, s) g_1(s, \widetilde{x}(s))ds \geq -\frac{b^*}{\widetilde{M}} \int_0^1 G(t, s)ds \geq -b^*.$$

From (19) we have that $g_1(t, \widetilde{x}(t)) = g(t, \widetilde{x}(t)), \ t \in [0, 1]$, then

$$\widetilde{x}(t) = \int_0^1 G(t, s) g_1(s, \widetilde{x}(s))ds = \int_0^1 G(t, s) g(s, \widetilde{x}(s))ds.$$

Thus $\widetilde{x}$ is the nontrivial solution of singular boundary value problems (1). ∎

Remark 3.1. In Theorems 3.1 and Corollary 3.2, we do not assume that $g(t, x) \geq 0$ for $x \geq 0$. And it is difficult to obtain those theorems using the theory of fixed point index on a cone. In order to obtain the existence of nontrivial solution, we make use of topological degree theory which is not confined in a cone.

Example Let $g(t, x) = \frac{1}{\sqrt{t(1-t)}}(x^3 + x^2)$. Let $h(t) = h_1(t) = h_2(t) = \frac{1}{\sqrt{t(1-t)}}$, $g(x) = g_1(x) = g_2(x) = x^3 + x^2$. It is clear that $h(t)$ is singular at both $t = 0$ and $t = 1$, and satisfies (H_3) by the convergence of Euler's integral.

It is easy to see that $g(t, x)$ is bounded below. In addition, $\lim_{x\to 0} |\frac{g(x)}{x}| = 0$ and $\lim_{x\to+\infty} \frac{g(x)}{x} = +\infty$. Thus by Theorem 3.1 one can obtain the existence of nontrivial solution of (1).

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