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A COINCIDENCE THEOREM FOR TWO KAKUTANI MAPS

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ABSTRACT. In this paper we prove the following theorem: Let X be a nonempty compact convex set in a locally convex Hausdorff topological vector space, D be the set of its extremal points and $F, T : X \multimap X$ two Kakutani maps; if for each nonempty finite subset A of D and for any $x \in \text{co}A$, $F(x) \cap \text{co}A \neq \emptyset$, then F and T have a coincidence point. The proof of this theorem is given first in the case when X is a simplex, then when X is a polytope and finally in the general case. Several reformulations of this result are given in the last part of the paper.

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1. INTRODUCTION AND PRELIMINARIES

A *multimap* (or simply a *map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of Y, that is a function with the *values* $F(x) \subset Y$ for $x \in X$. Given a map $F : X \multimap Y$, the set $\{(x, y) \in X \times Y | y \in F(x)\}$ is called the *graph* of F and the map $F^- : Y \multimap X$ defined by $F^-(y) = \{x \in X | y \in F(x)\}$ for $y \in Y$, is called the *(lower) inverse* of F; the sets $F^-(y)$ are called also the *fibers* of F. For $A \subset X$ let $F(A) = \cup\{F(x) : x \in A\}$.

Assume that X and Y are topological spaces. A map $F : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed subset B of Y the set $\{x \in X : F(x) \cap B \neq \emptyset\}$ is closed in X. If Y is a convex set in a topological vector space, a map $F : X \multimap Y$ is called a Kakutani map if it is u.s.c. with nonempty compact convex values. Let us denote by $\mathcal{K}(X, Y) = \{F : X \multimap Y | F \text{ is Kakutani map}\}$.

In the sequel, we shall use the following conventions and notations. Real Hausdorff topological vector spaces are abbreviated as t.v.s. and real locally convex Hausdorff topological vector spaces as l.c.s. If A is a set in a t.v.s., the standard abbreviations co$A$ and $\overline{A}$ denote the convex hull of A and the closure of A, respectively.

The famous Kakutani-Fan-Glicksberg fixed point theorem [10, 5, 7] asserts that if X is a compact convex subset of a l.s.c., then any map $F \in \mathcal{K}(X, X)$ has a fixed point. Lassonde [12] extends this result proving that if X is as above, any map $F : X \multimap X$ which can be factorized by a finite number of Kakutani maps has a fixed point. As a consequence of Lassonde's result, if $F \in \mathcal{K}(X, Y)$ and $T^- \in \mathcal{K}(Y, X)$ (X convex set in a l.c.s. and Y convex set in a t.v.s.) then F and T have a coincidence point, that is, there exists $x_0 \in X$ such that $F(x_0) \cap T(x_0) \neq \emptyset$.

Simple examples show that in general two maps $F, T \in \mathcal{K}(X, Y)$ have no coincidence point. However two such maps have a coincidence point if Y is a l.c.s., X is a compact convex subset of Y, and for each $x \in X$, $(F(x) - T(x)) \cap [\cup_{\lambda>0}(X - x)] \neq \emptyset$ (see [6]).

The aim of this paper is to prove the following coincidence theorem:

Theorem 1.1. *Let X be a compact convex set in a l.c.s., D be the set of its extremal points and $F \in \mathcal{K}(X, X)$. Suppose that for each non-empty finite subset A of D and for any $x \in coA$, $F(x) \cap coA \neq \emptyset$. Then for any $T \in \mathcal{K}(X, X)$ there is a point $x_0 \in X$ such that $F(x_0) \cap T(x_0) \neq \emptyset$.*

The case when X is a polytope endowed with the Euclidean topology is treated in Section 2. The proof of Theorem 1.1 is given in Section 3. In the same section we give several reformulations of Theorem 1.1.

2. FINITE DIMENSIONAL CASE

If X, Y are two topological spaces a continuous function $f : X \to Y$ is called *universal* [9] if for any continuous function $g : X \to Y$ there exists $x_0 \in X$ such that $f(x_0) = g(x_0)$.

Lemma 2.1. *Let $\Delta = [v_0 v_1 \dots v_n]$ be a simplex. Any continuous function $f : \Delta \to \Delta$ satisfying*

$$f([v_{i_0} v_{i_1} \dots v_{i_k}]) \subset [v_{i_0} v_{i_1} \dots v_{i_k}] \tag{2.1}$$

for each face $[v_{i_0} v_{i_1} \dots v_{i_k}]$ of Δ $(0 \le k \le n)$, is universal.

Proof. Let $g : \Delta \to \Delta$ be a continuous function. For each $t \in \Delta$, let $\lambda_0(t), \lambda_1(t), \dots, \lambda_n(t)$ be its barycentric coordinates. Then, $\lambda_i(t) \ge 0$, $0 \le i \le n$, $\sum_{i=0}^{n} \lambda_i(t) = 1$ and $t = \sum_{i=0}^{n} \lambda_i(t) v_i$.

For $0 \le i \le n$, consider the following closed set

$$M_i := \{t \in \Delta \ | \ \lambda_i(g(t)) \le \lambda_i(f(t))\}.$$

We show that for any face $[v_{i_0}v_{i_1}\dots v_{i_k}]$ of Δ $(0 \le k \le n)$

$$[v_{i_0}v_{i_1}\dots v_{i_k}] \subset \cup_{j=0}^{k} M_{i_j}. \tag{2.2}$$

Suppose to the contrary that for some face $[v_{i_0}v_{i_1}\dots v_{i_k}]$ there exists a point

$$t \in [v_{i_0}v_{i_1}\dots v_{i_k}] \setminus \cup_{j=0}^{k} M_{i_j}.$$

Since $t \notin \cup_{j=0}^{k} M_{i_j}$, $\lambda_{i_j}(f(t)) < \lambda_{i_j}(g(t))$ for $0 \le j \le k$. If $i \notin \{i_0, i_1, \dots, i_k\}$, by (2.1), $\lambda_i(f(t)) = 0$, hence in this case, $\lambda_i(f(t) \le \lambda_i(g(t)))$. It follows the following contradiction:

$$1 = \sum_{i=0}^{n} \lambda_i(f(t)) < \sum_{i=0}^{n} \lambda_i(g(t)) = 1,$$

which proves that (2.2) holds. By the Knaster-Kuratowski-Mazurkiewicz theorem [11] there exists a point $t_0 \in \cap_{i=0}^{n} M_i$. Then $\lambda_i(g(t_0)) \le \lambda_i(f(t_0))$ for $0 \le i \le n$, and since $\sum_{i=0}^{n} \lambda_i(g(t_0)) = \sum_{i=0}^{n} \lambda_i(f(t_0)) = 1$, we infer that $\lambda_i(g(t_0)) = \lambda_i(f(t_0)))$ for all i, hence $f(t_0) = g(t_0)$. ∎

Remark 2.1. For a point $t \in \Delta$ let us denote by $\text{car}_\Delta t$ the *carrier* of t, that is the lowest-dimensional face of Δ that contains t. It is easily seen that condition (2.1) is equivalent to the following: $f(t) \in \text{car}_\Delta t$, for each $t \in \Delta$.

For topological spaces X and Y, a map $F : X \multimap Y$ is said to be *closed* if its graph is a closed subset of $X \times Y$. For convenience we summarize the following facts (see [1]):

Lemma 2.2. *(i) If $F : X \multimap Y$ is u.s.c. with closed values and Y is regular, then F is closed.*
(ii) If $F : X \multimap Y$ is closed and Y is compact Hausdorff, then F is u.s.c.

Theorem 2.3. *Let $\Delta = [v_0v_1\dots v_n]$ be a simplex and $F \in \mathcal{K}(\Delta, \Delta)$. Suppose that for each face $[v_{i_0}v_{i_1}\dots v_{i_k}]$ of Δ $(0 \le k \le n)$ and any point $t \in [v_{i_0}v_{i_1}\dots v_{i_k}]$, $F(t) \cap [v_{i_0}v_{i_1}\dots v_{i_k}] \ne \emptyset$. Then any map $T \in \mathcal{K}(\Delta, \Delta)$ has a coincidence point with F, that is, there exists $\hat{t} \in \Delta$ such that $F(\hat{t}) \cap T(\hat{t}) \ne \emptyset$.*

Proof. For $p = 1, 2, \dots$ let Σ^p be a simplicial subdivision of Δ of mesh lower than $1/p$. Let V^p be the set of vertices of Σ^p. We shall define two functions $f_p, g_p : \Delta \to \Delta$, first on V^p and then on all points of Δ. For each $v^p \in V^p$ choose a point $y^p \in F(v^p) \cap \text{car}_\Delta v^p$ and a point $z^p \in T(v^p)$ and define $f_p(v^p) = y^p$ and $g_p(v^p) = z^p$.

We extend f_p and g_p to all points of Δ by extending them to each of the simplices of Σ^p. Let $\sigma \in \Sigma^p$ and $v_0^p, v_1^p, \dots, v_n^p$ its vertices. Each point $t \in \sigma$ can be written uniquely as $t = \sum_{j=0}^{n} \lambda_j^p v_j^p$, where $\lambda_j^p \ge 0$, $\sum_{j=0}^{n} \lambda_j^p = 1$. Put

$$\begin{cases} f_p(t) = \sum_{j=0}^{n} \lambda_j^p f_j(v_j^p) = \sum_{j=0}^{n} \lambda_j^p y_j^p, \\ g_p(t) = \sum_{j=0}^{n} \lambda_j^p g_j(v_j^p) = \sum_{j=0}^{n} \lambda_j^p z_j^p. \end{cases} \tag{2.3}$$

It is easily seen that f_p, g_p are well defined and continuous for all of the points in Δ. We need only observe that, if $t \in \sigma_i \cap \sigma_j$, $i \ne j$, then the value of f_p (resp., g_p) computed for t viewed as a point in σ_i is equal to the value of f_p (resp., g_p) for t viewed as a point in σ_j.

Note also that if t is an arbitrary point in Δ and σ is a simplex in Σ^p that contains t, then $f_p(t) \in \text{car}_\sigma t \subset \text{car}_\Delta t$. By Lemma 2.1, there exists a point $t^p \in \Delta$ such that

$$f_p(t^p) = g_p(t^p) =: u^p$$

Let $[v_0^p v_1^p \dots v_n^p]$ be a simplex in Σ^p that contains the point t^p. Thus, from (2.3), for some $\lambda_j^p \geq 0$ with $\sum_{j=0}^n \lambda_j^p = 1$, we have

$$u^p = \sum_{j=0}^n \lambda_j^p y_j^p = \sum_{j=0}^n \lambda_j^p z_j^p. \tag{2.4}$$

Without loss of generality we may assume that: $\lim_{p\to\infty} t^p = \hat{t} \in \Delta$, $\lim_{p\to\infty} u^p = \hat{u} \in \Delta$, and for each $j \in \{0,1,\dots,n\}$, $\lim_{p\to\infty} y_j^p = \hat{y}_j \in \Delta$, $\lim_{p\to\infty} \lambda_j^p = \hat{\lambda}_j$.

For any $j \in \{0,1,\dots,n\}$

$$\begin{aligned} \|v_j^p - \hat{t}\| &\leq \|v_j^p - t^p\| + \|t^p - \hat{t}\| \\ &\leq \frac{1}{p} + \|t^p - \hat{t}\| \to 0, \text{ when } p \text{ goes to infinity;} \end{aligned}$$

therefore, $\lim_{p\to\infty} v_j^p = \hat{t}$.

By Lemma 2.2 (i) the graph of F is closed in $\Delta \times \Delta$. Consequently, from $v_j^p \to \hat{t}$, $y_j^p \to \widehat{y_j}$, $y_j^p \in F(v_j^p)$, it follows that $\widehat{y_j} \in F(\hat{t})$ for each $j \in \{0,1,\dots,n\}$. By (2.4) $\hat{u} = \sum_{j=0}^n \hat{\lambda}_j \widehat{y_j}$, and since $F(\hat{t})$ is convex, $\hat{u} \in F(\hat{t})$. Similarly, it follows that $\hat{u} \in T(\hat{t})$, hence $\hat{u} \in F(\hat{t}) \cap T(\hat{t})$. ∎

If $D = \{x_0, x_1, \dots, x_n\}$ is a finite set in a vector space, then $X := \mathrm{co}D$ is called a *polytope*. Let $\Delta = [v_0 v_1 \dots v_n]$ be any n-simplex.

We define a function $\varphi : \Delta \to X$ in this way:

$$\text{if } t = \sum_{i=0}^n \lambda_i v_i \in S \quad (\lambda_i \geq 0, \sum_{i=0}^n \lambda_i = 1),$$

$$\text{then } \varphi(t) = \sum_{i=0}^n \lambda_i x_i.$$

Clearly φ is continuous and the following Lemma can be easily proven.

Lemma 2.4. *For each convex subset C of X, $\varphi^{-1}(C)$ is convex.*

The proof of the next lemma is also elementary.

Lemma 2.5. *Let X be a topological compact space and Y a Hausdorff one. If $f : X \to Y$ is a continuous function, then the inverse f^{-1} is a u.s.c. map with compact values.*

By means of the function φ defined above we can extend Theorem 2.3 from simplices to polytopes.

Theorem 2.6. *Let $X = co\{x_0, x_1, \dots, x_n\}$ be a polytope (endowed with the Euclidean topology) and $F \in \mathcal{K}(X,X)$. Suppose that for each nonempty finite subset A of $\{x_0, x_1, \dots, x_n\}$ and any $x \in coA$, $F(x) \cap coA \neq \emptyset$. Then any map $T \in \mathcal{K}(X,X)$ has a coincidence point with F.*

Proof. Considering Δ and φ as above we define the maps $\widetilde{F}, \widetilde{T} : \Delta \multimap \Delta$ by

$$\widetilde{F} = \varphi^{-1} \circ F \circ \varphi, \quad \widetilde{T} = \varphi^{-1} \circ T \circ \varphi.$$

Since the composite of finitely many u.s.c. maps is u.s.c, taking also into account Lemmas 2.4 and 2.5, it readily follows that $\widetilde{F}, \widetilde{T} \in \mathcal{K}(\Delta, \Delta)$.

Let $[v_{i_0}v_{i_1}\dots v_{i_k}]$ be a face of Δ and $t = \sum_{j=0}^{k}\lambda_j v_{i_j}$ be a point of this face ($\lambda_j \geq 0$, $\sum_{j=0}^{k}\lambda_j = 1$). We prove that $\widetilde{F}(t)\cap[v_{i_0}v_{i_1}\dots v_{i_k}]\neq\emptyset$. Let

$$x := \varphi(t) = \sum_{i=0}^{k}\lambda_j x_{i_j}.$$

By hypothesis, there exists a point $y \in F(x)\cap \mathrm{co}\{x_{i_0}, x_{i_1},\dots,x_{i_k}\}$. If $y = \sum_{j=0}^{k}\mu_j x_{i_j}$ ($\mu_j \geq 0$, $\sum_{j=0}^{k}\mu_j = 1$), let $u := \sum_{j=0}^{k}\mu_j v_{i_j}$.

Then

$$\begin{aligned} u &\in \varphi^{-1}(y)\cap \mathrm{co}\{v_{i_0}, v_{i_1},\dots,v_{i_k}\} \\ &\subset \varphi^{-1}(F(\varphi(t)))\cap \mathrm{co}\{v_{i_0}, v_{i_1},\dots,v_{i_k}\}, \end{aligned}$$

hence $\widetilde{F}(t)\cap \mathrm{co}\{v_{i_0}, v_{i_1},\dots,v_{i_k}\}\neq\emptyset$.

By Theorem 2.3, there exist $t_0, u_0 \in \Delta$ such that $u_0 \in \widetilde{F}(t_0)\cap\widetilde{T}(t_0)$. Then $\varphi(u_0) \in F(\varphi(t_0))\cap T(\varphi(t_0))$, hence F and T have a coincidence point. ∎

3. THE PROOF AND REFORMULATIONS OF THEOREM1.1

Let X be a convex set in a vector space. A point $x \in X$ is an *extremal point* of X if there is no way to express x as a convex combination $\lambda y + (1-\lambda)z$ such that $y, z \in X$ and $0 < \lambda < 1$, except by taking $y = z = x$. Throughout in this section, X stands for a nonempty compact convex set in a l.c.s. and D for the set of its external points.

We proceed now to prove Theorem 1.1. Let $\mathcal{V}$ be a base of closed convex symmetric neighborhoods of the origin of the space and $V \in \mathcal{V}$ arbitrarily fixed. According to Krein-Milman theorem (see [4, p. 65]), co$D$ is dense in X and since X is compact there is a finite set $\{y_1,\dots,y_k\}\subset \mathrm{co}D$ such that $X\subset\cup_{i=1}^{k}(y_i+V)$.

Let $\{x_0, x_1,\dots,x_n\}\subset D$ such that $\{y_1,\dots,y_k\}\subset C := \mathrm{co}\{x_0,x_1,\dots,x_n\}$. Then $X \subset C+V$. Consider the maps $\widetilde{F},\widetilde{T} : C \multimap C$ defined by

$$\widetilde{F}(x) = (F(x)+V)\cap C,\ \ \widetilde{T}(x) = (T(x)+V)\cap C,\ \text{for all } x\in C.$$

For each $x \in C$ we have $T(x)\subset X\subset C+V$ and, since V is symmetric, $(T(x)+V)\cap C\neq\emptyset$; hence $\widetilde{T}(x)\neq\emptyset$.

Now, let A be a subset of $\{x_0, x_1,\dots,x_n\}$ and $x \in \mathrm{co}A$. Since $F(x)\subset F(x)+V$, by hypothesis we get

$$\emptyset\neq F(x)\cap \mathrm{co}A\subset \widetilde{F}(x)\cap\mathrm{co}A.$$

Theorem 2.6 applied to the maps $\widetilde{F}$ and $\widetilde{T}$ gives points $x_V, y_V\in C$ such that $y_V\in\widetilde{F}(x_V)\cap\widetilde{T}(x_V)$. Thus, for each basis element V there exist $x_V, y_V \in X$ such that $y_V\in(F(x_V)+V)\cap(T(x_V)+V)$. Since X is compact we may assume that $x_V\to x_0\in X$, $y_V\to y_0\in X$. From $y_V\in F(x_V)+V$, it follows that there exists $z_V\in V$ such that $y_V - z_V\in F(x_V)$. Then $y_V - z_V$ also converges to y_0 and since the graph of F is closed in $X\times X$, it follows that $y_0\in F(x_0)$. In the same way one obtains that $y_0\in T(x_0)$, hence x_0 is a coincidence point for F and T.

Theorem 3.1. *Let $F, T : X \to X$ be two closed maps with nonempty convex fibers. Suppose that for each nonempty finite subset A of D, co$A \subset F(A)$. Then there exists $x_0 \in X$ such that $F(x_0)\cap T(x_0)\neq\emptyset$.*

Proof. The maps F, T being closed, F^-, T^- are also closed, and therefore u.s.c. and compact-valued by Lemma 2.2 (ii). By hypothesis F^-, T^- have nonempty convex values, hence $F^-, T^-\in\mathcal{K}(X,X)$.

Let A be a nonempty finite subset of D and $y \in \text{co}A$. By hypothesis there is a point $x \in \text{co}A$ such that $y \in F(x)$. Hence, $x \in F^{-}(y) \cap \text{co}A$.

Then F^{-} and T^{-} satisfy all the requirements of Theorem 1.1, and hence there exist $x_0, y_0 \in X$ such that $x_0 \in F^{-}(y_0) \cap T^{-}(y_0)$. Therefore, $y_0 \in F(x_0) \cap T(x_0)$ and the proof is complete. ∎

The next reformulation of Theorem 1.1 could be compared with von Neumann's intersection theorem [13].

Theorem 3.2. *Let M and N be two closed subsets of $X \times X$ such that for each $x \in X$ the sets $M_x = \{y \in X : (x, y) \in M\}$ and $N_x = \{y \in X : (x, y) \in N\}$ are nonempty and convex. Suppose that for each nonempty finite subset A of D and for each $x \in coA$, $coA \cap M_x \neq \emptyset$. Then $M \cap N \neq \emptyset$.*

Proof. Define the maps $F, T : X \multimap X$ by

$$F(x) = M_x \text{ and } T(x) = N_x.$$

Since M and N are closed, the maps F and T are closed, hence they are u.s.c. and compact-valued, by Lemma 2.2 (ii). By hypothesis, it follows on the one hand that for each $x \in X$, $F(x)$ and $T(x)$ are nonempty convex sets and on the other hand that for each nonempty finite set $A \subset D$ and each $x \in \text{co}A$ we have $F(x) \cap \text{co}A \neq \emptyset$. By Theorem 1.1 there exist $x_0, y_0 \in X$ such that $y_0 \in F(x_0) \cap T(x_0)$, that is $(x_0, y_0) \in M \cap N$. ∎

Finally we restate Theorem 1.1 as an analytic alternative that could be compared with many other such results (see [2, 3, 8, 14]).

Theorem 3.3. *Let $f, g : X \times X \to \mathbb{R}$ be two real-valued functions satisfying:*

(i) *f is lower semicontinuous and g is upper semicontinuous on $X \times X$;*
(ii) *for each $x \in X$, $f(x, \cdot)$ is quasiconvex and $g(x, \cdot)$ is quasiconcave.*

Then for any real numbers α, β at least one of the following situations holds:

(a) *there exist a nonempty finite subset A of D and $x_0 \in coA$ such that $f(x_0, y) > \alpha$ for all $y \in coA$;*
(b) *there exists $x_0 \in X$ such that $g(x_0, y) < \beta$ for all $y \in X$;*
(c) *there exists $(x_0, y_0) \in X \times X$ such that $f(x_0, y_0) \leq \alpha$ and $g(x_0, y_0) \geq \beta$.*

Proof. Suppose that (a) and (b) do not hold and define the maps $F, T : X \multimap X$ by

$$F(x) = \{y \in X : f(x, y) \leq \alpha\} \text{ and } T(x) = \{y \in X : g(x, y) \geq \beta\}.$$

By (i) it follows that the maps F and T are closed, hence by Lemma 2.2 (ii) they are u.s.c. and compact-valued. By (ii), F and T have convex values. Since (a) does not hold, for each nonempty finite subset A of D and for any $x \in \text{co}A$ there exists $y \in \text{co}A$ such that $f(x, y) \leq \alpha$, i.e., $y \in F(x_0) \cap \text{co}A$. Since (b) does not hold, for each $x \in X$, $T(x)$ is nonempty.

Applying Theorem 1.1 we get $(x_0, y_0) \in X \times X$ such that $y_0 \in F(x_0) \cap T(x_0)$, whence $f(x_0, y_0) \leq \alpha$ and $g(x_0, y_0) \geq \beta$. Hence (c) holds and the proof is complete. ∎

Corollary 3.4. *Let $f, g : X \times X \to \mathbb{R}$ be two real functions satisfying conditions (i) and (ii) in Theorem 3.3 and*

(iii) *$g(x, y) \leq f(x, y)$ fora all $(x, y) \in X \times X$.*

Then

$$\inf_{x \in X} \max_{y \in X} g(x, y) \leq \max_{A \subset D} \sup_{x \in coA} \min_{y \in coA} f(x, y),$$

where the maximum on the right-hand side is taken over all nonempty finite subsets A of D.

Proof. First let us observe that since f is l.s.c. on $X \times X$, then for each $x \in X$ $f(x, \cdot)$ is also l.s.c. function of y on X and therefore its minimum $\min_{y \in \mathrm{co}A} f(x,y)$ on the compact set co$A$ exists. Similarly we can prove that $\max_{y \in X} g(x,y)$ exists for each $x \in X$.

Let

(3.1) $$\alpha := \max_{A \subset D} \sup_{x \in \mathrm{co}A} \min_{y \in \mathrm{co}A} f(x,y) \text{ and} \beta := \inf_{x \in X} \max_{y \in X} g(x,y).$$

We may assume that $\alpha, \beta \in \mathbb{R}$. By (3.1) it follows that cases (a) and (b) in Theorem 3.3 cannot take place. Consequently, by Theorem 3.3, there exists $(x_0, y_0) \in X \times X$ such that $f(x_0, y_0) \le \alpha$ and $g(x_0, y_0) \ge \beta$. Taking into account (iii) we obtain

$$\beta \le g(x_0, y_0) \le f(x_0, y_0) \le \alpha,$$

and the proof is complete. ∎

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