The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 7, Issue 1, Article 13, pp. 1-27, 2010

GENERALIZED EFFICIENT SOLUTIONS TO ONE CLASS OF VECTOR OPTIMIZATION PROBLEMS IN BANACH SPACES

PETER I. KOGUT, ROSANNA MANZO, AND IGOR V. NECHAY

Received 2 November, 2008; accepted 6 March, 2009; published 16 November, 2010.
Communicated by: C. D'Apice

DEPARTMENT OF DIFFERENTIAL EQUATIONS, DNIPROPETROVSK NATIONAL UNIVERSITY, NAUKOVA STR., 13, 49050 DNIPROPETROVSK, UKRAINE
p.kogut@i.ua

UNIVERSITÀ DI SALERNO, DIPARTIMENTO DI INGEGNERIA DELL'INFORMAZIONE E MATEMATICA APPLICATA, VIA PONTE DON MELILLO, 84084 FISCIANO (SA), ITALY
manzo@diima.unisa.it

DEPARTMENT OF TECHNICAL CYBERNETICS, DNIPROPETROVSK TECHNICAL UNIVERSITY, ACAD. LAZARJAN STR., 2, 49010 DNIPROPETROVSK, UKRAINE
i.nechay@i.ua

ABSTRACT. In this paper, we study vector optimization problems in Banach spaces for essentially nonlinear operator equations with additional control and state constraints. We assume that an objective mapping possesses a weakened property of lower semicontinuity and make no assumptions on the interior of the ordering cone. Using the penalization approach we derive both sufficient and necessary conditions for the existence of efficient solutions of the above problems. We also prove the existence of the so-called generalized efficient solutions via the scalarization of some penalized vector optimization problems.

Key words and phrases: vector optimization, lower semicontinuity, control object, efficient solution.

2000 *Mathematics Subject Classification.* Primary 46B40, 49J45. Secondary 49N90, 90C29.

ISSN (electronic): 1449-5910

1. INTRODUCTION

The main goal of this paper is to study a class of vector optimization problems in Banach spaces for essentially nonlinear operator equations with additional control and state constraints. This topic, in the scalarized form, has been widely studied by many authors. We mainly mention Barbu [5], Fattorini [10], Fursikov [12], Ivanenko & Mel'nik [14], Ioffe & Tichomirov [13], Lions [23], Mel'nik & Zgurovsky [33] and the references cited therein. However, vector optimization problems are usually much harder than single-objective optimization problems. A systematic study of such problems in Banach spaces with different applications in engineering and game theory has been presented in an overwhelming amount of literature (see, for instance, Aubin & Ekeland [2], Chen & Huang & Yang Jahn [7], [16], Luc [24]).

An important aspect in vector optimization is to find conditions which guarantee the existence of so-called efficient solutions. The following result is well-known (see, for instance, [32]): *if the image of admissible solutions in an objective Banach space is compact then the set of efficient solutions is nonempty.* Since compactness is a very restrictive assumption, at least in an infinite-dimensional setting, many authors have tried to weaken it. The typical way is to endow the objective mapping with some lower semicontinuity properties. In the vector-valued case there are several possible ways to extend the "scalar" notion of lower semicontinuity (see, for example, [1, 3, 4, 6, 8, 17, 24, 28]). We could mention the lower semicontinuity, quasi lower semicontinuity, and order lower semicontinuity. However, as the following example indicates, these properties for the objective functions may fail at an efficient solution, even for simple vector optimization problems with a nonempty set of solutions. Indeed, let $\Xi = \left\{x \in L^2(\Omega) \,:\, \|x\|_{L^2(\Omega)} \le 1\right\}$, and $\Lambda = \mathbb{R}^2_+$ be the ordering cone in the objective space $\mathbb{R}^2$. Let the vector-valued mapping $I : \Xi \to \mathbb{R}^2$ be defined by

$$I(x) = \begin{bmatrix} 1 + \|x\|_{L^2(\Omega)} \\ 2 + \|x\|_{L^2(\Omega)} \end{bmatrix} \text{ if } x \neq 0, \quad I(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Then, it is easy to see that $x = 0$ is an efficient solution to the vector optimization problem $\Lambda\!-\min I(x)$, $x \in \Xi$. However, as we will see later, the quasi lower semicontinuity property for the mapping I does not hold at $x = 0$.

Our prime interest is to study vector optimization problems in the case when a control object is described by nonlinear operator equations, an objective function is a vector-valued mapping with a weakened property of lower semicontinuity, and the control and state constraints take the form of some operator inequalities and inclusions in Banach spaces. We consider the case when the objective mappings take values in a real Banach space Z partially ordered by a closed convex pointed cone Λ. Usually, the typical assumption in many papers is that the interior of the ordering cone Λ is non-empty. However, in many interesting and important cases, this property does not hold. For instance, in the case when $Z = L^p(\Omega)$, where Ω is an open bounded subset of $\mathbb{R}^n$, $p \in [1, +\infty)$, and Λ is the natural cone of positive elements of Z, we have $\operatorname{Int}\Lambda = \emptyset$. So, we make no additional assumptions on the cone Λ and its interior. We also extend the concept of lower semicontinuity to vector-valued mappings and discuss sufficient conditions of solvability of the corresponding vector optimization problems.

Let us describe the contents of the paper. Section 2 contains the statement of the vector optimization problem. In Section 3, we detail the ingredients needed in this work. We also introduce the notion of the Λ-lower sequential limit of vector-valued mappings, give the definition of Λ_w-efficient solutions to the corresponding vector optimization problems, and illustrate them with some examples. Section 4 includes the main assumptions about the control object. We introduce the so-called Λ_w-lower semicontinuity property for vector-valued mappings in Banach spaces with respect to the weak topology of the objective space, which can be viewed as an extension of the ones mentioned above and study its properties. In Section 5, we prove the main existence theorem of the Λ_w-efficient solutions for the vector optimization problem without any scalarization process. Instead of this, we propose to involve a penalized vector optimization problem which is much easier to manage, and to study its Λ_w-efficient solutions in the limit as the penalization parameter tends to zero. In Section 6 we discuss the scalarization of vector optimization problems with Λ_w-lower semicontinuous mappings, using the "simplest" method of the "weighted sum". We show that in this case one of the fundamental requirements on the scalarizing vector optimization problems (according to Sawaragi et al. [29]): *solutions to the scalarized optimization problem must also be efficient solutions to the original vector optimization problem*, may not hold. In view of this, Section 7 aims to extend the notion of Λ_w-efficient solutions to the so-called generalized Λ_w-solutions of the vector optimization problem. We study their main properties and derive sufficient conditions when the generalized solutions can be obtained via the penalized optimal control problems. Finally, Section 8 deals with the application of the main results to study a control object governed by nonlinear partial differential equations with inequality state and control constraints.

2. PROBLEM SETTING AND NOTATION

Let Y be a reflexive Banach space, and Y^* be its dual. Let Z be a Banach space partially ordered by a closed convex pointed reproducing cone $\Lambda \subset Z$. No assumption is required on the interior of Λ. Let U be a control space which is assumed to be dual to some separable Banach space V (that is $U = V^*$). Let U_∂ be a subset of admissible controls in U, and let K be a fixed subset of Y. The vector optimization problem we consider can be described as follows:

$$\text{Minimize } I(u, y) \text{ (with respect to the cone } \Lambda) \tag{2.1}$$

subject to

$$A(u, y) = f, \tag{2.2}$$

$$F(u, y) \geq_\Lambda 0, \tag{2.3}$$

$$y \in K, \quad u \in U_\partial \subset U, \tag{2.4}$$

where $A : U \times Y \to Y^*$, $F : U \times Y \to Z$ are (nonlinear) mappings, $I : U \times Y \to Z$ is an objective function, and f is a given element of Y^*.

Definition 2.1. *We say that the problem* (2.1)–(2.4) *is regular if for every* $f \in Y^*$ *there exists* $(u, y) \in U \times Y$*, where* $y = y(u)$ *is a corresponding solution of* (2.2)*, such that* (u, y) *satisfies the restrictions* (2.3)–(2.4) *and* $I(u, y) <_\Lambda z$ *for some element* z *of* Z*. In this case the pair* (u, y) *is said to be admissible (note that, in general, the mapping* $u \to y(u)$ *may be multi-valued).*

We denote by Ξ the set of all admissible pairs to the problem (2.1)–(2.4). Throughout this paper we will associate the vector optimization problem (2.1)–(2.4) with the triplet $\langle \Xi, I, \Lambda \rangle$. We denote by $\mathrm{int}_\tau\, Z_0$ and $\mathrm{cl}_\tau\, Z_0$ the interior of the set $Z_0 \subset Z$ and its closure with respect to the τ-topology, respectively. By default τ is always associated with the strong topology of Z.

To specify the definition of the efficient solutions to the vector optimization problem $\langle \Xi, I, \Lambda \rangle$, we outline the main notions of the vector-valued mappings.

3. PRELIMINARIES

Let Z_0 be a subset of Z. We say that an element $z^* \in Z_0$ is Λ-*minimal* for the set $Z_0 \subset Z$, if there is no $z \in Z_0$ such that $z \leq_\Lambda z^*$, $z \neq z^*$, that is

$$Z_0 \cap (z^* - \Lambda) = \{z^*\}.$$

We denote by $\Lambda - \mathrm{Min}(Z_0)$ the family of all such elements . We say that an element z^* is the Λ-*ideal minimal point* of the set Z_0, if $z^* \leq_\Lambda z$ for every $z \in Z_0$. By analogy we can introduce the sets of Λ-maximal elements and Λ-ideal maximal elements of the set Z_0.

Let $-\infty_\Lambda$ and $+\infty_\Lambda$ be two singular elements such that $-\infty_\Lambda \leq_\Lambda z \leq_\Lambda +\infty_\Lambda$ for all $z \in Z$. We use the notation $\overline{Z} = Z \cup \{\pm\infty_\Lambda\}$. Then $+\infty_\Lambda$ is the Λ-greatest element of the set $\overline{Z}$, and the element $-\infty_\Lambda$ is its Λ-smallest element. By $Z^\bullet$ we denote a semiextended Banach space: $Z^\bullet = Z \cup \{+\infty_\Lambda\}$ assuming that $\| + \infty_\Lambda\|_Z = +\infty$ and $z + \lambda(+\infty_\Lambda) = +\infty_\Lambda$ for all $z \in Z$ and $\lambda \in \mathbb{R}_+$.

The following concept is the crucial point of this paper.

Definition 3.1. *We say that a set E is the efficient Λ-infimum of a set $Z_0 \subset Z$ with respect to the weak topology of Z (or Λ_w-infimum for short) if E is the collection of all Λ-minimal elements of $\mathrm{cl}_w\, Z_0$ in the case when this set is non-empty, and otherwise $E = \{-\infty_\Lambda\}$.*

In what follows we always associate the objective mapping $I : \Xi \to Z$ with its natural extension $\widehat{I} : U \times Y \to Z^\bullet$ to the whole space $U \times Y$, where

$$\widehat{I}(u, y) = \begin{cases} I(u, y), & (u, y) \in \Xi \\ +\infty, & (u, y) \notin \Xi. \end{cases}$$

Following [31] we say that a function $\widehat{I} : U \times Y \to Z^\bullet$ is bounded below if there exists $z \in Z$ such that $z \leq_\Lambda \widehat{I}(u, y)$ for all $(u, y) \in U \times Y$. Hereinafter we denote the efficient Λ_w-infimum for $Z_0 \subset Z^\bullet$ by $\Lambda_w - \mathrm{Inf}\, Z_0$. Thus, in view of the above definition, we have

$$\Lambda_w - \mathrm{Inf}\, Z_0 := \begin{cases} \Lambda - \mathrm{Min}(\mathrm{cl}_w\, Z_0), & \Lambda - \mathrm{Min}(\mathrm{cl}_w\, Z_0) \neq \emptyset \\ -\infty_\Lambda, & \Lambda - \mathrm{Min}(\mathrm{cl}_w\, Z_0) = \emptyset. \end{cases}$$

Definition 3.2. *A subset E of $Z \cup \{\pm\infty_\Lambda\}$ is said to be a weak efficient Λ-infimum of a mapping $I : \Xi \to Z$ and is denoted by $\Lambda_w - \operatorname*{Inf}_{(u,y)\in\Xi} I(u, y)$, if E is an efficient Λ_w-infimum of the image $\widehat{I}(\Xi)$ of $\Xi \subset U \times Y$ in $Z^\bullet$, that is,*

$$\Lambda_w - \operatorname*{Inf}_{(u,y)\in\Xi} I(u, y) = \Lambda_w - \mathrm{Inf} \left\{ \widehat{I}(u, y) \ : \ (u, y) \in U \times Y \right\}.$$

Remark 3.3. *It is clear now that if $a \in \Lambda_w - \operatorname*{Inf}_{(u,y)\in\Xi} I(u, y)$, then*

$$\mathrm{cl}_w\, \{I(u, y) \ : \ \forall\, (u, y) \in \Xi\} \cap (a - \Lambda) = \{a\},$$

provided $\Lambda_w - \mathrm{Min}\, [\mathrm{cl}_w\, \{I(u, y) \ : \ \forall\, (u, y) \in \Xi\}] \neq \emptyset$.

Let $\{z_k\}_{k=1}^\infty$ be a sequence in Z. Let us denote by $\mathrm{L}\{z_k\}$ the set of all its cluster points with respect to the weak topology of Z, that is, $z \in \mathrm{L}\{z_k\}$ if there is a subsequence $\{z_{k_i}\}_{i=1}^\infty \subset \{z_k\}_{k=1}^\infty$ such that $z_{k_i} \rightharpoonup z$ in Z as $i \to \infty$. If this set is lower unbounded, i.e., $\Lambda_w - \mathrm{Inf}\, \mathrm{L}\{z_k\} = \{-\infty_\Lambda\}$, we assume that $\{-\infty_\Lambda\} \in \mathrm{L}\{z_k\}$. Let $(u_0, y_0) \in U \times Y$ be a fixed pair.

Definition 3.4. *A sequence $\{(u_k, y_k)\}_{k=1}^\infty \subset U \times Y$ is said to be w-convergent to the pair $(u_0, y_0) \in U \times Y$ (or $(u_k, y_k) \xrightarrow{w} (u_0, y_0)$ for short) if $u_k \overset{*}{\rightharpoonup} u_0$ in U and $y_k \rightharpoonup y_0$ in Y as k tends to $+\infty$.*

In what follows, for an arbitrary mapping $I : \ U \times Y \to Z^\bullet$ we make use of the following set:

$$\mathrm{L}_w(I, u_0, y_0) := \bigcup_{\{(u_k,y_k)\}_{k=1}^\infty \in \mathfrak{M}(u_0,y_0)} \mathrm{L}\{I(u_k, y_k)\},$$

where we denote by $\mathfrak{M}(u_0, y_0)$ the set of all sequences $\{(u_k, y_k)\}_{k=1}^\infty \subset U \times Y$ such that $(u_k, y_k) \overset{w}{\rightharpoonup} (u_0, y_0)$.

Definition 3.5. *We say that a subset $E \subset Z \cup \{\pm\infty_\Lambda\}$ is the Λ-lower sequential limit of the mapping $I : U \times Y \to Z^\bullet$ at the pair $(u_0, y_0) \in U \times Y$ with respect to the product of the w-topology of $U \times Y$ and the weak topology of Z, and we use the notation $E = \Lambda_w - \liminf\limits_{(u,y) \xrightarrow{w} (u_0,y_0)} I(u,y)$, if*

$$\Lambda_w - \liminf_{(u,y) \xrightarrow{w} (u_0,y_0)} I(u,y) := \mathrm{L}_w(I, u_0, y_0) \cap \Lambda_w - \operatorname*{Inf}_{(u,y)\in U\times Y} I(u,y), \tag{3.1}$$

in the case when $\mathrm{L}_w(I, u_0, y_0) \cap \Lambda_w - \operatorname*{Inf}\limits_{(u,y)\in U\times Y} I(u,y) \neq \emptyset$, and

$$\Lambda_w - \liminf_{(u,y) \xrightarrow{w} (u_0,y_0)} I(u,y) := \Lambda_w - \operatorname{Inf} \mathrm{L}_w(I, u_0, y_0), \tag{3.2}$$

otherwise (i.e. in the case when $\mathrm{L}_w(I, u_0, y_0) \cap \Lambda_w - \operatorname*{Inf}\limits_{(u,y)\in U\times Y} I(u,y) = \emptyset$).

In fact, relation (3.1) can be rewritten in the equivalent form

$$\Lambda_w - \liminf_{(u,y) \xrightarrow{w} (u_0,y_0)} I(u,y) := \Lambda_w - \operatorname{Inf} \mathrm{L}_w(I, u_0, y_0) \cap \Lambda_w - \operatorname*{Inf}_{(u,y)\in \Xi} I(u,y).$$

Remark 3.6. *Note that in the scalar case $(I : U \times Y \longrightarrow \overline{\mathbb{R}})$ the sets*

$$\Lambda_w - \operatorname*{Inf}_{(u,y)\in U\times Y} I(u,y) \quad \text{and} \quad \Lambda_w - \operatorname{Inf} \mathrm{L}_w(I, u_0, y_0)$$

are singletons. Thus, if $\mathrm{L}_w(I, u_0, y_0) \cap \Lambda_w - \operatorname*{Inf}\limits_{(u,y)\in U\times Y} I(u,y) \neq \emptyset$, then

$$\mathrm{L}_w(I, u_0, y_0) \cap \Lambda_w - \operatorname*{Inf}_{(u,y)\in U\times Y} I(u,y) \equiv \Lambda_w - \operatorname{Inf} \mathrm{L}_w(I, u_0, y_0),$$

and therefore the rules (3.1) *and* (3.2) *coincide and give the classical definition of the lower limit.*

To illustrate the crucial role of the conditions

$$\mathrm{L}_w(I, u_0, y_0) \cap \Lambda_w - \operatorname*{Inf}_{(u,y)\in U\times Y} I(u,y) \neq \emptyset \text{ and } \mathrm{L}_w(I, u_0, y_0) \cap \Lambda_w - \operatorname*{Inf}_{(u,y)\in U\times Y} I(u,y) = \emptyset$$

in Definition 3.5, we give the following example.

Example 3.7. [20] *Let $U \times Y = Z = \mathbb{R}^2$, $\Xi = \Xi^1 \cup \Xi^2$,*

$$\Xi^1 = \left\{ x \in \mathbb{R}^2 \ : \ (x_1 - 6)^2 + (x_2 - 6)^2 \le 25,\ x_1 + x_2 \le 7 \right\}, \tag{3.3}$$

$$\Xi^2 = \left\{ x \in \mathbb{R}^2 \ : \ x_1 + x_2 > 7,\ x_1 + x_2 \le 8,\ x_1 \ge 1,\ x_2 \ge 1 \right\}, \tag{3.4}$$

and let $\Lambda = \mathbb{R}^2_+$ be the cone of positive elements. We define a vector-valued mapping $I : \Xi \to Z$ as follows:

$$I(x) = \begin{cases} x, & x \notin X_0, \\ \left[\begin{smallmatrix} 6 \\ 2 \end{smallmatrix}\right], & x \in X_0' \cup \{A, C\}, \\ \left[\begin{smallmatrix} 2 \\ 6 \end{smallmatrix}\right], & x \in X_0'' \cup \{B, D\}, \end{cases} \tag{3.5}$$

where $A = \left[\begin{smallmatrix} 1 \\ 7 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 \\ 6 \end{smallmatrix}\right]$, $C = \left[\begin{smallmatrix} 6 \\ 1 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 7 \\ 1 \end{smallmatrix}\right]$, $X_0 = X_0' \cup X_0'' \cup \{A, B, C, D\}$,

$$X_0' = \left\{ x \in \Xi \ : \ (x_1 - 6)^2 + (x_2 - 6)^2 = 25,\ 1 < x_1 \le x_2 \right\},$$
$$X_0'' = \left\{ x \in \Xi \ : \ (x_1 - 6)^2 + (x_2 - 6)^2 = 25,\ x_2 < x_1 < 6 \right\}.$$

Let us find the Λ-lower sequential limit of $I : \Xi \to Z$ at $x_0 = A$, and at $x_0 = C$. To begin with, we note that $\Lambda_w - \operatorname*{Inf}\limits_{x\in\Xi} I(x) = X_0' \cup X_0'' \cup \{B, C\}$ (see Fig. 1). Then, in the case when

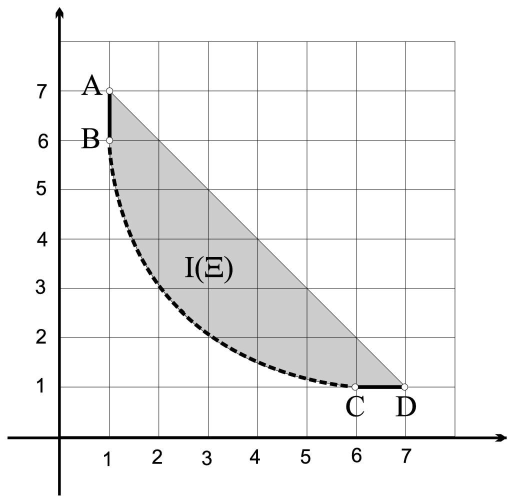

Figure 1: The image of Ξ in Example 3.7

$x_0 = A$, we have $\mathrm{L}_w(I, x_0) = \left\{A, \left[\begin{smallmatrix}6\\2\end{smallmatrix}\right]\right\}$. Hence, since $\mathrm{L}_w(I, x_0) \cap \Lambda_w - \operatorname*{Inf}_{x\in\Xi} I(x) = \emptyset$, by Definition 3.5, we conclude that

$$\Lambda_w - \liminf_{x\to x_0} \widehat{I}(x) = \Lambda_w - \operatorname{Inf} \mathrm{L}_w(I, x_0) = \left\{A, \begin{bmatrix}6\\2\end{bmatrix}\right\}.$$

On the other hand, if we take $x_0 = C$, then $\mathrm{L}_w(I, x_0) = \left\{C, \left[\begin{smallmatrix}6\\2\end{smallmatrix}\right]\right\}$. Hence,

$$\Lambda_w - \liminf_{x\to x_0} \widehat{I}(x) = \mathrm{L}_w(I, x_0) \cap \Lambda_w - \operatorname*{Inf}_{x\in\Xi} I(x) = \{C\}.$$

We are now able to give the definition of efficient solutions to the vector optimization problem (2.1)–(2.4). Let us recall the standard definitions (see, for instance, [7, 16, 24, 28]). A pair $(u^0, y^0) \in \Xi$ is an efficient solution of the problem $\langle \Xi, I, \Lambda\rangle$ if $(I(\Xi) - I(u^0, y^0)) \cap (-\Lambda) = \{0\}$, whereas a pair $(\widehat{u}, \widehat{y}) \in \Xi$ is said to be a weakly efficient solution of this problem if $\operatorname{int} \Lambda \neq \emptyset$ and there is no $z \in I(\Xi)$ such that $I(\widehat{u}, \widehat{y}) \neq z$ and $I(\widehat{u}, \widehat{y}) - z \in \operatorname{int} \Lambda$.

Further to the notions of efficient and weakly efficient solutions, the following concept will be used in this work.

Definition 3.8. *We say that $(u^*, y^*) \in \Xi$ is a Λ_w-efficient solution of the problem $\langle \Xi, I, \Lambda\rangle$ if $(\mathrm{cl}_w\, I(\Xi) - I(u^*, y^*)) \cap (-\Lambda) = \{0\}$, where $\mathrm{cl}_w\, I(\Xi)$ is the closure of the set $I(\Xi)$ with respect to the weak topology of Z.*

In other words, $(u^*, y^*) \in \Xi$ is a Λ_w-efficient solution of the problem $\langle \Xi, I, \Lambda\rangle$ if (u^*, y^*) realizes the weak efficient Λ-infimum of the mapping $I : \Xi \to Z$, that is,

$$I(u^*, y^*) \in \Lambda_w - \operatorname*{Inf}_{(u,y)\in\Xi} I(u, y).$$

We denote by $\mathrm{Eff}\,(\Xi;\, I;\, \Lambda)$, $\mathrm{Eff}_w\,(\Xi;\, I;\, \Lambda)$, and $\mathrm{Sol}_w(\Xi;\, I;\, \Lambda)$, respectively, the sets of all efficient solutions, all weakly efficient ones, and all Λ_w-efficient ones to the above vector problem.

Remark 3.9. *The difference between the notion of Λ_w- efficient solutions to the vector optimization problem* (2.1)–(2.4) *and the standard definition of efficient solutions should be noted.*

It is easy to show that each Λ_w-efficient solution is an efficient solution to this problem since $\mathrm{cl}_w\, I(\Xi) \supseteq I(\Xi)$*, i.e.,*

$$\mathrm{Sol}_w(\Xi;\, I;\, \Lambda) \subset \mathrm{Eff}\,(\Xi;\, I;\, \Lambda),$$

and $\mathrm{Sol}_w(\Xi;\, I;\, \Lambda) \subset \mathrm{Eff}\,(\Xi;\, I;\, \Lambda) \subset \mathrm{Eff}_w\,(\Xi;\, I;\, \Lambda)$*, provided the cone* Λ *has a non-empty interior.*

So, the sets $\mathrm{Eff}\,(\Xi;\, I;\, \Lambda)$, $\mathrm{Eff}_w\,(\Xi;\, I;\, \Lambda)$, and $\mathrm{Sol}_w(\Xi;\, I;\, \Lambda)$ do not coincide in general. To illustrate this fact more precisely, we give the following example.

Example 3.10. [21] *Let* $Z = \mathbb{R}^2$ *and let* $\Lambda = \mathbb{R}^2_+$ *be the cone of positive elements. We suppose that a vector-valued mapping* $I : U \times Y \to Z$ *and a set of admissible pairs* Ξ *are such that* $I(\Xi) = \bigcup_{i=1}^4 \Omega_i$*, where*

$$\Omega_1 = \left\{z \in \mathbb{R}^2 \;:\; z_1 \geq 1,\; z_2 > 3,\; z_1 + z_2 \leq 5\right\},$$
$$\Omega_2 = \left\{z \in \mathbb{R}^2 \;:\; z_1 > 2,\; z_2 > 2,\; z_1 + z_2 \leq 5\right\},$$
$$\Omega_3 = \left\{z \in \mathbb{R}^2 \;:\; z_1 > 3,\; z_2 \geq 4,\; z_1 + z_2 \leq 5\right\}, \quad \Omega_4 = \{(2;3), (3;2), (3;1)\}.$$

Then straightforward calculations show that

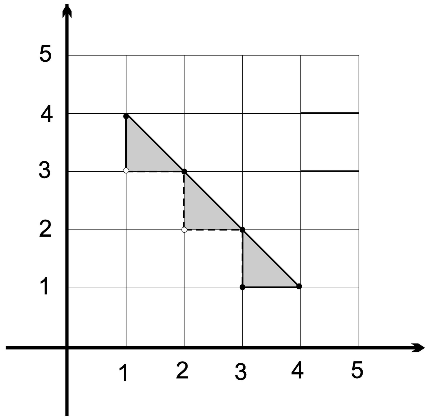

Figure 2: The image of the set Ξ *in Example 3.10*

$$\mathrm{Eff}\,(\Xi;\, I;\, \Lambda) = I^{-1}(\Omega_{eff}),\; \mathrm{Eff}_w\,(\Xi;\, I;\, \Lambda) = I^{-1}(\Omega_w),\;\; \textit{and}\;\; \mathrm{Sol}_w(\Xi;\, I;\, \Lambda) = I^{-1}(\Omega_0),$$

where

$$\Omega_{eff} = \{(2;3), (3;1)\}, \quad \Omega_0 = \{(3;1)\},$$
$$\Omega_w = \left\{z \in \mathbb{R}^2 \;:\; 3 < z_2 \leq 4, z_1 = 1\right\} \cup \left\{z \in \mathbb{R}^2 \;:\; 3 \leq z_1 \leq 4, z_2 = 1\right\} \cup \{(2;3), (3;2)\}.$$

4. THE MAIN ASSUMPTIONS

Since our main aim is an existence theorem for the Λ_w-efficient solutions to the vector optimization problem $\langle \Xi, I, \Lambda\rangle$, that is, to find sufficient conditions which guarantee the relation $\mathrm{Sol}_w(\Xi;\, I;\, \Lambda) \neq \emptyset$ without any scalarization process of the original optimization problem, we begin with the following assumptions (see for comparison [19]):

(A1) U_∂ is a bounded sequentially weakly-$*$ closed subset of U;

(A2) the operator $A : U \times Y \to Y^*$ is coercive in the following sense

$$\inf_{u \in G} \frac{\langle A(u,y), y\rangle_{Y^*;Y}}{\|y\|_Y} \to +\infty \quad \text{as} \quad \|y\|_Y \to +\infty$$

for any bounded subset $G \subset U$;

(A3) the operator $A : U \times Y \to Y^*$ possesses the property $(\mathfrak{M})$, i.e., for any w-convergent sequence $\{(u_k, y_k)\}_{k=1}^{\infty}$ the conditions

$$A(u_k, y_k) \rightharpoonup d \text{ in } Y^*, \quad \limsup_{k\to\infty} \langle A(u_k, y_k), y_k\rangle_{Y^*;Y} \le \langle d, y\rangle_{Y^*;Y} \tag{4.1}$$

imply $d = A(u, y)$;

(A4) the operator $F : U \times Y \to Z$ is sequentially continuous in the following sense: $F(u_k, y_k) \rightharpoonup F(u, y)$ in Z whenever $(u_k, y_k) \overset{w}{\rightharpoonup} (u, y)$;

(A5) K is a weakly closed subset of Y;

(A6) the objective function $I : \Xi \to Z$ is sequentially Λ_w-lower semicontinuous (Λ_w-lsc) with respect to the w-convergence in $U \times Y$ in the following sense (see [18, 20]):

$$I(\widehat{u}, \widehat{y}) \in \Lambda_w - \liminf_{(u,y) \overset{w}{\rightharpoonup} (\widehat{u},\widehat{y})} \widehat{I}(u, y), \quad \forall\, (\widehat{u}, \widehat{y}) \in \Xi. \tag{4.2}$$

Before proceeding further, we note that the concept of sequential Λ_w-lower semicontinuity for vector-valued mappings, given above, is more general than well known extensions of the "scalar" notion of lower semicontinuity to the vector-valued case (see, for example, [1, 3, 4, 6, 8, 17, 24, 28]). We now recall a few main definitions of lower semicontinuity of vector-valued mappings with respect to the product of the w-topology of $U \times Y$ and the weak topology of Z, introduced in [6, 8, 11, 30].

Definition 4.1. [8] *A mapping $I : U \times Y \to Z^\bullet$ is said to be sequentially lower semicontinuous (s-lsc) at $(u^0, y^0) \in U \times Y$, if for any $z \in Z$ satisfying $z \le_\Lambda I(u^0, y^0)$ and for any sequence $\{(u_k, y_k)\}_{k=1}^{\infty}$ of $U \times Y$ w-convergent to (u^0, y^0), there exists a sequence $\{z_k\}_{k=1}^{\infty}$ (in Z) weakly converging to z in Z and satisfying condition $z_k \le_\Lambda I(u_k, y_k)$, for any $k \in \mathbb{N}$.*

Remark 4.2. *For $(u^0, y^0) \in U \times Y$, Definition 4.1 can be expressed simply as follows. For each sequence $\{(u_k, y_k)\}_{k=1}^{\infty}$ w-converging to (u^0, y^0), there exists a sequence $\{z_k\}_{k=1}^{\infty}$ weakly converging to $I(u^0, y^0)$ in Z such that $z_k \le_\Lambda I(u_k, y_k)$ for all $k \in \mathbb{N}$.*

Definition 4.3. [6] *A mapping $I : U \times Y \to Z^\bullet$ is said to be quasi lower semicontinuous (q-lsc) at $(u^0, y^0) \in U \times Y$, if for each $z \in Z$ such that $z \not\ge_\Lambda I(u^0, y^0)$, there exists a neighborhood $\mathcal{O}$ of (u^0, y^0) in the w-topology of $U \times Y$ such that $z \not\ge_\Lambda I(u, y)$ for each (u, y) in $\mathcal{O}$.*

A mapping I is s-lsc (resp., q-lsc) if I is s-lsc (resp., q-lsc) at each point of $U \times Y$. It is clear that the s-lsc-property of I at (u, y) implies it is q-lsc at this pair. To characterize the properties of Λ_w-lower semicontinuity more precisely, we give the following result. The other properties concerning these notions can be found in [18].

Lemma 4.4. *If a mapping $I : \Xi \to Z$ is q-lower semicontinuous at $(u^0, y^0) \in \Xi$ with respect to the weak topology of Z and the w-topology of $U \times Y$, then I is Λ_w-lower semicontinuous at this pair.*

Proof. Let $I : \Xi \to Z$ be a q-lower semicontinuous mapping at $(u^0, y^0) \in \Xi$, and let $\widehat{I} : U \times Y \to Z^\bullet$ be its natural extension to the whole space $U \times Y$. Let $\{(u_k, y_k)\}_{k=1}^{\infty}$ be a sequence w-converging to (u^0, y^0). Hence $\{(u_k, y_k)\}_{k=1}^{\infty} \in \mathfrak{M}(u^0, y^0)$. Let us assume that there exists a subsequence $\{I(u_{k_i}, y_{k_i})\}_{i=1}^{\infty}$ such that $I(u_{k_i}, y_{k_i}) \not\ge_\Lambda I(u^0, y^0)$. Then, in view of the definition of the quasi-lower semicontinuity, we just conclude $\{+\infty_\Lambda\} \in \mathrm{L}_w(I, u^0, y^0)$.

So, to characterize the set $\Lambda_w - \liminf_{(u,y)\xrightarrow{w}(u^0,y^0)} \widehat{I}(u,y)$, we suppose that the corresponding image sequence $\{I(u_k,y_k)\}_{k=1}^{\infty}$ is bounded above with respect to the cone Λ. Then there exists an integer k^* such that

$$I(u_k,y_k) \geq_\Lambda I(u^0,y^0), \quad \forall k \geq k^*.$$

Hence, for any $z^* \in \mathrm{L}_w(I,u^0,y^0$, we have $I(u^0,y^0) \leq_\Lambda z^*$. This means that

$$\{I(u^0,y^0)\} \in \Lambda_w - \mathrm{Inf}\, \mathrm{L}_w(I(u^0,y^0)).$$

Thus, due to Definition 3.5, we deduce: $I(u^0,y^0) \in \Lambda_w - \liminf_{(u,y)\xrightarrow{w}(u^0,y^0)} \widehat{I}(u,y)$. This concludes the proof. ∎

As a consequence of this result and the properties of quasi-lower semicontinuity, we have: if I is s-lsc then I is Λ_w-lsc. However, in general, for vector-valued mappings, Λ_w-ls continuity does not imply q-lsc. Indeed, let us consider the following example when I depends on one variable (see also Example 6.5).

Example 4.5. [18] *Let $\Xi \subset \mathbb{R}$, $Z = \mathbb{R}^2$, and let $\Lambda = \mathbb{R}^2_+$ be the cone of positive elements. To state a vector optimization problem $\langle \Xi, I, \Lambda \rangle$, we define the set of admissible solutions Ξ and the mapping $I : \Xi \to Z$ as follows:*

$$\Xi = \{x \in \mathbb{R}^1 : \ -3 \leq x \leq -1\}, \tag{4.3}$$

$$I(x) = \begin{bmatrix} -x \\ 2 \end{bmatrix}, \text{ for all } x \neq -1, \quad I(-1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{4.4}$$

Let $x_0 = -1$. Then

$$I(x_0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \Lambda_w - \liminf_{x \to x_0} \widehat{I}(x) = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \tag{4.5}$$

(see Fig. 3). Let us take $z = \left[\begin{smallmatrix} 1{,}5 \\ 3 \end{smallmatrix}\right]$. Obviously $z \not\geq_\Lambda I(x_0)$ and there is no neighborhood of the

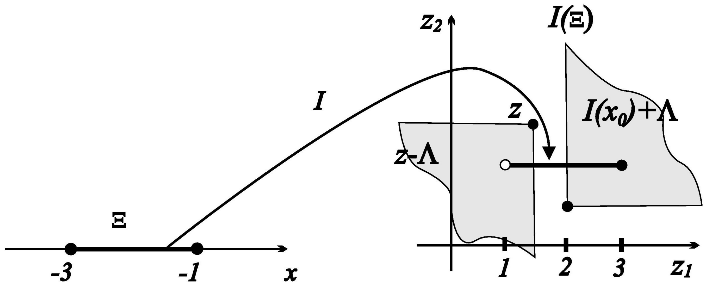

Figure 3: An example of Λ_w-lsc mapping which is neither lsc nor q-lsc

point x_0 such that $z \not\geq_\Lambda I(x)$ for all x from this neighborhood. Hence, I is neither q-lsc nor lsc at the point x_0.

However, by (4.5)*, I is a Λ_w-lower semicontinuous mapping at $x_0 = -1$.*

We note also that if sequential Λ_w-lower semicontinuity for an objective mapping fails, the set of Λ_w- efficient solutions $\mathrm{Sol}_w(\Xi;\, I;\, \Lambda)$ to the corresponding problem may be empty. Indeed, let us consider the following example.

Example 4.6. *Let $Z=\mathbb{R}^2$, $\Xi=[1,2]$, and let $\Lambda=\mathbb{R}^2_+$ be the cone of positive elements. We define the objective mapping $I:\Xi\to Z$ as follows: $I(x)=\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right]$ if $x\in(1,2]$, and $I(x)=\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ at the point $x=1$ (see Fig. 4). It is clear that in this case* $\mathrm{Eff}\,(\Xi;\,I;\,\Lambda)=\mathrm{Eff}_w\,(\Xi;\,I;\,\Lambda)=$

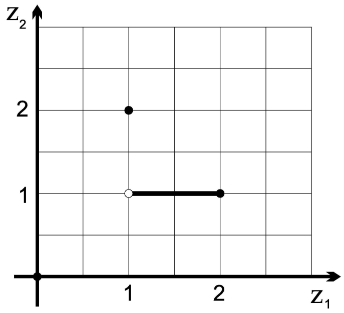

Figure 4: The example of the problem for which $\mathrm{Sol}_w(\Xi;\,I;\,\Lambda)=\emptyset$

$\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$. *However, straightforward calculations show that $\Lambda_w-\liminf\limits_{x\to1}\widehat{I}(x)=\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, and hence* $\mathrm{Sol}_w(\Xi;\,I;\,\Lambda)=\emptyset$. *Moreover, I is not Λ_w-lsc at $x=1$.*

To conclude this section we give the following observation concerning the property of two Λ_w-lsc mappings. It is well known that the sum of two q-lsc mappings does not give a q-lsc mapping in general. Due to the following example, we can give a similar conclusion for Λ_w-lsc mappings.

Example 4.7. *Let $\Lambda=\mathbb{R}^2_+$ be the cone of positive elements in $\mathbb{R}^2$. Let us consider the mappings $I:\mathbb{R}\to\mathbb{R}^2$ and $G:\mathbb{R}\to\mathbb{R}^2$ defined by*

$$I(x)=\begin{cases}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \text{if } x=0,\\ \left[\begin{smallmatrix}-2+|x|^{-1}\\-2|x|^{-1}\end{smallmatrix}\right] & \text{if } x\neq0,\end{cases},\quad G(x)=\begin{cases}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] & \text{if } x=0,\\ \left[\begin{smallmatrix}-|x|^{-1}\\2|x|^{-1}\end{smallmatrix}\right] & \text{if } x\neq0.\end{cases}$$

It is easy to see that each of these mappings is q-lsc at $x_0=0$. So, due to Lemma 4.4 these mappings are Λ_w-lsc at 0. However, for the mapping $I+G$, we have

$$\mathrm{L}_w(I(0)+G(0))=\left\{\begin{bmatrix}-2\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right\}\quad\text{and}\quad\Lambda-\operatorname*{Inf}_{x\in\mathbb{R}}\,[I(x)+G(x)]=\left\{\begin{bmatrix}-2\\0\end{bmatrix}\right\}.$$

Hence, $\Lambda_w-\liminf\limits_{x\to0}\,[I(x)+G(x)]=\left\{\left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right]\right\}\not\ni I(0)+G(0)$, and we obtain the required conclusion: in general the sum of two Λ_w-lsc mappings is not a Λ_w-lsc mapping.

5. EXISTENCE THEOREM

We begin with the following supposition: assume that the ordering cone Λ possesses the so-called D-property, that is, every decreasing sequence in Z is weakly convergent if and only if it has a Λ-lower bound. The typical ordering cone with this property is the so-called natural ordering cone in $L^p(\Omega)$ $(1<p<+\infty)$ which is defined as (see [15])

$$\Lambda_{L^p(\Omega)}=\{f\in L^p(\Omega)\ :\ f(x)\geq0\text{ almost everywhere on }\ \Omega\}\,.$$

Definition 5.1. *We say that $\{(u_k,y_k)\}_{k=1}^\infty\subset\ \Xi$ is a minimizing sequence for the mapping $I:\Xi\to Z$ if there exists an element $\xi\in\Lambda_w-\operatorname*{Inf}\limits_{(u,y)\in\Xi}I(u,y)$ such that $I(u_k,y_k)\rightharpoonup\xi$ in Z.*

First of all we establish the following result:

Theorem 5.2. *Assume that the properties (A1)–(A6) hold true and the ordering cone $\Lambda \subset Z$ satisfies the D-property. Then the vector optimization problem* (2.1)–(2.4) *has a nonempty set of Λ_w-efficient solutions.*

Proof. To begin with, we show that the mapping $I : \Xi \to Z$ satisfies the property: for any sequence $\{(u_k, y_k)\}_{k=1}^{\infty} \subset \Xi$ for which its image $\{I(u_k, y_k)\}_{k=1}^{\infty} \subset Z$ is a decreasing sequence in Z there exists an element $z \in Z$ such that $z \leq_\Lambda I(u_k, y_k)$ for all $k \in \mathbb{N}$. Let us assume the converse. Then there exist sequences $\{(\widehat{u}_k, \widehat{y}_k)\}_{k=1}^{\infty} \subset \Xi$ and $\{\widehat{z}_k\}_{k=1}^{\infty} \subset Z$ such that $\widehat{z}_{k+1} \leq_\Lambda \widehat{z}_k$ $\forall\, k \in \mathbb{N}$, and

$$\Lambda_w - \operatorname{Inf} \{\widehat{z}_k\}_{k=1}^{\infty} = -\infty_\Lambda, \quad I(\widehat{u}_k, \widehat{y}_k) <_\Lambda \widehat{z}_k, \quad \forall\, k \in \mathbb{N}. \tag{5.1}$$

Due to the initial assumptions, we have $\{\widehat{u}_k\}_{k=1}^{\infty} \subset U_\partial$ and the sequence $\{\widehat{u}_k\}_{k=1}^{\infty}$ is bounded in U. So, we may assume that $\widehat{u}_k \overset{*}{\rightharpoonup} \widehat{u}$ in U and $\widehat{u} \in U_\partial$. Since

$$\langle A(\widehat{u}_k, \widehat{y}_k), \widehat{y}_k \rangle_{Y^*;Y} = \langle f, \widehat{y}_k \rangle_{Y^*;Y} \leq \|f\|_{Y^*} \|\widehat{y}_k\|_Y, \quad \forall\, k \in \mathbb{N}$$

and the operator A is coercive (see (A2)), it follows that

$$\sup_{k \in \mathbb{N}} \frac{\langle A(\widehat{u}_k, \widehat{y}_k), \widehat{y}_k \rangle_{Y^*;Y}}{\|\widehat{y}_k\|_Y} \leq \|f\|_{Y^*}.$$

Hence, $\{\widehat{y}_k\}_{k=1}^{\infty} \subset K$ is a bounded sequence in Y and there exists an element $\widehat{y} \in K$ such that, passing to a subsequence if necessary, we get $\widehat{y}_k \rightharpoonup \widehat{y}$ in Y. Thus for a given sequence of pairs, we have $(\widehat{u}_k, \widehat{y}_k) \overset{w}{\rightharpoonup} (\widehat{u}, \widehat{y})$.

Now we can pass to the limit in (5.1) as $k \to \infty$. Using the D-property of ordering cone Λ, we obtain

$$\xi \leq_\Lambda -\infty_\Lambda, \quad \forall \xi \in L\{I(\widehat{u}_k, \widehat{y}_k)\}, \tag{5.2}$$

where $L\{I(\widehat{u}_k, \widehat{y}_k)\}$ is the set of all cluster points of $\{I(\widehat{u}_k, \widehat{y}_k)\}_{k=1}^{\infty}$ with respect to the weak topology of Z. So, in view of Definition 3.5 and the Λ_w-lower semicontinuity of I, we have

$$I(\widehat{u}, \widehat{y}) \in \Lambda_w - \liminf_{(u,y) \overset{w}{\rightharpoonup} (\widehat{u}, \widehat{y})} \widehat{I}(u, y), \text{ and hence } I(\widehat{u}, \widehat{y}) \not>_\Lambda \xi, \quad \forall \xi \in L\{I(\widehat{u}_k, \widehat{y}_k)\}.$$

Combining this result with (5.2), we obtain

$$I(\widehat{u}, \widehat{y}) \not>_\Lambda -\infty_\Lambda.$$

However this contradicts (5.1). Hence $\Lambda_w - \operatorname*{Inf}_{(u,y) \in \Xi} I(u, y) \not\ni -\infty_\Lambda$.

Let ξ be any element of $\Lambda_w - \operatorname*{Inf}_{(u,y) \in \Xi} I(u, y)$. Then, by definition of the Λ_w-efficient infimum, there exists a sequence $\{(u_k, y_k)\}_{k=1}^{\infty} \subset \Xi$ such that $I(u_k, y_k) \rightharpoonup \xi$ in Z as $k \to \infty$. By the previous arguments this sequence is uniformly bounded in $U \times Y$. Since $\Xi \subset U_\partial \times K$ and the set $U_\partial \times K$ is sequentially closed with respect to the w-convergence, we may assume that there exists a pair $(u^0, y^0) \in U_\partial \times K$ such that $(u_k, y_k) \overset{w}{\rightarrow} (u^0, y^0)$. Let us show that (u^0, y^0) is an admissible pair to the problem (2.1)–(2.4).

Indeed, taking into account that $A(u_k, y_k) = f$ for all $k \in \mathbb{N}$ and passing to the limit in the equality

$$\langle A(u_k, y_k), y_k \rangle_{Y^*;Y} = \langle f, y_k \rangle_{Y^*;Y}$$

as $k \to \infty$, we obtain

$$\lim_{k \to \infty} \langle A(u_k, y_k), y_k \rangle_{Y^*;Y} = \left\langle f, y^0 \right\rangle_{Y^*;Y}.$$

Hence $A(u^0, y^0) = f$ by $(\mathfrak{M})$-property of the operator $A : U \times Y \to Y^*$.

By the initial assumptions, the cone $\Lambda \subset Z$ is reproducing and uniquely determined by its conjugate semigroup (the dual cone) Λ^*, i.e.,

$$\Lambda = \left\{ \xi \in Z \ : \ \langle \varphi, \xi \rangle_{Z^*;Z} \geq 0 \quad \forall\, \varphi \in \Lambda^* \right\}. \tag{5.3}$$

Here we denote by Z^* the dual space of Z with duality pairing $\langle \cdot, \cdot \rangle_{Z^*;Z}$. In view of the continuity property of the operator $F : U \times Y \to Z$ with respect to the w-convergence, we have $F(u_k, y_k) \rightharpoonup F(u^0, y^0)$ in Z. Since the elements of the conjugate semigroup Λ^* are linear continuous functionals on Z, it follows that

$$0 \leq \langle \psi, F(u_k, y_k) \rangle_{Z^*;Z} \to \left\langle \psi, F(u^0, y^0) \right\rangle_{Z^*;Z} \ \text{ as } \ k \to \infty, \ \forall\, \psi \in \Lambda^*.$$

Therefore $\langle \psi, F(u^0, y^0) \rangle_{Z^*;Z} \geq 0$, and hence $F(u^0, y^0) \geq_\Lambda 0$.

Thus, the limit pair (u^0, y^0) is an admissible solution to the problem (2.1)–(2.4), i.e., $(u^0, y^0) \in \Xi$. As a result, we have

$$\xi \in \mathrm{L}_w(I(u^0, y^0)) \ \text{ and, therefore, } \ \mathrm{L}_w(I(u^0, y^0)) \cap \Lambda_w - \operatorname*{Inf}_{(u,y) \in \Xi} I(u, y) \neq \emptyset. \tag{5.4}$$

Let us show that $(u^0, y^0) \in \Xi$ is a Λ_w-efficient solution of this problem. In view of Definition 3.5 and the Λ_w-lower semicontinuity of I, we obtain

$$I(u^0, y^0) \in \Lambda_w - \liminf_{(u,y) \overset{w}{\rightharpoonup} (u^0, y^0)} \widehat{I}(u, y) = \mathrm{L}_w(I(u^0, y^0)) \cap \Lambda_w - \operatorname*{Inf}_{(u,y) \in \Xi} I(u, y).$$

Hence, $I(u^0, y^0) \in \mathrm{L}_w(I(u^0, y^0))$, which implies

$$I(u^0, y^0) = \xi, \quad \text{ and } \quad \xi \in \Lambda_w - \operatorname*{Inf}_{(u,y) \in \Xi} I(u, y).$$

Thus, (u^0, y^0) is a Λ_w-efficient solution of the vector optimization problem (2.1)–(2.4) and this concludes the proof. ∎

Let us denote by $\mathbb{P}_+(\Lambda^*)$ the set of all equivalence classes with respect to the binary relation

$$\varphi_1 \backsim \varphi_2 \Longleftrightarrow \exists\, t \in \mathbb{R}_+ \setminus \{0\} \ : \ \varphi_1 = t \varphi_2, \ \varphi_1, \varphi_2 \in \Lambda^*.$$

Let $\Pi^* : \Lambda^* \setminus 0 \to \mathbb{P}_+(\Lambda^*)$ be the corresponding canonical quotient-mapping. We assume that $\mathbb{P}_+(\Lambda^*)$ is endowed with the quotient-topology. It is clear that in this case the mapping

$$\Pi^*|_{S_1^* \cap \Lambda^*} : S_1^* \cap \Lambda^* \to \mathbb{P}_+(\Lambda^*)$$

is a continuous surjection, i.e., every equivalence class can be interpreted as the image of some element of Λ^* belonging to the unit sphere S_1^* in Z^*. Hence, if $F(v, \xi) \notin \Lambda$ for some pair $(v, \xi) \in U_\partial \times Y$, then there is an element $\psi \in S_1^* \cap \Lambda^*$ such that $\langle \psi, F(v, \xi) \rangle_{Z^*;Z} < 0$. Taking into account these observations, we introduce the following penalized problem to the original one (2.1)–(2.4)

$$I_\varepsilon(u, y) = I(u, y) + \varepsilon^{-1} \sup_{\psi \in S_1^* \cap \Lambda^*} \left[\mu \left(\langle \psi, F(u, y) \rangle_{Z^*;Z} \right) \right] b \to \Lambda_w - \mathrm{Inf}, \tag{5.5}$$

$$A(u, y) = f, \tag{5.6}$$

$$y \in K, \quad u \in U_\partial \subset U, \tag{5.7}$$

where $b \in \Lambda$ is a fixed element, $\mu : \mathbb{R} \to \mathbb{R}_+$ is a lower semicontinuous monotone decreasing function such that $\mu(0) = 0$ and μ is strictly monotone on $\mathbb{R}_-$. We denote by $\mathcal{F}(\mathbb{R}, \mathbb{R}_+)$ the set of all functions μ with the properties mentioned above.

Let $\overline{\Xi}$ be the set of admissible solutions to the penalized problem (5.5)–(5.7), that is

$$\overline{\Xi} = \left\{ (u, y) \in U_\partial \times K \ : \ A(u, y) = f \right\}.$$

It is clear that $\Xi \subset \overline{\Xi}$ for every $\varepsilon > 0$.

Lemma 5.3. *Assume that the properties (A1)–(A6) hold true. Then for every $\varepsilon > 0$ and every fixed $\mu \in \mathcal{F}(\mathbb{R}, \mathbb{R}_+)$ the problem* (5.5)–(5.7) *has a nonempty set of Λ_w-efficient solutions.*

Proof. By analogy with Theorem 5.2, it can be proved that $\Lambda_w - \operatorname*{Inf}_{(u,y)\in\overline{\Xi}} I_\varepsilon(u,y) \not\ni -\infty_\Lambda$. Let ξ be any element of $\Lambda_w - \operatorname*{Inf}_{(u,y)\in\overline{\Xi}} I_\varepsilon(u,y)$. Then, by definition of the Λ_w-efficient infimum, there exists a sequence $\{(u_k,y_k)\}_{k=1}^\infty \subset \overline{\Xi}$ such that $I_\varepsilon(u_k,y_k) \rightharpoonup \xi$ in Z as $k\to\infty$. By arguments in the previous proof, this sequence is uniformly bounded in $U\times Y$. Since $\overline{\Xi}\subset U_\partial\times K$ and the set $U_\partial \times K$ is sequentially closed with respect to the w-convergence, we may assume that there exists a pair $(u^0_\varepsilon, y^0_\varepsilon)\in U_\partial\times K$ such that $(u_k,y_k)\overset{w}{\to}(u^0_\varepsilon,y^0_\varepsilon)$. Then, taking into account that $A(u_k,y_k)=f$ for all $k\in\mathbb{N}$ and passing to the limit in the equality

$$\langle A(u_k,y_k), y_k\rangle_{Y^*;Y} = \langle f, y_k\rangle_{Y^*;Y}$$

as $k\to\infty$, we obtain

$$\lim_{k\to\infty}\langle A(u_k,y_k), y_k\rangle_{Y^*;Y} = \left\langle f, y^0_\varepsilon\right\rangle_{Y^*;Y}.$$

Hence $A(u^0_\varepsilon,y^0_\varepsilon)=f$ by the $(\mathfrak{M})$-property of the operator $A: U\times Y\to Y^*$. Thus, the limit pair $(u^0_\varepsilon,y^0_\varepsilon)$ is an admissible solution to the problem (5.5)–(5.7), i.e., $(u^0_\varepsilon,y^0_\varepsilon)\in\overline{\Xi}$. As a result, we have

$$\xi\in \mathrm{L}_w(I_\varepsilon(u^0_\varepsilon,y^0_\varepsilon)) \text{ and, therefore, } \mathrm{L}_w(I_\varepsilon(u^0_\varepsilon,y^0_\varepsilon))\cap\Lambda_w-\operatorname*{Inf}_{(u,y)\in\overline{\Xi}} I_\varepsilon(u,y)\neq\emptyset. \tag{5.8}$$

Let us show that $(u^0_\varepsilon,y^0_\varepsilon)\in\overline{\Xi}$ is a Λ_w-efficient solution of this problem. Indeed, in view of the continuity property of the operator $F: U\times Y\to Z$ with respect to the w-convergence, we have $F(u_k,y_k)\rightharpoonup F(u^0_\varepsilon,y^0_\varepsilon)$ in Z. Since the elements of the conjugate semigroup Λ^* are linear continuous functionals on Z, it follows that

$$\langle\psi, F(u_k,y_k)\rangle_{Z^*;Z}\to\left\langle\psi, F(u^0_\varepsilon,y^0_\varepsilon)\right\rangle_{Z^*;Z} \text{ as } k\to\infty,\ \forall\,\psi\in\Lambda^*.$$

By initial assumptions, $Z\ni\eta\mapsto\mu\left(\langle\psi,\eta\rangle_{Z^*;Z}\right)\in\mathbb{R}$ is a lower semicontinuous function. Hence $\eta\mapsto\sup_{\psi\in S_1^*\cap\Lambda^*}\left[\mu\left(\langle\psi,\eta\rangle_{Z^*;Z}\right)\right]$ is a semicontinuous non-negative function with respect to the weak topology of Z. Therefore, for every fixed $b\in\Lambda$, the mapping $G: U\times Y\to Z$, where

$$G(u,y)=\varepsilon^{-1}\sup_{\psi\in S_1^*\cap\Lambda^*}\left[\mu\left(\langle\psi, F(u,y)\rangle_{Z^*;Z}\right)\right]b,$$

is sequentially lower semicontinuous in the sense of Definition 4.1. So, we get

$$\varepsilon^{-1}\sup_{\psi\in S_1^*\cap\Lambda^*}\left[\mu\left(\left\langle\psi, F(u^0_\varepsilon,y^0_\varepsilon)\right\rangle_{Z^*;Z}\right)\right]b\leq_\Lambda\zeta,\quad\forall\,\zeta\in L\{G(u_k,y_k)\}. \tag{5.9}$$

Due to the initial assumptions, we have

$$I(u_k,y_k)+\varepsilon^{-1}\sup_{\psi\in S_1^*\cap\Lambda^*}\left[\mu\left(\langle\psi, F(u_k,y_k)\rangle_{Z^*;Z}\right)\right]=I_\varepsilon(u_k,y_k)\rightharpoonup\xi \text{ in } Z.$$

We claim that (5.9) implies

$$\xi=\text{weak}-\lim_{k\to\infty} I_\varepsilon(u_k,y_k)\not<_\Lambda a+\varepsilon^{-1}\sup_{\psi\in S_1^*\cap\Lambda^*}\left[\mu\left(\left\langle\psi, F(u^0_\varepsilon,y^0_\varepsilon)\right\rangle_{Z^*;Z}\right)\right]b \tag{5.10}$$

for every $a\in\Lambda_w-\liminf_{(u,y)\overset{w}{\rightharpoonup}(u^0_\varepsilon,y^0_\varepsilon)}\widehat{I}(u,y)$.

Assume the converse. Namely, there exists an element

$$a^* \in \Lambda_w - \liminf_{(u,y) \xrightarrow{w} (u^0_\varepsilon, y^0_\varepsilon)} \widehat{I}(u,y)$$

such that

$$\xi = \text{weak} - \lim_{k\to\infty} I_\varepsilon(u_k, y_k) <_\Lambda a^* + \varepsilon^{-1} \sup_{\psi \in S_1^* \cap \Lambda^*} \left[\mu \left(\left\langle \psi, F(u^0_\varepsilon, y^0_\varepsilon) \right\rangle_{Z^*;Z} \right) \right] b. \tag{5.11}$$

Taking into account (5.8), we can deduce that in this case there are two possibilities for the representation of ξ: either $\xi = a_1 + G(u^0_\varepsilon, y^0_\varepsilon)$ or $\xi = a_2 + g^*$, where $a_1, a_2 \in I(\overline{\Xi})$, and $g^* \in L\{G(u_k, y_k)\}$. By (5.9) $g^* \geq_\Lambda G(u^0_\varepsilon, y^0_\varepsilon)$. The first case gives

$$a_1 + G(u^0_\varepsilon, y^0_\varepsilon) <_\Lambda a^* + G(u^0_\varepsilon, y^0_\varepsilon) \quad \Longrightarrow \quad a_1 <_\Lambda a^*,$$

and this contradicts the condition $a^* \in \Lambda_w - \liminf_{(u,y) \xrightarrow{w} (u^0_\varepsilon, y^0_\varepsilon)} \widehat{I}(u,y)$.

In the second case, we have $a_2 + g^* <_\Lambda a^* + G(u^0_\varepsilon, y^0_\varepsilon)$. Since $g^* \geq_\Lambda G(u^0_\varepsilon, y^0_\varepsilon)$ it follows that $a_2 <_\Lambda a^*$. But this is impossible by the previous argument. Thus (5.11) was erroneous.

Hence (5.10) holds for every $a \in \Lambda_w - \liminf_{(u,y) \xrightarrow{w} (u^0_\varepsilon, y^0_\varepsilon)} \widehat{I}(u,y)$. Since the objective function I is Λ_w-lower semicontinuous, we have $I(u^0_\varepsilon, y^0_\varepsilon) \in \Lambda_w - \liminf_{(u,y) \xrightarrow{w} (u^0_\varepsilon, y^0_\varepsilon)} \widehat{I}(u,y)$. Setting $a = I(u^0_\varepsilon, y^0_\varepsilon)$ in (5.10), we get

$$\xi \not<_\Lambda I_\varepsilon(u^0_\varepsilon, y^0_\varepsilon) \quad \text{for every} \quad \xi \in \Lambda_w - \operatorname*{Inf}_{(u,y)\in\overline{\Xi}} I_\varepsilon(u,y).$$

Hence $I_\varepsilon(u^0_\varepsilon, y^0_\varepsilon) \in \Lambda_w - \operatorname*{Inf}_{(u,y)\in\overline{\Xi}} I_\varepsilon(u,y)$, i.e., $(u^0_\varepsilon, y^0_\varepsilon)$ is a Λ_w-efficient solution of the penalized problem (5.5)–(5.7). ∎

Let $\left\{(u^0_\varepsilon, y^0_\varepsilon) \in \overline{\Xi}\right\}_{\varepsilon>0}$ be any sequence of Λ_w-efficient solutions to the problem (5.5)–(5.7). The next step of our analysis is to study the asymptotic behaviour of this sequence as ε tends to zero.

Lemma 5.4. *Let $\left\{(u^0_\varepsilon, y^0_\varepsilon) \in \overline{\Xi}\right\}_{\varepsilon>0}$ be a sequence of Λ_w-efficient solutions of the problem* (5.5)–(5.7) *(when $\varepsilon > 0$ varies in a strictly decreasing sequence of positive numbers which converge to* 0*) such that the set $\{I(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon>0}$ is bounded below in Z. Then under assumptions (A1)–(A6), a subsequence of $\{(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon>0}$ still denoted by ε can be extracted such that*

$$\begin{aligned} &(u^0_\varepsilon, y^0_\varepsilon) \xrightarrow{w} (u^0, y^0) \text{ in } U \times Y \text{ as } \varepsilon \to 0, \\ &A(u^0, y^0) = f, \quad y^0 \in K, \quad u^0 \in U_\partial \subset U. \end{aligned}$$

Proof. In the same way as in the proof of Lemma 5.3, we can conclude that the sequence $\{(u^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ is relatively w-compact in $U\times Y$ and, passing to a subsequence if necessary, we get

$$u^0_\varepsilon \overset{*}{\rightharpoonup} u^0 \text{ in } U, \quad y^0_\varepsilon \rightharpoonup y^0 \text{ in } Y, \text{ where } (u^0, y^0) \in \overline{\Xi}.$$

Let us prove that $F(u^0, y^0) \geq_\Lambda 0$. Let (u, y) be any admissible pair to the original problem, that is, $(u, y) \in \Xi$. Then, by the initial assumptions, $\mu\Big(\langle \psi, F(u,y)\rangle_{Z^*;Z}\Big) = 0$. Therefore

$$I_\varepsilon(u^0_\varepsilon, y^0_\varepsilon) \not>_\Lambda I_\varepsilon(u,y) \equiv I(u,y).$$

Setting $g_\varepsilon = \sup_{\psi \in S_1^* \cap \Lambda^*} \left[\mu \left(\langle \psi, F(u^0_\varepsilon, y^0_\varepsilon) \rangle_{Z^*;Z} \right) \right]$, we have $\varepsilon^{-1} g_\varepsilon b + I(u^0_\varepsilon, y^0_\varepsilon) \not>_\Lambda I(u,y)$. By the initial assumptions the set $\{I(u^0_\varepsilon, y^0_\varepsilon)\}_{\varepsilon>0}$ is bounded below, say by $z \in Z$. The latter yields $\varepsilon^{-1} g_\varepsilon b \not>_\Lambda w$, with $w = I(u,y) - z$, i.e. $g_\varepsilon b \not>_\Lambda \varepsilon w$. On the other hand, $g_\varepsilon b \geq_\Lambda 0_Z \; \forall\, b \in \Lambda$

and $\forall\,\varepsilon > 0$. Hence, passing to the limit as $\varepsilon \to 0$ in the above relations and using the fact that $\varepsilon w \to 0_Z$, we come to the inequality $0_Z \le \liminf_{\varepsilon\to 0} g_\varepsilon b \not>_\Lambda 0_Z$. Since g_ε takes the scalar values and this relation holds for any $b \in \Lambda$, one gets

$$\liminf_{\varepsilon\to 0} g_\varepsilon = \liminf_{\varepsilon\to 0} \sup_{\psi\in S_1^*\cap\Lambda^*} \left[\mu\left(\left\langle \psi, F(u_\varepsilon^0, y_\varepsilon^0)\right\rangle_{Z^*;Z}\right)\right] = 0.$$

Then, in view of the lower semicontinuity property, we obtain

$$\sup_{\psi\in S_1^*\cap\Lambda^*} \left[\mu\left(\left\langle \psi, F(u^0, y^0)\right\rangle_{Z^*;Z}\right)\right] \le \liminf_{\varepsilon\to 0} \sup_{\psi\in S_1^*\cap\Lambda^*} \left[\mu\left(\left\langle \psi, F(u_\varepsilon^0, y_\varepsilon^0)\right\rangle_{Z^*;Z}\right)\right] = 0.$$

Since this is equivalent to the inequality $F(u^0, y^0) \ge_\Lambda 0$, it follows that the limit pair (u^0, y^0) is an admissible solution to the original problem (2.1)–(2.4). ∎

The following assertion is an obvious consequence of Lemmas 5.3 and 5.4.

Corollary 5.5. *Under assumptions (A1)–(A6) the sets of admissible pairs Ξ and $\overline{\Xi}$ are bounded and sequentially compact with respect to the w-convergence.*

We are now in a position to prove the following result:

Theorem 5.6. *Assume that the properties (A1)–(A6) hold true. Then under assumptions of Lemma 5.4,* $\mathrm{Sol}_w(\Xi;\, I;\, \Lambda) \ne \emptyset$ *if and only if the vector optimization problem* (2.1)–(2.4) *is regular.*

Proof. We have to prove only the sufficient conditions of this theorem. Assume that $\Xi \ne \emptyset$. Then, taking into account Lemmas 5.3–5.4, we can construct a sequence of Λ_w-efficient solutions $\left\{(u_\varepsilon^0, y_\varepsilon^0) \in \overline{\Xi}\right\}_{\varepsilon>0}$ to the penalized problem (5.5)–(5.7) such that $(u_\varepsilon^0, y_\varepsilon^0) \overset{w}{\rightharpoonup} (u^0, y^0)$, where (u^0, y^0) is some admissible pair to the original problem (2.1)–(2.4). Let us show that (u^0, y^0) is a Λ_w-efficient solution, that is, $(u^0, y^0) \in \mathrm{Sol}_w(\Xi;\, I;\, \Lambda)$. To do so, we assume the converse. Namely, there is a pair $(\widehat{u}, \widehat{y}) \in \Xi$ such that $I(\widehat{u}, \widehat{y}) <_\Lambda I(u^0, y^0)$. Then this pair is also admissible to the penalized problem (5.5)–(5.7), i.e. $(\widehat{u}, \widehat{y}) \in \overline{\Xi}$. Hence

$$I(\widehat{u}, \widehat{y}) \equiv I_\varepsilon(\widehat{u}, \widehat{y}) \not<_\Lambda I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0), \quad \forall\,\varepsilon > 0. \tag{5.12}$$

From this we immediately conclude that

$$I_\varepsilon(\widehat{u}, \widehat{y}) \not<_\Lambda \xi \quad \text{for every } \xi \in L\{I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0)\}. \tag{5.13}$$

However for all $\varepsilon > 0$ we have the obvious inequality

$$I_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) \ge_\Lambda I(u_\varepsilon^0, y_\varepsilon^0). \tag{5.14}$$

Hence, in view of the relations (5.12)–(5.14), we obtain

$$I(\widehat{u}, \widehat{y}) \not<_\Lambda \eta \quad \text{for all } \eta \in L\{I(u_\varepsilon^0, y_\varepsilon^0)\}. \tag{5.15}$$

Since the objective mapping $I : U \times Y \to Z$ is sequentially Λ_w-lower semicontinuous with respect to the w-convergence, it follows from (5.15) that $I(\widehat{u}, \widehat{y}) \not<_\Lambda I(u^0, y^0)$. As a result, we come to the contradiction with the inequality $I(\widehat{u}, \widehat{y}) <_\Lambda I(u^0, y^0)$. This concludes the proof. ∎

Remark 5.7. *Note that Theorems 5.2 and 5.6 still hold if instead of (A3) we assume that the operator $A : U \times Y \to Y^*$ is quasi-monotone. We recall that an operator $A : U \times Y \to Y^*$ is said to be quasi-monotone if for any sequence $\{(u_k, y_k)\}_{k=1}^\infty$ which is w-convergent to some pair (u, y), the condition*

$$\limsup_{k\to\infty} \langle A(u_k, y_k), y_k - y\rangle_{Y^*,Y} \le 0 \tag{5.16}$$

implies the relation

$$\liminf_{k\to\infty} \langle A(u_k, y_k), y_k - \xi\rangle_{Y^*,Y} \geq \langle A(u, y), y - \xi\rangle_{Y^*,Y}\,, \quad \forall\, \xi \in Y. \tag{5.17}$$

Indeed, let us prove the implication: "A is quasi-monotone" $\Longrightarrow$ "A possesses the property $(\mathfrak{M})$*". Let $\{(u_k, y_k)\}_{k=1}^{\infty}$ be a sequence such that*

$$(u_k, y_k) \overset{w}{\rightharpoonup} (u, y),\ A(u_k, y_k) \rightharpoonup \zeta \ \textit{in}\ Y^*,\ \textit{and}\ \limsup_{k\to\infty} \langle A(u_k, y_k), y_k\rangle_{Y^*,Y} \leq \langle \zeta, y\rangle_{Y^*,Y}\,.$$

This immediately leads us to the inequality (5.16)*. Hence, by the quasi-monotonicity property, we have*

$$\begin{aligned}\langle A(u, y), y - \xi\rangle_{Y^*,Y} &\leq \liminf_{k\to\infty} \langle A(u_k, y_k), y_k - \xi\rangle_{Y^*,Y}\\ &\leq \limsup_{k\to\infty} \langle A(u_k, y_k), y_k - \xi\rangle_{Y^*,Y}\\ &\leq \langle \zeta, y - \xi\rangle_{Y^*,Y} \quad \forall\, \xi \in Y.\end{aligned}$$

Thus $A(u, y) = \zeta$, and we come to the required conclusion: the operator A possesses the property $(\mathfrak{M})$.

6. Λ_w-LOWER SEMICONTINUOUS MAPPINGS AND THE PROBLEM OF THEIR SCALARIZATION

The traditional approach to solving vector optimization problems is by scalarization, which involves the formulation of a single objective optimization problem that is related to the original one. Among various scalarization procedures known in the literature (see, for instance, [9, 25, 27] and the references therein), we consider the problem of the scalar representation of vector problem with Λ_w-lower semicontinuous mappings, using the "simplest" method of the "weighted sum". Then the scalar problem associated with the penalized vector optimization problem (5.5)–(5.7) has the following representation

$$\langle \lambda^*, I_\varepsilon(u, y)\rangle_{Z^*;Z} = \langle \lambda^*, I(u, y)\rangle_{Z^*;Z} + \varepsilon^{-1} \sup_{\psi\in S_1^*\cap\Lambda^*} \Big[\mu\Big(\langle \psi, F(u, y)\rangle_{Z^*;Z}\Big)\Big] \langle \lambda^*, b\rangle_{Z^*;Z} \to \inf, \tag{6.1}$$

$$A(u, y) = f, \tag{6.2}$$

$$y \in K, \quad u \in U_\partial \subset U, \tag{6.3}$$

where λ^* is an element of the dual cone $\Lambda^* = \Big\{w \in Z^*\ :\ \langle w, b\rangle_{Z^*;Z} \geq 0\ \forall\, b \in \Lambda\Big\}$.

We begin with the following concept [22, 26, 30].

Definition 6.1. *We say that $\lambda^* \in Z^*$ is a quasi-interior point of the dual cone Λ^* if $\lambda^* \in \Lambda^*$ and $\langle \lambda^*, b\rangle_{Z^*;Z} > 0$ for all $b \in \Lambda \setminus \{0\}$.*

We denote by $\Lambda^\sharp$ the set of all quasi-interior points to Λ^*. Note that, in general, we have the inclusion $\operatorname{int}\Lambda \subseteq \Lambda^\sharp$ (we refer for instance to [26]). However, we suppose that $\Lambda^\sharp \neq \emptyset$ even if $\operatorname{int}\Lambda = \emptyset$ (see [22]).

Now we can give the main property of the scalar problems associated with the vector problem by the rule (6.1).

Theorem 6.2. *Assume that the vector optimization problem* (2.1)–(2.4) *is regular. Let λ^* be any element of $\Lambda^\sharp$, and let $(u^0, y^0) \in \operatorname*{Argmin}_{(u,y)\in\Xi} \langle \lambda^*, I(u, y)\rangle_{Z^*;Z}$. Then*

$$(u^0, y^0) \in \mathrm{Sol}_w(\Xi;\ I;\ \Lambda). \tag{6.4}$$

Proof. By assumption, we have

$$\big\langle \lambda^*, I(u^0,y^0) - I(u,y)\big\rangle_{Z^*;Z} \le 0, \quad \forall\,(u,y)\in\Xi. \tag{6.5}$$

Let z be any fixed element of the set $\mathrm{cl}_w\, I(\Xi)$. Then there exists a sequence $\{(u_k,y_k)\}_{k=1}^{\infty} \subset \Xi$ such that $I(u_k,y_k) \rightharpoonup z$ in Z as $k\to\infty$. Hence, in view of (6.5), we get

$$\big\langle \lambda^*, I(u^0,y^0) - I(u_k,y_k)\big\rangle_{Z^*;Z} \le 0, \quad \forall\,k\in\mathbb{N}. \tag{6.6}$$

Passing to the limit in (6.6) as $k\to\infty$, we obtain

$$\big\langle \lambda^*, I(u^0,y^0) - z\big\rangle_{Z^*;Z} \le 0, \quad \forall\,z\in \mathrm{cl}_w\, I(\Xi). \tag{6.7}$$

Let us assume the converse, that is, $(u^0,y^0)\not\in \mathrm{Sol}_w(\Xi;\,I;\,\Lambda)$. Then we may find an element $h\in \mathrm{cl}_w\, I(\Xi)$ such that $h <_\Lambda I(u^0,y^0)$, i.e., $I(u^0,y^0)-h\in\Lambda\setminus\{0\}$. Hence, by Definition 6.1, $\langle\lambda^*, I(u^0,y^0)-h\rangle_{Z^*;Z} > 0$, and we come to the contradiction with (6.7). So, $(u^0,y^0)\in \mathrm{Sol}_w(\Xi;\,I;\,\Lambda)$ and this concludes the proof. ∎

Remark 6.3. *Note that Theorem 6.2 generally fails when $\lambda\in\Lambda^*\setminus\Lambda^\sharp$. Indeed, if we take $\lambda^* = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ in Example 4.6, then $\langle\lambda^*, I(x)\rangle_{Z^*;Z} = x$ and hence $\operatorname*{Argmin}_{x\in[1,2]}\langle\lambda^*, I(x)\rangle_{Z^*;Z} = \{1\}$. However, as it was shown before, $\mathrm{Sol}_w(\Xi;\,I;\,\Lambda)=\emptyset$ and $\mathrm{Eff}\,(\Xi;\,I;\,\Lambda) = \mathrm{Eff}_w\,(\Xi;\,I;\,\Lambda) = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$.*

In fact, Theorem 6.2 immediately leads us to the conclusion

$$\bigcup_{\lambda^*\in\Lambda^\sharp} \operatorname*{Argmin}_{(u,y)\in\Xi}\langle\lambda^*, I(u,y)\rangle_{Z^*;Z} \subseteq \mathrm{Sol}_w(\Xi;\,I;\,\Lambda),$$

which does not seem to be an important result from a practical point of view. Indeed, as the following examples show, for Λ_w-lower semicontinuous mappings $I : U\times Y\to Z$ it is possible to have a situation when none of the scalar functions $(u,y)\mapsto\langle\lambda^*, I(u,y)\rangle_{Z^*;Z}$ is lower semicontinuous for any $\lambda^*\in\Lambda^\sharp$.

Example 6.4. [21] *Let $\Xi=[1,2]\subset\mathbb{R}$, and let $\Lambda=\mathbb{R}^2_+$ be the cone of positive elements in $\mathbb{R}^2$. Consider the mapping $I:\Xi\to\mathbb{R}^2$ defined by (see Fig. 5)*

$$I(x) = \begin{cases} \left[\begin{smallmatrix}x\\1\end{smallmatrix}\right], & \text{if } x\in[1,2]\setminus\{1+1/k,\ k\in\mathbb{N}\},\\ \left[\begin{smallmatrix}0\\1+k\end{smallmatrix}\right], & \text{if } x = 1+1/k,\ k\in\mathbb{N}. \end{cases}$$

Straightforward calculations show that

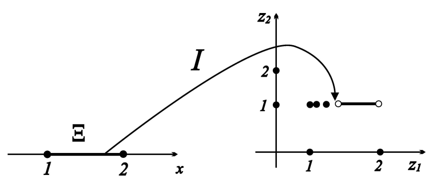

Figure 5: The vector-valued mapping in Example 6.4

$$\Lambda_w - \liminf_{x\to 1}\widehat{I}(x) = \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}, \quad \Lambda_w - \liminf_{x\to(1+1/k)}\widehat{I}(x) = \left\{\begin{bmatrix}0\\1+k\end{bmatrix}, \begin{bmatrix}1+1/k\\1\end{bmatrix}\right\}.$$

Since $I(1) \in \Lambda_w - \liminf_{x \to 1} \widehat{I}(x)$ *and* $I(1+1/k) \in \Lambda_w - \liminf_{x \to (1+1/k)} \widehat{I}(x)$, *it means that the mapping* $I : \Xi \to \mathbb{R}^2$ *is* Λ_w*-lower semicontinuous at these points and in fact on the whole domain* Ξ. *Let* $\lambda^* = \left[\begin{smallmatrix} \lambda_1 \\ \lambda_2 \end{smallmatrix}\right]$ *be any vector with non-negative components, i.e.* $\lambda^* \in \Lambda^*$. *Then the scalar function* I_{λ^*}, *associated with the vector-valued mapping* I *by the scheme of the "weighted sum", can be represented in the form*

$$I_{\lambda^*}(x) := \langle \lambda^*, I(x) \rangle_{Z^*;Z} = \begin{cases} \lambda_1 x + \lambda_2, & \text{if } x \neq 1 + 1/k, \\ \lambda_2(1+k), & \text{if } x = 1 + 1/k, \end{cases} \quad \forall\, k \in \mathbb{N}\, \forall\, x \in \Xi. \tag{6.8}$$

To be sure that the lower semicontinuity property for this function at the points $x_k = 1 + 1/k$ *is valid, we have to choose the parameters* λ_1 *and* λ_2 *so that the inequality*

$$\lambda_2(1+k) \leq \lambda_1(1+1/k) + \lambda_2 \tag{6.9}$$

holds true for every $k \in \mathbb{N}$.

However, taking into account the non-negativeness of λ_i *and passing in* (6.9) *to the limit as* $k \to \infty$, *we obtain* $\lambda_2 = 0$. *As a result, we have*

$$I_{\lambda^*}(x) = \begin{cases} \lambda_1 x, & \text{if } x \neq 1 + 1/k, \\ 0, & \text{if } x = 1 + 1/k, \end{cases} \quad \forall\, k \in \mathbb{N}\, \forall\, x \in \Xi. \tag{6.10}$$

Nevertheless, as follows from (6.10), *the inequality* $I_{\lambda^*}(1) \leq \liminf_{k \to \infty} I_{\lambda^*}(x_k)$ *does not hold for any* $\lambda_1 > 0$ *with the exception of* $\lambda_1 = 0$. *Thus, there is a unique scalar function in the collection* (6.8) *satisfying the lower semicontinuity property in the domain* $\Xi = [1, 2]$. *This function is* $I_{\lambda^*}(x) = 0$.

The next example shows a vector-value mapping $I : \Xi \to Z$ not quasi lower semicontinuous at any point of Ξ, whereas it is Λ_w-lower semicontinuous on Ξ.

Example 6.5. *Let* Ξ *be a bounded closed convex subset of a reflexive infinite-dimensional Banach space* X, *let* $Z = \mathbb{R}^2$, *and let* $\Lambda = \mathbb{R}^2_+$ *be the cone of positive elements in* $\mathbb{R}^2$. *Let us consider the mapping* $I : \Xi \to \mathbb{R}^2$ *defined as follows*

$$I(x) = \begin{bmatrix} \|x\| \\ -\|x\| \end{bmatrix}, \quad \forall\, x \in \Xi.$$

Then $I(\Xi)$ *is a segment*

$$\mathfrak{D} = \left\{ z \in \mathbb{R}^2 \ : \ z = \alpha \begin{bmatrix} m \\ -m \end{bmatrix} + (1-\alpha) \begin{bmatrix} M \\ -M \end{bmatrix},\ \alpha \in [0,1] \right\},$$

where $m = \min_{x \in \Xi} \|x\|$ *and* $m = \max_{x \in \Xi} \|x\|$. *Hence each element of* Ξ *is a* Λ_w*-efficient solution to the corresponding problem* $\langle \Xi, I, \Lambda \rangle$ *since* $\Lambda_w - \operatorname{Inf}_{x \in \Xi} I(x) = \mathfrak{D}$. *However, since*

$$\liminf_{k \to \infty} \|x_k\| \geq \|x\|, \quad \forall\, x_k \rightharpoonup x \ \text{ in } \ X, \tag{6.11}$$

and equality sometimes fails, the lower and quasi lower semicontinuity properties for $I : \Xi \to \mathbb{R}^2$ *do not hold at any points* $x \in \Xi$.

At the same time for every $x_0 \in \Xi$ *we have* $\mathrm{L}_w(I, x_0) := \bigcup_{x_k \rightharpoonup x_0} \mathrm{L}\{I(x_k)\} \subset \Lambda_w - \operatorname{Inf}_{x \in \Xi} I(x)$, *and* $I(x_0) \in \Lambda_w - \operatorname{Inf}_{x \in \Xi} I(x)$ *due to* (6.11). *Thus, the objective function* $I : \Xi \to Z$ *is sequentially* Λ_w*-lower semicontinuous at each point of* Ξ. *Let* $\lambda^* = \left[\begin{smallmatrix} \lambda_1 \\ \lambda_2 \end{smallmatrix}\right]$ *be a vector such that* $\lambda_2 > \lambda_1 > 0$, *whence* $\lambda^* \in \Lambda^\sharp$. *Then the scalar function* I_{λ^*}, *associated with the vector-valued mapping* I, *takes the form* $I_{\lambda^*}(x) = (\lambda_1 - \lambda_2)\|x\|$. *As a result, we come to the same conclusion as in the* *previous example: none of these scalar functions is lower semicontinuous with respect to the weak topology of X.*

7. Generalized Solutions and Well-Posed Scalarized Problems

Let us rewrite the scalar minimization problem (6.1)–(6.3) as follows

$$\min J_{\varepsilon,\lambda^*}(u,y), \quad (u,y)\in\overline{\Xi}, \tag{7.1}$$

where $J_{\varepsilon,\lambda^*}(u,y)=\langle\lambda^*,I_\varepsilon(u,y)\rangle_{Z^*;Z}$ and $\overline{\Xi}$ is the set of admissible solutions.

Denote by

$$\mathrm{Sol}(\overline{\Xi};\ J_{\varepsilon,\lambda^*}):=\operatorname*{Argmin}_{(u,y)\in\overline{\Xi}} J_{\varepsilon,\lambda^*}(u,y)$$

the solution set to the problem (7.1). We recall that the problem (7.1) is said to be well-posed in the generalized sense when every minimizing sequence $\{(u_k,y_k)\}_{k=1}^\infty\subset\overline{\Xi}$ (i.e. such that $J_{\varepsilon,\lambda^*}(u_k,y_k)\to\inf_{(u,y)\in\overline{\Xi}}J_{\varepsilon,\lambda^*}(u,y)$) has a subsequence w-converging to some pair of $\mathrm{Sol}(\overline{\Xi};\ J_{\varepsilon,\lambda^*})$. We recall also a generalization of the above mentioned notion. The problem (7.1) is said to be well-set when every minimizing sequence contained in $\overline{\Xi}\setminus\mathrm{Sol}(\overline{\Xi};\ J_{\varepsilon,\lambda^*})$ has a w-cluster pair in $\mathrm{Sol}(\overline{\Xi};\ J_{\varepsilon,\lambda^*})$. However, as follows from the arguments of the previous section, the problem (6.1)–(6.3) is neither well-posed nor well-set, in general. The main reason is the Λ_w-lower semicontinuity property of the objective function I.

In view of this we introduce the following notion.

Definition 7.1. *We say that a pair $(u^*,y^*)\in\Xi$ is a generalized Λ_w-solution of the vector optimization problem $\langle\Xi,I,\Lambda\rangle$ if there exist a sequence $\{(u_k,y_k)\}_{k=1}^\infty\subset\overline{\Xi}$ and an element $\xi\in\Lambda_w-\operatorname*{Inf}_{(u,y)\in\Xi}I(u,y)$ such that $(u_k,y_k)\overset{w}{\rightharpoonup}(u^*,y^*)$ and $I(u_k,y_k)\rightharpoonup\xi$ in Z.*

We denote by $\mathrm{GenSol}_w(\Xi;I;\Lambda)$ the set of all generalized Λ_w-solutions to problem $\langle\Xi,I,\Lambda\rangle$. It is clear that $\mathrm{Sol}_w(\Xi;\ I;\ \Lambda)\subseteq\mathrm{GenSol}_w(\Xi;I;\Lambda)$. However, as the following example indicates, the inclusion $\mathrm{GenSol}_w(\Xi;I;\Lambda)\subset\mathrm{Eff}(\Xi;\ I;\ \Lambda)$ does not generally hold.

Example 7.2. *Let Ξ be a unit ball in some normed space X centered at the origin, that is, $\Xi=\{x\in X\ :\ \|x\|\le1\}$. Let $S=\partial\Xi$ be the unit sphere in X, let $Z=\mathbb{R}^2$, and let $\Lambda=\mathbb{R}^2_+$ be the cone of positive elements in $\mathbb{R}^2$. Let us consider the mapping $I:\Xi\to\mathbb{R}^2$ defined by*

$$I(x)=\begin{bmatrix}1+\|x\|\\1+\|x\|\end{bmatrix}\ \textit{if}\ x\in\Xi\setminus\{0\cup S\},\ I(x)=\begin{bmatrix}1\\2\end{bmatrix}\ \textit{if}\ x\in S,\ I(0)=\begin{bmatrix}2\\1\end{bmatrix}.$$

Then $\Lambda-\mathrm{Min}(I(\Xi))=\left\{\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right],\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]\right\}$, $\mathrm{Eff}(\Xi;\ I;\ \Lambda)=\{0\}\cup S$, and $\Lambda_w-\operatorname*{Inf}_{x\in\Xi}I(x)=\left\{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}$. Hence, $\mathrm{Sol}_w(\Xi;\ I;\ \Lambda)=\emptyset$. However, the set of generalized Λ_w-solutions of the problem $\langle\Xi,I,\Lambda\rangle$ is nonempty. Indeed, let us fix a sequence $\{x_k\}_{k=1}^\infty\subset\Xi$ such that $x_k\rightharpoonup0$ in X and $I(x_k)\to\left\{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}$. Then, in view of Definition 7.1, we have $0\in\mathrm{GenSol}_w(\Xi;I;\Lambda)$ and, in fact, $\mathrm{GenSol}_w(\Xi;I;\Lambda)=\{0\}$.

Having taken $\lambda^*=\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, we consider the following scalar problem associated with the vector problem $\langle\Xi,I,\Lambda\rangle$:

$$I_{\lambda^*}(x):=\langle\lambda^*,I(x)\rangle_{Z^*;Z}=\begin{cases}1+\|x\|, & \textit{if}\ \|x\|<1\ \textit{and}\ x\neq0,\\ 1, & \textit{if}\ \|x\|=1,\\ 2, & \textit{if}\ x=0\end{cases} \tag{7.2}$$

It is a matter of direct verification to show that $\operatorname*{Argmin}_{x\in\Xi}I_{\lambda^*}(x)=\{x\in\Xi\ :\ \|x\|=1\}$. As a result, we have $\mathrm{GenSol}_w(\Xi;I;\Lambda)\cap\operatorname*{Argmin}_{x\in\Xi}I_{\lambda^*}(x)=\emptyset$. Thus, any solution of the scalar

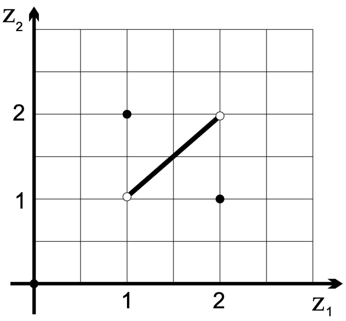

Figure 6: The set $I(\Xi)$ to Example 7.2

problem (7.2) *is neither a Λ_w-efficient solution nor a generalized one to the vector problem $\langle \Xi, I, \Lambda\rangle$.*

We begin with the following result.

Lemma 7.3. *Assume that the properties (A1)–(A6) hold true. Let $\mathrm{sc}_w \langle \lambda^*, I(u,y)\rangle_{Z^*;Z}$ be the lower semicontinuous envelope of the functional $\langle \lambda^*, I(u,y)\rangle_{Z^*;Z}$ with respect to the w-convergence in $U \times Y$, where $\lambda^* \in \Lambda^*$. Then the optimal control problem*

$$\widetilde{J}_{\varepsilon,\lambda^*}(u,y) = \mathrm{sc}_w \langle \lambda^*, I(u,y)\rangle_{Z^*;Z} + \varepsilon^{-1} \sup_{\psi \in S_1^* \cap \Lambda^*} \Big[\mu\Big(\langle \psi, F(u,y)\rangle_{Z^*;Z}\Big)\Big] \langle \lambda^*, b\rangle_{Z^*;Z} \to \inf, \tag{7.3}$$

$$A(u,y) = f, \tag{7.4}$$

$$y \in K, \quad u \in U_\partial \subset U, \tag{7.5}$$

has a nonempty set of solutions for every $\varepsilon > 0$, $\forall\, \mu \in \mathcal{F}(\mathbb{R}, \mathbb{R}_+)$, and $\forall\, b \in \Lambda \setminus 0$.

Proof. First of all, we show that the cost functional $\widetilde{J}_{\varepsilon,\lambda^*} : U_\partial \times Y \to \overline{\mathbb{R}}$ is bounded below on the set $\overline{\Xi}$. Let us assume the converse. Then there exists a sequence $\{(u_k, y_k)\}_{k=1}^\infty \subset \overline{\Xi}$ such that $\widetilde{J}_{\varepsilon,\lambda^*}(u_k, y_k) < -k$ for all $k \in \mathbb{N}$. Due to the initial assumptions and using the same arguments as in Lemma 5.3, it can be shown that there exists a pair $(\widehat{u}, \widehat{y}) \in \overline{\Xi}$ such that, passing to a subsequence if necessary, we obtain $(u_k, y_k) \xrightarrow{w} (\widehat{u}, \widehat{y})$. Then, having used the sequentially lower semi-continuity of the cost functional $\widetilde{J}_{\varepsilon,\lambda^*}$ with respect to the w-convergence and non-negativeness of the term

$$\sup_{\psi \in S_1^* \cap \Lambda^*} \Big[\mu\Big(\langle \psi, F(u,y)\rangle_{Z^*;Z}\Big)\Big] \langle \lambda^*, b\rangle_{Z^*;Z},$$

we come to the contradiction

$$\widetilde{J}_{\varepsilon,\lambda^*}(\widehat{u}, \widehat{y}) \le \liminf_{k\to\infty} \widetilde{J}_{\varepsilon,\lambda^*}(u_k, y_k) \le \liminf_{k\to\infty} \widetilde{J}_{\varepsilon,\lambda^*}(u_k, y_k) < -\infty.$$

Thus the cost functional $\widetilde{J}_{\varepsilon,\lambda^*} : U_\partial \times Y \to \overline{\mathbb{R}}$ is bounded below on the set $\overline{\Xi}$.

Let $\{(u_k, y_k)\}_{k=1}^\infty \subset \overline{\Xi}$ be a minimizing sequence of admissible pairs to the problem (7.3)–(7.5). By the previous arguments, this sequence is bounded in $U \times Y$. Since $\overline{\Xi} \subset U_\partial \times K$ and the set $U_\partial \times K$ is sequentially closed with respect to the w-convergence, we may assume that there exists a pair $(u_\varepsilon^0, y_\varepsilon^0) \in U_\partial \times K$ such that $(u_k, y_k) \xrightarrow{w} (u_\varepsilon^0, y_\varepsilon^0)$. Then, in view of the $(\mathfrak{M})$-property of the operator $A : U \times Y \to Y^*$, and taking into account that $A(u_k, y_k) = f$ for all $k \in \mathbb{N}$, we just conclude: $A(u_\varepsilon^0, y_\varepsilon^0) = f$. Thus, the limit pair $(u_\varepsilon^0, y_\varepsilon^0)$ is an admissible pair to the problem (7.3)–(7.5).

Let us show that $(u_\varepsilon^0, y_\varepsilon^0) \in \overline{\Xi}$ is an optimal pair to this problem. Indeed, as was noted in Lemma 5.3, the function

$$\sup_{\psi \in S_1^* \cap \Lambda^*} \left[\mu \left(\langle \psi, F(u,y) \rangle_{Z^*;Z} \right) \right]$$

is semi-continuous with respect to the w-convergence. Therefore,

$$\begin{aligned}
\inf_{(u,y)\in\overline{\Xi}} \widetilde{J}_{\varepsilon,\lambda^*}(u,y) &= \liminf_{k\to\infty} \widetilde{J}_{\varepsilon,\lambda^*}(u_k,y_k) \\
&\geq \mathrm{sc}_w \left\langle \lambda^*, I(u_\varepsilon^0, y_\varepsilon^0) \right\rangle_{Z^*;Z} \\
&\quad + \varepsilon^{-1} \liminf_{k\to\infty} \sup_{\psi \in S_1^*\cap\Lambda^*} \left[\mu \left(\langle \psi, F(u_k,y_k) \rangle_{Z^*;Z} \right) \right] \langle \lambda^*, b \rangle_{Z^*;Z} \\
&\geq \widetilde{J}_{\varepsilon,\lambda^*}(u_\varepsilon^0, y_\varepsilon^0),
\end{aligned}$$

and we obtain the required conclusion: $(u_\varepsilon^0, y_\varepsilon^0)$ is an optimal pair to the penalized problem (7.3)–(7.5). ∎

Let us denote by $\operatorname*{Argmin}_{(u,y)\in\overline{\Xi}} \widetilde{J}_{\varepsilon,\lambda^*}(u,y)$ the set of optimal pairs to the problem (7.3)–(7.5) for fixed $\varepsilon > 0$, $\mu \in \mathcal{F}(\mathbb{R}, \mathbb{R}_+)$, $\lambda^* \in \Lambda^*$, and $b \in \Lambda$.

Lemma 7.4. *Under the assumptions of Lemma 7.3, the following inclusion is valid:*

$$\bigcup_{\lambda^*\in\Lambda^\sharp} \operatorname*{Argmin}_{(u,y)\in\overline{\Xi}} \widetilde{J}_{\varepsilon,\lambda^*}(u,y) \subseteq \mathrm{GenSol}_w\left(\overline{\Xi}; I_\varepsilon; \Lambda\right). \tag{7.6}$$

Proof. For an arbitrary $\lambda^* \in \Lambda^\sharp$, let us fix a pair

$$(u_\varepsilon^*, y_\varepsilon^*) \in \operatorname*{Argmin}_{(u,y)\in\overline{\Xi}} \widetilde{J}_{\varepsilon,\lambda^*}(u,y). \tag{7.7}$$

Since $\mathrm{sc}_w \langle \lambda^*, I(u,y) \rangle_{Z^*;Z}$ is the lower w-semicontinuous envelope of the functional

$$\langle \lambda^*, I(u,y) \rangle_{Z^*;Z},$$

it follows that there exists a sequence $\{(u_k,y_k)\}_{k=1}^\infty \subset \Xi$ such that $(u_k,y_k) \xrightarrow{w} (u_\varepsilon^*, y_\varepsilon^*)$ and

$$\begin{aligned}
\lim_{k\to\infty} \langle \lambda^*, I_\varepsilon(u_k,y_k) \rangle_{Z^*;Z} = \widetilde{J}_{\varepsilon,\lambda^*}(u_\varepsilon^*, y_\varepsilon^*) &\leq \text{ (by condition (7.7)) } \\
\leq \widetilde{J}_{\varepsilon,\lambda^*}(u,y) \leq \langle \lambda^*, I_\varepsilon(u,y) \rangle_{Z^*;Z} &\quad \forall (u,y) \in \Xi.
\end{aligned} \tag{7.8}$$

Since $\lambda^* \in \Lambda^\sharp$, by (7.8) the sequence $\{I_\varepsilon(u_k,y_k)\}_{k=1}^\infty$ is bounded in Z. So, we may suppose the existence of an element $\eta \in Z$ such that $I_\varepsilon(u_k,y_k) \rightharpoonup \eta$ in Z as $k \to \infty$.

For now we assume that

$$(u_\varepsilon^*, y_\varepsilon^*) \notin \mathrm{GenSol}_w\left(\overline{\Xi}; I_\varepsilon; \Lambda\right). \tag{7.9}$$

Then, as follows from Definition 7.1, $\eta \notin \Lambda_w - \operatorname*{Inf}_{(u,y)\in\overline{\Xi}} I_\varepsilon(u,y)$. Hence, there can be found an element $\xi \in \Lambda_w - \operatorname*{Inf}_{(u,y)\in\overline{\Xi}} I_\varepsilon(u,y)$ such that $\xi <_\Lambda \eta$. Therefore $\eta - \xi \in \Lambda \setminus \{0\}$, and using the fact that $\lambda \in \Lambda^\sharp$, we come to the inequality

$$\langle \lambda^*, \eta \rangle_{Z^*;Z} > \langle \lambda^*, \xi \rangle_{Z^*;Z} \text{ which is equivalent to } \lim_{k\to\infty} \langle \lambda^*, I_\varepsilon(u_k,y_k) \rangle_{Z^*;Z} > \langle \lambda^*, \xi \rangle_{Z^*;Z}. \tag{7.10}$$

On the other hand, there exists a sequence $\{(v_k, p_k)\}_{k=1}^{\infty} \subset \overline{\Xi}$ such that $I_\varepsilon(v_k, p_k) \rightharpoonup \xi$ in Z. Since the set $\overline{\Xi}$ is sequentially compact with respect to the w-convergence (see Lemma 5.3), we may suppose that $(v_k, p_k) \overset{w}{\rightharpoonup} (v^*, p^*) \in \overline{\Xi}$. Then, by inequality (7.8), we deduce

$$\lim_{k\to\infty} \langle \lambda^*, I_\varepsilon(u_k, y_k)\rangle_{Z^*;Z} \le \langle \lambda^*, I_\varepsilon(v_i, p_i)\rangle_{Z^*;Z}\,, \quad \forall\, i \in \mathbb{N}. \tag{7.11}$$

Passing to the limit in (7.11) as $i \to \infty$, we get

$$\lim_{k\to\infty} \langle \lambda^*, I_\varepsilon(u_k, y_k)\rangle_{Z^*;Z} \le \langle \lambda^*, \xi\rangle_{Z^*;Z}\,.$$

However, this contradicts (7.10) and (7.9), concluding the proof. ∎

Before proceeding further, we recall the concept of the upper limit in the sense of Kuratowski of a set sequence $\{C_\varepsilon\}_{\varepsilon\to 0} \subset U \times Y$. Then the weak upper limit, in the sense of Kuratowski of the set sequence $\{C_\varepsilon\}_{\varepsilon>0} \subset U \times Y$ with respect to the w-convergence in $U \times Y$, is defined by

$$w\!-\limsup_{\varepsilon\to 0} C_\varepsilon = \left\{ (u, y) \in U \times Y \ :\ \exists (u_{\varepsilon_k}, y_{\varepsilon_k}) \in C_{\varepsilon_k} \text{ such that } \left| \begin{array}{l} \varepsilon_k \to 0 \text{ as } k \to \infty, \\ (u_{\varepsilon_k}, y_{\varepsilon_k}) \overset{w}{\rightharpoonup} (u, y) \end{array} \right. \right\}.$$

We are now in a position to prove our main result.

Theorem 7.5. *Assume that (A1)–(A6) hold and the vector optimization problem* (2.1)–(2.4) *is regular. Let* ε *be a small scalar parameter varying in a strictly decreasing sequence of positive numbers which converge to* 0*, and let* λ^* *be an element of* $\Lambda^\sharp$*. Then*

$$w\!-\limsup_{\varepsilon\to 0} \left[\operatorname*{Argmin}_{(u,y)\in\overline{\Xi}} \widetilde{J}_{\varepsilon,\lambda^*}(u, y) \right] \subseteq \mathrm{GenSol}_w\,(\Xi; I; \Lambda). \tag{7.12}$$

Proof. Let $\left\{ (u^*_\varepsilon, y^*_\varepsilon) \in \operatorname*{Argmin}_{(u,y)\in\overline{\Xi}} \widetilde{J}_{\varepsilon,\lambda^*}(u, y) \right\}_{\varepsilon>0}$ be a sequence of optimal pairs to the corresponding minimization problems (7.3)–(7.5). Then, in view of Lemma 7.4, we have $(u^*_\varepsilon, y^*_\varepsilon) \in \mathrm{GenSol}_w\,(\overline{\Xi}; I_\varepsilon; \Lambda)$ for every $\varepsilon > 0$. Our aim is to show that each w-cluster point of this sequence is a generalized Λ_w-solution of the original vector optimization problem (2.1)–(2.4).

By analogy with Lemma 5.4, it can be shown that the sequence $\{(u^*_\varepsilon, y^*_\varepsilon)\}_{\varepsilon>0}$ is relatively w-compact. Hence a subsequence of $\{(u^*_\varepsilon, y^*_\varepsilon)\}_{\varepsilon>0}$, still denoted by the suffix ε, can be extracted such that $(u^*_\varepsilon, y^*_\varepsilon) \overset{w}{\rightharpoonup} (u^*, y^*)$ as ε tends to zero, where $(u^*, y^*) \in \overline{\Xi}$.

Let us show that this pair is admissible for the problem (2.1)–(2.4). To do so, it is enough to prove the inequality $F(u^*, y^*) \ge_\Lambda 0$. By the initial assumptions, we have

$$\mu\left(\langle \psi, F(u, y)\rangle_{Z^*;Z} \right) = 0 \ \text{ for any } \ (u, y) \in \Xi.$$

Therefore

$$\widetilde{J}_{\varepsilon,\lambda^*}(u^*_\varepsilon, y^*_\varepsilon) \le \widetilde{J}_{\varepsilon,\lambda^*}(u, y) \equiv \mathrm{sc}_w \,\langle \lambda^*, I(u, y)\rangle_{Z^*;Z} \quad \text{for every } \varepsilon > 0.$$

Whence

$$\sup_{\psi\in S_1^*\cap\Lambda^*} \left[\mu\left(\langle \psi, F(u^*_\varepsilon, y^*_\varepsilon)\rangle_{Z^*;Z} \right) \right] \langle \lambda^*, b\rangle_{Z^*;Z} \le \varepsilon C,$$

where the constant C is independent of both ε and $\psi \in S_1^* \cap \Lambda^*$. Then, using the w-lower semi-continuity property of the scalar function $\sup_{\psi\in S_1^*\cap\Lambda^*}\left[\mu\left(\langle\psi, F(u,y)\rangle_{Z^*;Z}\right)\right]$, we have

$$\begin{aligned}&\sup_{\psi\in S_1^*\cap\Lambda^*}\left[\mu\left(\langle\psi, F(u^*,y^*)\rangle_{Z^*;Z}\right)\right]\langle\lambda^*, b\rangle_{Z^*;Z}\\ &\le \liminf_{\varepsilon\to 0}\sup_{\psi\in S_1^*\cap\Lambda^*}\left[\mu\left(\langle\psi, F(u_\varepsilon^*,y_\varepsilon^*)\rangle_{Z^*;Z}\right)\right]\langle\lambda^*, b\rangle_{Z^*;Z} = 0.\end{aligned}$$

Since this is equivalent to the inequality $F(u^*,y^*) \ge 0$, it follows that the w-cluster pair (u^*,y^*) is admissible to the original problem (2.1)–(2.4).

Our next step is to prove that the pair $(u^*,y^*) \in \Xi$ is an optimal one to the minimization problem

$$\widetilde{J}_{\lambda^*}(u,y) := \mathrm{sc}_w\,\langle\lambda^*, I(u,y)\rangle_{Z^*;Z} \to \inf, \tag{7.13}$$

$$A(u,y) = f, \quad F(u,y) \ge_\Lambda 0, \tag{7.14}$$

$$y \in K, \quad u \in U_\partial \subset U. \tag{7.15}$$

Let us assume the converse. Namely, there is a pair $(\widehat{u},\widehat{y}) \in \Xi$ such that $\widetilde{J}_{\lambda^*}(\widehat{u},\widehat{y}) < \widetilde{J}_{\lambda^*}(u^*,y^*)$. Then this pair is admissible to the penalized problem (7.3)–(7.5). Hence

$$\widetilde{J}_{\lambda^*}(\widehat{u},\widehat{y}) \equiv \widetilde{J}_{\varepsilon,\lambda^*}(\widehat{u},\widehat{y}) \ge \inf_{(u,y)\in\overline{\Xi}}\widetilde{J}_{\varepsilon,\lambda^*}(u,y) = \widetilde{J}_{\varepsilon,\lambda^*}(u_\varepsilon^*,y_\varepsilon^*), \quad \forall\,\varepsilon > 0, \tag{7.16}$$

$$\widetilde{J}_{\lambda^*}(\widehat{u},\widehat{y}) \ge \liminf_{\varepsilon\to 0}\widetilde{J}_{\varepsilon,\lambda^*}(u_\varepsilon^*,y_\varepsilon^*) \ge \liminf_{\varepsilon\to 0}\mathrm{sc}_w\,\langle\lambda^*, I(u_\varepsilon^*,y_\varepsilon^*)\rangle_{Z^*;Z} \ge \widetilde{J}_{\lambda^*}(u^*,y^*), \tag{7.17}$$

and it leads us to the contradiction.

Thus $(u^*,y^*) \in \operatorname*{Argmin}_{(u,y)\in\Xi}\widetilde{J}_{\lambda^*}(u,y)$. As a result, using the arguments of the proof in Lemma 7.4, we come to the required conclusion $(u^*,y^*) \in \mathrm{GenSol}_w\,(\Xi; I; \Lambda)$. ∎

Remark 7.6. *In spite of the result of Theorem 7.5, we should note that the inclusion*

$$w\!-\limsup_{\varepsilon\to 0}\left[\mathrm{GenSol}_w\,(\overline{\Xi}; I_\varepsilon; \Lambda)\right] \subseteq \mathrm{GenSol}_w\,(\Xi; I; \Lambda) \tag{7.18}$$

can be wrong in general. Indeed, let $\left\{(u_\varepsilon^*,y_\varepsilon^*) \in \mathrm{GenSol}_w\,(\overline{\Xi}; I_\varepsilon; \Lambda)\right\}_{\varepsilon>0}$ be a sequence of generalized Λ_w-solutions to the penalized vector optimization problem (5.5)–(5.7)*. In view of Lemma 7.4, we can assume that there exists a sequence $\left\{\lambda_\varepsilon^* \in \Lambda^\sharp\right\}_{\varepsilon>0}$ such that*

$$(u_\varepsilon^*,y_\varepsilon^*) \in \operatorname*{Argmin}_{(u,y)\in\overline{\Xi}}\widetilde{J}_{\varepsilon,\lambda_\varepsilon^*}(u,y) \quad \textit{for all } \varepsilon > 0.$$

Moreover, taking into account the structure of the cost functionals (7.3)*, we can suppose that this sequence is compact with respect to the strong topology of Z. Closely following the line of the previous proof, it can be shown that $\{(u_\varepsilon^*,y_\varepsilon^*)\}_{\varepsilon>0}$ is relatively w-compact, and every w-cluster pair (u^*,y^*) belongs to the set Ξ. Then, passing in* (7.3) *(where λ^* should be replaced by λ_ε^*) to the limit as $\varepsilon \to 0$ and using the inequalities* (7.16)–(7.17)*, it is easy to prove that (u^*,y^*) is an optimal pair to the minimization problem* (7.13)–(7.15)*, where the vector λ^* is a strong limit of the sequence $\left\{\lambda_\varepsilon^* \in \Lambda^\sharp\right\}_{\varepsilon>0}$. However, in this case we cannot assert that the vector λ^* is a quasi-interior point of the dual cone Λ^*. So, in general, we only have $\lambda^* \in \Lambda^*$. But, as Example 7.2 indicates, the solutions of the scalar problem* (7.13)–(7.15) *when $\lambda^* \in \Lambda^* \setminus \Lambda^\sharp$ are neither Λ_w-efficient nor generalized solutions to the vector optimization problem $\langle\Xi, I, \Lambda\rangle$.*

8. APPLICATION AND FINAL REMARKS

In this section we illustrate the results obtained above with some examples. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a Lipschitz boundary $\partial\Omega$, and let $\mathcal{D}$ be its subdomain with the characteristic function $\chi_{\mathcal{D}}$. For a given function $f \in L^q(\Omega)$, $\xi \in W_0^{1,p}(\Omega)$, and $v^* \in L^q(\partial\Omega)$, we consider the following control object

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial y}{\partial x_i}\right|^{p-2} \frac{\partial y}{\partial x_i}\right) = f(x), \quad x \in \Omega, \tag{8.1}$$

$$\left.\frac{\partial y}{\partial \nu_A}\right|_{\partial\Omega} = u(x), \quad x \in \partial\Omega, \tag{8.2}$$

$$u \in U_\partial = \left\{v \in L^q(\partial\Omega) \;:\; \|v - v^*\|_{L^q(\partial\Omega)} \leq \alpha\right\}, \tag{8.3}$$

$$|(y(x) - \xi(x))| \leq \beta \quad \text{almost everywhere in } \mathcal{D}. \tag{8.4}$$

Here u is a control function, y is a state of the control object,

$$\frac{\partial y}{\partial \nu_A} = \sum_{i=1}^{n} \left|\frac{\partial y}{\partial x_i}\right|^{p-2} \frac{\partial y}{\partial x_i}\, \nu_i(x),$$

$\alpha > 0$, $\beta > 0$, $p \geq 2$, and $p^{-1} + q^{-1} = 1$.

Let Λ be the cone of positive elements in $L^p(\Omega)$, i.e., $\eta \in \Lambda$ if $\eta(x) \geq 0$ almost everywhere in Ω. It is clear that $\Lambda^\sharp \neq \emptyset$ whereas $\operatorname{int} \Lambda = \operatorname{int} \Lambda^* = \emptyset$. Clearly, this cone is reproducing, since each function $x \in L^p(\Omega)$ can be represented as $x = x_+ - x_-$, where $x_+ = \max\{x, 0\}$, $x_- = \max\{-x, 0\}$, and $x_+, x_- \in \Lambda$. Moreover, as it is shown in [15], this cone possesses the D-property. For every pair $(u, y) \in L^q(\partial\Omega) \times W^{1,p}(\Omega)$ we define an objective mapping $I : L^q(\partial\Omega) \times W^{1,p}(\Omega) \to L^p(\Omega)$ by the rule

$$I(u, y) = \sum_{i=1}^{n} \left|\frac{\partial y(x)}{\partial x_i}\right|. \tag{8.5}$$

Then the vector optimization problem for the object (8.1)–(8.4) we consider can be stated as follows:

$$\text{Minimize } I(u, y) \text{ (with respect to the cone } \Lambda) \tag{8.6}$$

subject to the restrictions (8.1)–(8.4). From the physical point of view it means that we try "to minimize the total oscillation" of the function $y \in W^{1,p}(\Omega)$ which has be pointwise close to the given function ξ upon the domain $\mathcal{D} \subset \Omega$.

To rewrite this problem in the form of the vector optimization problem (2.1)–(2.4), we use the following notations. Let

$$U = L^q(\partial\Omega),\ Y = W^{1,p}(\Omega),\ F(u, y) = (\beta - |y(x) - \xi(x)|)\chi_{\mathcal{D}}(x),\ K = W^{1,p}(\Omega), \tag{8.7}$$

and let $A : L^q(\partial\Omega) \times W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ be the nonlinear operator associated with the boundary value problem (8.1)–(8.2). We define the w-convergence in $U \times Y$ as the weak convergence in $L^q(\partial\Omega) \times W^{1,p}(\Omega)$.

It is easy to see that the boundary value problem (8.1)–(8.2) is non-coercive. Hence we cannot assert that this problem has a solution $y(u) \in W^{1,p}(\Omega)$ for every $u \in U_\partial$. So, we deal with an incorrect problem from the point of view of partial differential equations theory. Nevertheless, if we take $u = \partial\xi/\partial\nu_A|_{\partial\Omega}$ and $\xi \in W_0^{1,p}(\Omega)$ as the unique solution of (8.1) with the Dirichlet boundary condition $y|_{\partial\Omega} = 0$, then the pair (u, ξ) will be admissible to the problem (8.1)–(8.6). So, we may suppose that $\Xi \neq \emptyset$ for the given initial data, and hence the vector optimization problem (8.1)–(8.6) is regular.

Let us verify the hypotheses (A1)–(A6). Taking into account the notations (8.7), the fulfilment of the hypotheses (A1), (A2), (A4), and (A5) is obvious. The quasi-monotonicity property of the operator A (which implies the $(\mathfrak{M})$-property due to Remark 5.7) has been proved in [14]. As for the Λ_w-lower semicontinuity property of the objective mapping (8.5), it immediately follows from the weak continuity of this mapping in $L^p(\Omega)$ with respect to the w-convergence.

Thus, in view of Theorems 5.6–7.5, we can give the following conclusion: the set of Λ_w-efficient solutions to the vector optimization problem (8.1)–(8.6) is nonempty,

$$\mathrm{Sol}_w(\Xi;\ I;\ \Lambda) \subseteq \mathrm{GenSol}_w\,(\Xi; I; \Lambda),$$

and the generalized Λ_w-solutions can be obtained as cluster points of the solutions sequence $\{(u^*_\varepsilon, y^*_\varepsilon)\}_{\varepsilon>0}$ to the following penalized optimal control problem

$$\widetilde{J}_{\varepsilon,\lambda^*}(u,y) = \int_\Omega \lambda^*(x) \left(\sum_{i=1}^n \left| \frac{\partial y(x)}{\partial x_i} \right| \right) dx$$

$$+\varepsilon^{-1} \sup_{\varphi\in S_1^*\cap\Lambda^*} \left[\mu \left(\int_{\mathcal{D}} \varphi(x)\,(\beta - |y(x) - \xi(x)|)\, dx \right) \right] \int_\Omega \lambda^*(x)\, dx \to \inf$$

subject to the restrictions (8.1)–(8.3), where λ^* is any element of $\Lambda^\sharp \subset L^q(\Omega)$.

We conclude the paper with the following observation. As follows from definition of the Λ_w-lower semicontinuity for vector-valued mappings $I : \Xi \to Z$, this property essentially depends on the domain $\Xi \subset U \times Y$. In fact, the assertion: "if $I : U \times Y \to Z$ is a Λ_w-lower semicontinuous mapping then its restriction on any bounded subset $\Xi \subset U \times Y$ preserves this property at every point of Ξ" can be wrong in general. However such a situation is both natural and typical in the vectorial case. Indeed, for different sets of admissible solutions Ξ_1, Ξ_2 $(\Xi_1 \cap \Xi_2 \neq \emptyset)$ and any pair (u_0, y_0) such that $(u_0, y_0) \in \Xi \cap \Xi_2$, the sets $\Lambda_w - \mathrm{Inf}\,\mathrm{L}_w(I(u_0, y_0))$ and $\Lambda_w - \operatorname*{Inf}_{(u,y)\in\Xi_i} I(u,y)$ are not singletons in general. So, the sets

$$\Lambda_w - \mathrm{Inf}\,\mathrm{L}_w(I(u_0, y_0)) \cap \Lambda_w - \operatorname*{Inf}_{(u,y)\in\Xi_1} I(u,y),$$

$$\Lambda_w - \mathrm{Inf}\,\mathrm{L}_w(I(u_0, y_0)) \cap \Lambda_w - \operatorname*{Inf}_{(u,y)\in\Xi_2} I(u,y),$$

can be drastically different as well. Thus, in view of Definition 3.5 and condition (4.2), the mappings $I : \Xi_1 \to Z$ and $I : \Xi_2 \to Z$ can be distinguished by the Λ_w-lower semicontinuity property at the point $(u_0, y_0) \in \Xi \cap \Xi_2$.

REFERENCES

[1] M. AKIAN and I. SINGER, Topologies on lattice ordered groups, separations from closed downwards and conjugations of type Lau, *Optimization*, **52** (2003), No. 6, pp. 629–673.

[2] J. P. AUBIN and I. EKELAND, *Applied Nonlinear Analysis*, John Wiley & Son, N.Y., 1984.

[3] J. P. AUBIN and H. FRANKOWSKA, *Set-Valued Analysis*, Birkhäuser, Cambridge, MA, 1990.

[4] M. AIT MANSOUR, A. METRANE and M. THÉRA, Lower semicontinuous regularization for vector-valued mappings, *Université de Limoges, Rapport de recherche* n° *2004-06*, 2004.

[5] V. BARBU, *Analysis and Control of Infinite Dimensional Systems*, Ser. Math. Sci. Engrg., Vol. 190, Academic Press, N.Y., 1993.

[6] J. M. BORWEIN, J. P. PENOT and , M. THÉRA, Conjugate convex operators, *Math. Anal. & Appl.*, **102** (1984), pp. 399–414.

[7] G. CHEN, X. HUANG and X. YANG, *Vector Optimization. Set-value and Variational Analysis*, Springer, Serie: Lecture Notes in Economics and Mathematical Systems, Vol. 541, 2007.

[8] C. COMBARI, M. LAGHDIR and L. THIBAULT, Sous-différentiel de fonctions convexes composées, *Ann. Sci. Math. Québec*, **18** (1994), No. 2, pp. 119–148.

[9] M. EHRGOTT, *Multicriteria Optimization*, Springer, Berlin, 2005.

[10] H. O. FATTORINI, *Infinite Dimensional Optimization and Control Theory*, Serie: Encyclopedia of Mathematics and its Applications, Vol. 62, Cambridge University Press, 1999.

[11] C. FINET, L. QUARTA and C. TROESTLER, Vector-value variational principles, *Preprint* ♯4, January 19, Institute de Mathématique et d'Informatique, Université de Mons-Hainaut, 2001.

[12] A. V. FURSIKOV, *Optimal Control of Distributed Systems. Theory and Applications*, AMS, N.Y., 2000.

[13] A. D. IOFFE and V. M. TICHOMIROV, *Theory of Extremal Problems*, North Holland, Amsterdam, 1974.

[14] V. I. IVANENKO and V. S. MEL'NIK, *Variational Methods in Optimal Control Problem for Distributed Systems*, Naukova Dumka, Kyiv, 1988 (in Russian).

[15] J. JAHN, *Vector Optimization. Theory, Applications, and Extensions*, Springer-Verlag, Berlin, 2004.

[16] J. JAHN, *Mahtematical Vector Optimization in Partially Ordered Spaces*, Peter Lang, Frankfurt am Main, 1986.

[17] P. I. KOGUT, Topological convergence of epigraphs for vector value mappings in semiordered vector spaces, *Matematytchni Studii*, **12** (1999), No. 2, pp. 161–170.

[18] P. I. KOGUT, R. MANZO and I. V NECHAY, On existence of efficient solutions to vector optimization problems in Banach spaces, *Note di Matematica*, (to appear).

[19] P. I. KOGUT and V. S. MEL'NIK, On one class of extremum problems for nonlinear operator systems, *Cybern. Syst. Anal.*, **34** (1998), No. 6, pp. 894-904.

[20] P. I. KOGUT and I. V NECHAY, On solvability of one class of vector optimization problem, *Naukovi Visti HTUU-KPI*, **4** (2006), pp. 68–79 (in Ukraine).

[21] P. I. KOGUT and I. V NECHAY, On scalarization of one class of vector optimization problems in Banach spaces, *J. Automation and Information Sciences*, **6** (2008), pp. 56–73.

[22] M. A. KRASNOSELSKI, *Positive Solutions of Operator Equations*, Fizmatlit, Moskow, 1962 (in Russian).

[23] J.-L. LIONS, *Control of Distributed Singular Systems*, Gauthier-Villars, Paris, 1985.

[24] D. T. LUC, *Theory of Vector Optimization*, Springer-Verlag, New York, 1989.

[25] D. T. LUC, Scalarization of vector optimization problems, *Journal of Optim. Theory and Appl.*, **55** (2005), No. 1, pp. 85–102.

[26] E. MUSELLI, Upper and lower semicontinuity for set-valued mappings involving constraints, *Journal of Optim. Theory and Appl.*, **106** (2000), No. 3, pp. 527–550.

[27] E. MIGLIERINA and E. MOLHO, Scalarization and its stability in vector optimization, *Journal of Optim. Theory and Appl.*, **114** (2002), pp. 657–670.

[28] J. P. PENOT and M. THÉRA, Semi-continuous mappings in general topology, *Arch. Math.*, **38** (1982), pp. 158–166.

[29] Y. SAWARAGI, H. NAKAYAMA and T. TANINO, *Theory of Multyobjective Optimization*, Academic Press, Orlando, 1985.

[30] M. THÉRA, *Études des fonctions convexes vectorielles semi-continues*, Thèse de 3^e cycle., Université de Pau, 1978.

[31] B. Z. VULICH, *Introduction to Functional Analysis*, Nauka, Moskow, 1967 (in Russian).

[32] P. L. YU, Cone convexity, cone extrem points and non-dominated solutions in decision problems with multiobjectives, *Journal of Optim. Theory and Appl.*, **14** (1974), pp. 319–377.

[33] M. Z. ZGUROVSKY and V. S. MEL'NIK, *Nonlinear Analysis and Control of Physical Processes and Fields*, Springer–Verlag, Berlin, 2004.