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ASYMPTOTIC DISTRIBUTION OF PRODUCTS OF WEIGHTED SUMS OF DEPENDENT RANDOM VARIABLES

Y. MIAO AND J. F. LI

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COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE, HENAN NORMAL UNIVERSITY, 453007 HENAN, CHINA.
yumiao728@yahoo.com.cn

COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE, HENAN NORMAL UNIVERSITY, 453007 HENAN, CHINA.
junfen_li@yahoo.com.cn

ABSTRACT. In this paper we establish the asymptotic distribution of products of weighted sums of dependent positive random variable, which extends the results of Rempała and Wesołowski (2002).

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1. INTRODUCTION

Recently there are several studies to the products of partial sums of independent identically distributed (i.i.d.) positive random variables. It is well known that the products of i.i.d. positive, square integrable random variables are asymptotically log-normal. This fact is an immediate consequence of the classical central limit theorem (CLT). This point, up to the knowledge of the author, was first argued by Arnold and Villaseñr [1], who considered the limiting properties of the sums of records. In their paper Arnold and Villaseñr obtained the following version of the CLT for a sequence of i.i.d. exponential r.v.'s $(X_n)_{n\geq 1}$ with the mean equal to one:

$$\frac{\sum_{k=1}^{n}\log S_k - n\log n + n}{\sqrt{2n}} \xrightarrow{d} \Phi,\ \text{as } n\to\infty,$$

where $S_k=\sum_{j=1}^{k}X_j$, $1\leq k\leq n$, and Φ is a standard normal random variable. Rempała and Wesołowski [8] have noted that this limit behavior of a product of partial sums has a universal character and holds for any sequence of square integrable, positive i.i.d. random variables. Namely, they have proved the following

Theorem 1.1 (Theorem RW). *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1=\mu$, $Var(X_1)=\sigma^2>0$ and the coefficient of variation $\gamma=\sigma/\mu$. Then*

$$\left(\frac{\prod_{k=1}^{n}S_k}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})} \xrightarrow{d} e^{\sqrt{2}\Phi}.$$

Recently, Gonchigdanzan and Rempała [2] discussed an almost sure limit theorem for the product of the partial sums of i.i.d. positive random variables as follows.

Theorem 1.2 (Theorem GR). *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1=\mu>0$, $Var(X_1)=\sigma^2$. Denote $\gamma=\sigma/\mu$ the coefficient of variation. Then for any real x,*

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left(\left(\frac{\prod_{k=1}^{n}S_k}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})}\leq x\right)=F(x),\quad a.s.$$

where $F(\cdot)$ is the distribution function of the random variable $e^{\sqrt{2}\Phi}$.

For further discussions of the CLT, the authors refer to [5, 6, 7]. Zhang and Huang [10] obtained the the invariance principle of the product of sums of random variables. It is perhaps worth to notice that by the strong law of large numbers and the property of the geometric mean it follows directly that

$$\left(\frac{\prod_{k=1}^{n}S_k}{n!}\right)^{1/n} \xrightarrow{a.s.} \mu \tag{1.1}$$

if only existence of the first moment is assumed.

In the present paper we are interested in the asymptotic distribution of products of general weighted sums of dependent positive random variables, which extends the above Theorem RW.

2. MAIN RESULTS

2.1. **Some notations and assumptions.** Given probability space $(\Omega,\mathcal{F},\mathbb{P})$, let $(X_n,\mathcal{F}_n)_{n\geq 1}$ be a sequence of dependent positive random variables with the common expectation $\mu=\mathbb{E}(X_1)>0$ and $\mathcal{F}_0=\{\emptyset,\Omega\}$, $\mathcal{F}_i\subseteq\mathcal{F}_{i+1}\subseteq\mathcal{F}$ for all $i=1,\cdots$. Denote

$$S_{k,k}=a_{1,k}X_1+\cdots+a_{k,k}X_k,\, k=1,2,\ldots$$

where $\{a_{i,k}, 1 \leq i \leq k, k \geq 1\}$ is a triangular array of positive real numbers with $\sum_{i=1}^k a_{i,k} = 1$ for all $k \geq 1$. In addition for any $i = 1, \cdots,$ let

$$b_{i,n} = \sum_{k=i}^n a_{i,k},\ Y_i = X_i - \mu.$$

Assume that the following conditions are satisfied: for any $\varepsilon \in (0,1)$, as $n \to \infty$

$$\sum_{i=1}^n \mathbb{P}\left(b_{i,n}|Y_i| > \varepsilon\big(\sum_{i=1}^n b_{i,n}^2\big)^{1/2}\Big|\mathcal{F}_{i-1}\right) \xrightarrow{\mathbb{P}} 0; \tag{2.1}$$

$$\sum_{i=1}^n \mathbb{E}\left(\frac{b_{i,n}|Y_i|}{\sqrt{\sum_{i=1}^n b_{i,n}^2}} I\Big\{b_{i,n}|Y_i| \leq \big(\sum_{i=1}^n b_{i,n}^2\big)^{1/2}\Big\}\Big|\mathcal{F}_{i-1}\right) \xrightarrow{\mathbb{P}} 0; \tag{2.2}$$

$$\sum_{i=1}^n Var\left(\frac{b_{i,n}|Y_i|}{\sqrt{\sum_{i=1}^n b_{i,n}^2}} I\Big\{b_{i,n}|Y_i| \leq \varepsilon\big(\sum_{i=1}^n b_{i,n}^2\big)^{1/2}\Big\}\Big|\mathcal{F}_{i-1}\right) \xrightarrow{\mathbb{P}} \sigma^2 \geq 0; \tag{2.3}$$

$$\frac{S_{n,n}}{\mu} - 1 \xrightarrow{a.e.} 0; \tag{2.4}$$

$$\frac{1}{\sqrt{\sum_{i=1}^n b_{i,n}^2}} \sum_{i=1}^n (S_{k,k} - 1)^2 \xrightarrow{\mathbb{P}} 0. \tag{2.5}$$

2.2. **Main results.**

Theorem 2.1. *Under the above notations and assumptions, for any A_n satisfying*

$$\frac{A_n}{\sum_{i=1}^n b_{i,n}^2} \to 1, \quad as \ \ n \to \infty, \tag{2.6}$$

we have

$$\left(\prod_{k=1}^n (S_{k,k}/\mu)\right)^{\frac{\mu}{\sqrt{A_n}}} \xrightarrow{d} e^{\mathcal{N}}, \quad as \ \ n \to \infty, \tag{2.7}$$

where $\mathcal{N}$ is a normal random variable with mean zero and variation σ^2.

Remark 2.2. *In the above theorem, we give up the assumption of independence and even that of finiteness of the absolute values of the $(1+\delta)$-order moments, where $\delta > 0$ is arbitrary constant. However negligibility in the limit of the terms will be retained.*

In condition (2.3) the case $\sigma^2 = 0$ is not excluded. Hence, in particular, the above theorem yields a convergence condition for degenerate distributions.

Remark 2.3. *It is well known that (see also Shiryaev* [9]*) condition (2.1) is equivalent to the following (2.1′)*

$$\max_{1\leq i\leq n} \frac{b_{i,n}|Y_i|}{\big(\sum_{i=1}^n b_{i,n}^2\big)^{1/2}} \xrightarrow{\mathbb{P}} 0. \tag{2.1′}$$

Assuming (2.1) or (2.1′), condition (2.3) is equivalent to the following (2.3′)

$$(2.3') \quad \sum_{i=1}^{n}\left(\frac{b_{i,n}|Y_i|}{\sqrt{\sum_{i=1}^{n} b_{i,n}^2}} - \mathbb{E}\left(\frac{b_{i,n}|Y_i|}{\sqrt{\sum_{i=1}^{n} b_{i,n}^2}} I\Big\{b_{i,n}|Y_i| \le \varepsilon\big(\sum_{i=1}^{n} b_{i,n}^2\big)^{1/2}\Big\}\right)\Big|\mathcal{F}_{i-1}\right)^2 \xrightarrow{\mathbb{P}} \sigma^2.$$

3. PROOF OF THE RESULTS IN SECTION 2

Before proving Theorem 2.1 we need mention the following key lemma.

Lemma 3.1. *Under the conditions (2.1)-(2.3), for any A_n satisfying (2.6), we have*

$$(3.1) \qquad \frac{\mu}{\sqrt{A_n}}\sum_{k=1}^{n}\left(\frac{S_{k,k}}{\mu} - 1\right) \xrightarrow{d} \mathcal{N} \ \ as \ \ n \to \infty.$$

Proof. Using the notations in Section 2, (3.1) becomes

$$\frac{1}{\sqrt{A_n}}\sum_{k=1}^{n}\sum_{i=1}^{k} a_{i,k}Y_i \xrightarrow{d} \mathcal{N}.$$

Observe that

$$\frac{\mu}{\sqrt{A_n}}\sum_{k=1}^{n}\left(\frac{S_{k,k}}{\mu} - 1\right) = \frac{1}{\sqrt{A_n}}\sum_{k=1}^{n}\sum_{i=1}^{k} a_{i,k}Y_i = \frac{1}{\sqrt{A_n}}\sum_{i=1}^{n}\left(\sum_{k=i}^{n} a_{i,k}\right)Y_i,$$

recall $b_{i,n} = \sum_{k=i}^{n} a_{i,k}$ and define now

$$Z_{i,n} = \frac{1}{\sqrt{A_n}} b_{i,n}Y_i,$$

then

$$\frac{\mu}{\sqrt{A_n}}\sum_{k=1}^{n}\left(\frac{S_{k,k}}{\mu} - 1\right) = \sum_{i=1}^{n} Z_{i,n}.$$

From the Theorem 1 (p. 541 in Shiryaev [9]), the desired result is obtained by the conditions (2.1)-(2.3). ∎

Proof of Theorem 2.1. Here we will use the delta-method expansion to prove our results as Rempała and Wesołowski [8]. Denote $C_k = S_{k,k}/\mu$. By the condition (2.4), for any $\delta > 0$, there exists a number R such that for any $r > R$

$$\mathbb{P}\left(\sup_{k\ge r}|C_k - 1| > \delta\right) \le \delta.$$

Consequently, there exist two sequences $\{\delta_m\} \downarrow 0$ ($\delta_1 = 1/2$) and $(R_m) \uparrow \infty$ such that

$$\mathbb{P}\left(\sup_{k\ge R_m}|C_k - 1| > \delta_m\right) \le \delta_m.$$

Taking now any real x and any m, we have

$$\begin{aligned}\mathbb{P}\left(\frac{\mu}{\sqrt{A_n}}\sum_{k=1}^{n}\log C_k \le x\right) =& \mathbb{P}\left(\frac{\mu}{\sqrt{A_n}}\sum_{k=1}^{n}\log C_k \le x,\ \sup_{k\ge R_m}|C_k - 1| \ge \delta_m\right)\\ &+ \mathbb{P}\left(\frac{\mu}{\sqrt{A_n}}\sum_{k=1}^{n}\log C_k \le x,\ \sup_{k\ge R_m}|C_k - 1| < \delta_m\right)\\ :=& A_{m,n} + B_{m,n}\end{aligned}$$

where $A_{m,n} \le \delta_m$. Next we will control the term $B_{m,n}$. By the following logarithm:

$$\log(1+x) = x + \frac{x^2}{(1+\theta x)^2},$$

where $\theta \in (0,1)$ depends on $x \in (-1,1)$, we have

$$\begin{aligned}
B_{m,n} =& \mathbb{P}\Big\{ \frac{\mu}{\sqrt{A_n}} \sum_{k=1}^{R_m} \log C_k + \frac{\mu}{\sqrt{A_n}} \sum_{k=R_m+1}^{n} \log(1+(C_k-1)) \le x, \sup_{k\ge R_m} |C_k - 1| < \delta_m \Big\} \\
=& \mathbb{P}\Big\{ \frac{\mu}{\sqrt{A_n}} \sum_{k=1}^{R_m} \log C_k + \frac{\mu}{\sqrt{A_n}} \sum_{k=R_m+1}^{n} (C_k-1) \\
& + \frac{\mu}{\sqrt{A_n}} \sum_{k=R_m+1}^{n} \frac{(C_k-1)^2}{(1+\theta_k(C_k-1))^2} \le x, \sup_{k\ge R_m} |C_k - 1| < \delta_m \Big\} \\
=& \mathbb{P}\Big\{ \frac{\mu}{\sqrt{A_n}} \sum_{k=1}^{R_m} \log C_k + \frac{\mu}{\sqrt{A_n}} \sum_{k=R_m+1}^{n} (C_k-1) \\
& + \Big[\frac{\mu}{\sqrt{A_n}} \sum_{k=R_m+1}^{n} \frac{(C_k-1)^2}{(1+\theta_k(C_k-1))^2} \Big] I(\sup_{k\ge R_m} |C_k - 1| < \delta_m) \le x \Big\} \\
& - \mathbb{P}\Big\{ \frac{\mu}{\sqrt{A_n}} \sum_{k=1}^{R_m} \log C_k + \frac{\mu}{\sqrt{A_n}} \sum_{k=R_m+1}^{n} (C_k-1) \le x, \sup_{k\ge R_m} |C_k - 1| \ge \delta_m \Big\} \\
:=& D_{m,n} + F_{m,n},
\end{aligned}$$

where $\theta_k, k = 1, \cdots, n$ are $(0,1)$-valued random variables and $F_{m,n} \le \delta_m$. To estimate the term $D_{m,n}$, we rewrite it as

$$\begin{aligned}
D_{m,n} =& \mathbb{P}\Big\{ \frac{\mu}{\sqrt{A_n}} \sum_{k=1}^{R_m} [\log C_k - (C_k-1)] + \frac{\mu}{\sqrt{A_n}} \sum_{k=1}^{n} (C_k-1) \\
& + \Big[\frac{\mu}{\sqrt{A_n}} \sum_{k=R_m+1}^{n} \frac{(C_k-1)^2}{(1+\theta_k(C_k-1))^2} \Big] I(\sup_{k\ge R_m} |C_k - 1| < \delta_m) \le x \Big\}.
\end{aligned}$$

For any fixed m,

$$\frac{\mu}{\sqrt{A_n}} \sum_{k=1}^{R_m} [\log C_k - (C_k-1)] \xrightarrow{\mathbb{P}} 0$$

as $n \to \infty$. By the following elementary inequality: for $|x| < 1$ and any $\theta \in (0,1)$ it follows that $x^2/(1+\theta x)^2 \le 4x^2$. Then for any m, by condition (2.5),

$$\Big[\frac{\mu}{\sqrt{A_n}} \sum_{k=R_m+1}^{n} \frac{(C_k-1)^2}{(1+\theta_k(C_k-1))^2} \Big] I(\sup_{k\ge R_m} |C_k - 1|) \le \frac{4\mu}{\sqrt{A_n}} \sum_{k=1}^{n} (C_k-1)^2 \xrightarrow{\mathbb{P}} 0,$$

as $n \to \infty$. Since

$$\mathbb{P}\left(\log \left(\prod_{k=1}^{n} (S_{k,k}/\mu) \right)^{\frac{\mu}{\sqrt{A_n}}} \le x \right) = \mathbb{P}\left(\frac{\mu}{\sqrt{A_n}} \sum_{k=1}^{n} \log C_k \le x \right) = A_{m,n} + D_{m,n} + F_{m,n}$$

and from Lemma 3.1, we have $A_{m,n} + F_{m,n} \le 2\delta_m \to 0$ as $m \to \infty$ and $\lim_{n\to\infty} D_{m,n} \to \Phi(x)$ as $m \to \infty$, which imply the desired results. ∎

Remark 3.2. From the proof of Theorem 2.1, we know that condition (2.4) can be replaced by the following: for any $\delta > 0$, there exists a constants R such that

$$\mathbb{P}\left(\max_{n\geq R}|S_{n,n}/\mu - 1| > \delta\right) \leq \delta. \tag{3.2}$$

4. FURTHER DISCUSSIONS

4.1. **Independent case.** When $(X_n)_{n\geq 1}$ is a sequence of positive independent random variables, conditions (2.1)-(2.3) become

$$\sum_{i=1}^{n}\mathbb{P}\left(b_{i,n}|Y_i| > \varepsilon\big(\sum_{i=1}^{n} b_{i,n}^2\big)^{1/2}\right) \to 0;$$

$$\sum_{i=1}^{n}\mathbb{E}\left(\frac{b_{i,n}|Y_i|}{\sqrt{\sum_{i=1}^{n} b_{i,n}^2}} I\Big\{b_{i,n}|Y_i| \leq \big(\sum_{i=1}^{n} b_{i,n}^2\big)^{1/2}\Big\}\right) \to 0;$$

$$\sum_{i=1}^{n} Var\left(\frac{b_{i,n}|Y_i|}{\sqrt{\sum_{i=1}^{n} b_{i,n}^2}} I\Big\{b_{i,n}|Y_i| \leq \varepsilon\big(\sum_{i=1}^{n} b_{i,n}^2\big)^{1/2}\Big\}\right) \to \sigma^2;$$

These are well known (see Gnedenko and Kolmogorov [4]) that

$$\frac{\mu}{\sqrt{A_n}}\sum_{k=1}^{n}\left(\frac{S_{k,k}}{\mu} - 1\right) \xrightarrow{d} \mathcal{N}.$$

Furthermore it is easy to check that when $(X_n)_{n\geq 1}$ is a sequence of positive square integrable i.i.d random variables and $a_{i,k} = 1/k$ for all $1 \leq i \leq k, k \geq 1$, Theorem 2.1 induces the Theorem RW in Section 1.

4.2. **Martingale case.** Let

$$Z_{i,n} = \frac{\mu}{\sqrt{\sum_{i=1}^{n} b_{i,n}^2}} b_{i,n} Y_i$$

and $(Z_{i,n}, \mathcal{F}_i)_{n\geq 1, 1\leq i\leq n}$ be a square integrable martingale difference:

$$\mathbb{E}Z_{i,n}^2 < \infty, \ \ \mathbb{E}(Z_{i,n}|\mathcal{F}_{n-1}) = 0.$$

Suppose that the Lindeberg condition is satisfied: for any $\varepsilon > 0$

$$(L) \quad \sum_{i=1}^{n}\mathbb{E}(Z_{i,n}^2 I(|Z_{i,n}| > \varepsilon)|\mathcal{F}_{i-1}) \xrightarrow{\mathbb{P}} 0.$$

Then from Shiryaev [9], we have

$$\sum_{k=0}^{n}\mathbb{E}(Z_{i,n}^2|\mathcal{F}_{i-1}) \xrightarrow{\mathbb{P}} \sigma^2 \Rightarrow \frac{\mu}{\sqrt{A_n}}\sum_{k=1}^{n}\left(\frac{S_{k,k}}{\mu} - 1\right) \xrightarrow{d} \mathcal{N}, \tag{4.1}$$

$$\sum_{k=0}^{n} Z_{i,n}^2 \xrightarrow{\mathbb{P}} \sigma^2 \Rightarrow \frac{\mu}{\sqrt{A_n}}\sum_{k=1}^{n}\left(\frac{S_{k,k}}{\mu} - 1\right) \xrightarrow{d} \mathcal{N}. \tag{4.2}$$

In addition assume that the conditions (2.4), (2.5) are satisfied then (2.7) holds.

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