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ON THE GENERALIZED STABILITY AND ASYMPTOTIC BEHAVIOR OF QUADRATIC MAPPINGS

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ABSTRACT. We extend the stability of quadratic mappings to the stability of general quadratic mappings with several variables, and then obtain an improved asymptotic property of quadratic mappings on restricted domains.

Key words and phrases: Hyers-Ulam stability, Quadratic mappings, Alternative of fixed point.

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1. INTRODUCTION

In 1940, S.M. Ulam [23] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

In 1941, D.H. Hyers [9] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies the *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

In 1978, Th.M. Rassias [17] proved a theorem for the stability of the linear mapping, which allows the Cauchy difference to be controlled by a sum of powers of norms. Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. Further, if for each $x \in E$ the mapping $t \to f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$.

In 1990, Th.M. Rassias during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for all real values of p that are greater than or equal to one. In 1991, Z. Gajda [7], following the same approach as in the Th.M. Rassias stability theorem, provided an affirmative solution to the question for all real values of p that are strictly greater than one. This was made possible by just replacing n by $-n$ throughout the proof of Th.M. Rassias' theorem. The inequality (1.1) that was introduced by Th.M.Rassias has provided much influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept of stability is known today as the generalized Hyers–Ulam stability or Hyers–Ulam–Rassias stability or Cauchy–Rassias stability of functional equations (cf. the books of S. Czerwik [6], D.H. Hyers, G. Isac and Th.M. Rassias [12]). It was shown by Z. Gajda [7], as well as by Th.M. Rassias and P. Šemrl [18] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. P. Găvruta [8] provided a further generalization of Th.M. Rassias' Theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations and mappings (see [2, 3, 6, 10, 12, 20]).

Now, a square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for all vectors x, y. The following functional equation, which was motivated by the last identity,

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \tag{1.3}$$

is called a quadratic functional equation, and every solution of the equation (1.3) is said to be a quadratic mapping. In 1983 F. Skof [22] was the first author to solve the Ulam problem for additive mappings on a restricted domain. In 1998 S. Jung [13] and in 2002 J.M. Rassias [15] investigated the Hyers–Ulam stability for additive and quadratic mappings on restricted domains. The stability problems of several quadratic functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [1, 4, 10, 11, 14, 19, 21].

Now we are going to extend the equation (1.3) to a more generalized equation with several $(d+1)$-variables. For this purpose, we employ the operator $\biguplus_{x_2} f(x_1)$ as follows

$$\biguplus_{x_2} f(x_1) = f(x_1 + x_2) + f(x_1 - x_2)$$

for a given mapping $f : E_1 \to E_2$ between vector spaces. Similarly, we define $\biguplus^2_{x_2,x_3} f(x_1) = \biguplus_{x_3} \left(\biguplus_{x_2} f(x_1)\right)$ and inductively

$$\biguplus^{d}_{x_2,\ldots,x_{d+1}} f(x_1) = \biguplus_{x_{d+1}} \left(\biguplus^{d-1}_{x_2,\ldots,x_d} f(x_1) \right)$$

for all natural numbers d. Here, we consider the following new functional equation,

$$\sum_{1\le i<j\le d+1} \left(\biguplus_{x_j} f(x_i) \right) = 2d \sum_{i=1}^{d+1} f(x_i) \tag{1.4}$$

for all $(d+1)$-variables $x_1, \ldots, x_{d+1} \in E_1$, where $d \ge 1$ is a natural number. As a special case, the equation (1.4) reduces to the equation (1.3) for the case $d = 1$. In this paper, it will be verified that the general solutions of the above functional equations (1.4) are quadratic mappings in the class of functions between vector spaces. In addition, we establish new theorems about the Hyers–Ulam–Rassias stability for general equations and apply our results on restricted domains to the asymptotic behavior of functional equations. Thus, we may obtain the stability result of the equation (1.4) utilizing independently direct methods without using the fixed point theorem of the alternative for contractions.

2. APPROXIMATELY QUADRATIC MAPPINGS

Lemma 2.1. *Let E_1 and E_2 be vector spaces. A mapping $f : E_1 \to E_2$ satisfies the functional equation (1.4) if and only if the mapping f satisfies the functional equation (1.3).*

Proof. We first assume that f is a solution of the functional equation (1.4). Set $x_i := 0$ in (1.4) for all $i = 1, \ldots, d+1$ to get $f(0) = 0$. Putting $x_i := 0$ in (1.4) for all $i = 3, \ldots, d+1$, we get $f(x_1 + x_2) + f(x_1 - x_2) = 2[f(x_1) + f(x_2)]$ for all $x_1, x_2 \in E_1$.

Conversely, assume that the mapping f satisfies the functional equation (1.3). By induction, we first assume that f satisfies the equation

$$\sum_{1\le i<j\le d} \left(\biguplus_{x_j} f(x_i) \right) = 2(d-1) \sum_{i=1}^{d} f(x_i) \tag{2.1}$$

for all d-variables $x_1, \dots, x_d \in E_1$. Then we get

$$\begin{aligned}(2.2)\qquad \sum_{1\le i<j\le d+1}\left(\biguplus_{x_j} f(x_i)\right) &= \sum_{1\le i<j\le d}\left(\biguplus_{x_j} f(x_i)\right) + \sum_{i=1}^{d}[f(x_i+x_{d+1})+f(x_i-x_{d+1})] \\ &= 2(d-1)\sum_{i=1}^{d} f(x_i) + \sum_{i=1}^{d}[2f(x_i)+2f(x_{d+1})] \\ &= 2d\sum_{i=1}^{d+1} f(x_i)\end{aligned}$$

for all $(d+1)$-variables $x_1, \dots, x_{d+1} \in E_1$. Thus f satisfies the equation (1.4). This completes the proof of the lemma. ∎

From now on, let X be a normed space and Y a Banach space unless otherwise specified. Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and d a positive integer with $d \ge 1$. Now before taking up the main subject, given $f : X \to Y$, we define the difference operator $Df : X^{d+1} \to Y$ by

$$Df(x_1, x_2, \dots, x_{d+1}) := \sum_{1\le i<j\le d+1}\left(\biguplus_{x_j} f(x_i)\right) - 2d\sum_{i=1}^{d+1} f(x_i)$$

for all $(d+1)$-variables $x_1, \dots, x_{d+1} \in X$, which acts as a perturbation of the equation (1.4). We now investigate the generalized Hyers–Ulam stability problem for the equation (1.4). Thus we give conditions in order for a true mapping near an approximate mapping of the equation (1.4) to exist.

Theorem 2.2. *Assume that there exists a mapping $\varepsilon : X^{d+1} \to [0,\infty)$ for which a mapping $f : X \to Y$ satisfies the functional inequality*

$$(2.3)\qquad \|Df(x_1, x_2, \dots, x_{d+1})\| \le \varepsilon(x_1, \dots, x_{d+1})$$

for all $(d+1)$-variables $x_1, \dots, x_{d+1} \in X$. If there exists a real number $L(0 < L < 1)$ such that the mapping ε satisfies

$$(2.4)\qquad \varepsilon(2x_1, \dots, 2x_{d+1}) \le 4L\varepsilon(x_1, \dots, x_{d+1})$$

for all $x_1, \dots, x_{d+1} \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality

$$(2.5)\qquad \left\| f(x) - \frac{f(0)}{3} - Q(x)\right\| \le \frac{\varepsilon(x, \dots, x)}{2d(d+1)(1-L)}$$

for all $x \in X$. The mapping Q is defined by

$$Q(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^{2n}}$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Proof. It follows from (2.4) that

$$\varepsilon(2^i x_1, \dots, 2^i x_{d+1}) \le (4L)^i \varepsilon(x_1, \dots, x_{d+1})$$

for all $x \in X$. If we take $(x, \ldots, x)$ instead of $(x_1, \ldots, x_{d+1})$ in (2.11), we obtain

$$\left\| \binom{d+1}{2}[f(2x) + f(0)] - 2d(d+1)f(x) \right\| \leq \varepsilon(x, \ldots, x), \tag{2.6}$$

which can be rewritten in the form

$$\left\| \left[f(2x) - \frac{f(0)}{3}\right] - 4\left[f(x) - \frac{f(0)}{3}\right] \right\| \leq \frac{2}{d(d+1)}\varepsilon(x, \ldots, x), \tag{2.7}$$
$$\text{or,} \qquad \left\| \frac{q(2x)}{4} - q(x) \right\| \leq \frac{1}{2d(d+1)}\varepsilon(x, \ldots, x)$$

for all $x \in X$, where $q(x) := f(x) - \frac{f(0)}{3}$, $x \in X$. We claim that a sequence defined by $\left\{\frac{q(2^n x)}{4^n}\right\}$, $x \in X$, is Cauchy in the Banach space Y. By (2.7), we get

$$\begin{aligned} \left\| \frac{q(2^l x)}{4^l} - \frac{q(2^n x)}{4^n} \right\| &\leq \sum_{i=l}^{n-1} \left\| \frac{q(2^i x)}{4^i} - \frac{q(2^{i+1} x)}{4^{i+1}} \right\| \\ &\leq \frac{1}{2d(d+1)} \sum_{i=l}^{n-1} \frac{\varepsilon(2^i x, \ldots, 2^i x)}{4^i} \\ &= \frac{1}{2d(d+1)} \sum_{i=l}^{n-1} L^i \, \varepsilon(x, \ldots, x) \end{aligned} \tag{2.8}$$

for all integers with $n > l \geq 0$. Hence a mapping $Q : X \to Y$ given by the formula

$$Q(x) = \lim_{n\to\infty} \frac{q(2^n x)}{4^n} = \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}}$$

is well defined for all $x \in X$. In addition it is clear from (2.3) that the following inequality

$$\begin{aligned} \frac{1}{4^n} \left\| Df(2^n x_1, \ldots, 2^n x_{d+1}) \right\| &\leq \frac{1}{4^n}\varepsilon(2^n x_1, \ldots, 2^n x_{d+1}) \\ &\leq L^n \, \varepsilon(x_1, \ldots, x_{d+1}) \end{aligned}$$

holds for all $x_1, \ldots, x_{d+1} \in X$ and all $n \in \mathbb{N}$. Taking the limit $n \to \infty$, we see that the mapping Q satisfies the equation (1.4) and so Q is quadratic. Letting $l := 0$ in the inequality (2.8) and taking the limit $n \to \infty$, we find that there exists a quadratic mapping $Q : X \to Y$ satisfying the inequality (2.5).

To prove the afore-mentioned uniqueness, we assume that there exists a quadratic mapping $Q_1 : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\left\| f(x) - \frac{f(0)}{3} - Q_1(x) \right\| \leq \frac{\varepsilon(x, \ldots, x)}{2d(d+1)(1-L)} \tag{2.9}$$

for all $x \in X$. Since Q and Q_1 are quadratic, we see the identities $Q(x) = 2^{-2n}Q(2^n x)$, $Q_1(x) = 2^{-2n}Q_1(2^n x)$ hold for all $x \in X$ and all $n \in \mathbb{N}$. Thus it follows from inequalities (2.5) and (2.9) that

$$\begin{aligned}
&\|Q(x)-Q_1(x)\| \\
&= \frac{1}{2^{2n}}\|Q(2^n x)-Q_1(2^n x)\| \\
&\le \frac{1}{2^{2n}}\left(\left\|Q(2^n x)-f(2^n x)+\frac{f(0)}{3}\right\|+\left\|f(2^n x)-\frac{f(0)}{3}-Q_1(2^n x)\right\|\right) \\
&\le \frac{\varepsilon(2^n x,\ldots,2^n x)}{d(d+1)(1-L)2^{2n}} \le \frac{L^n\varepsilon(x,\ldots,x)}{d(d+1)(1-L)}
\end{aligned}$$

for all $x\in X$ and all $n\in\mathbb{N}$. Therefore letting $n\to\infty$, one has $Q(x)-Q_1(x)=0$ for all $x\in X$, completing the proof of uniqueness.

The proof of last assertion in the theorem follows in the same manner as the proof of Theorem 3 in [5]. The proof is complete. ∎

Theorem 2.3. *Assume that there exists a mapping $\varepsilon: X^{d+1}\to[0,\infty)$ for which a mapping $f: X\to Y$ satisfies the functional inequality*

$$\|Df(x_1,x_2,\ldots,x_{d+1})\|\le\varepsilon(x_1,\ldots,x_{d+1})$$

for all $(d+1)$-variables $x_1,\ldots,x_{d+1}\in X$. If there exists a real number $L(0<L<1)$ such that the mapping ε satisfies

$$\varepsilon\left(\frac{x_1}{2},\ldots,\frac{x_{d+1}}{2}\right)\le\frac{L}{4}\varepsilon(x_1,\ldots,x_{d+1}) \tag{2.10}$$

for all $x_1,\ldots,x_{d+1}\in X$, then there exists a unique quadratic mapping $Q: X\to Y$ which satisfies the equation (1.4) and the inequality

$$\|f(x)-Q(x)\|\le\frac{L\,\varepsilon(x,\ldots,x)}{2d(d+1)(1-L)}$$

for all $x\in X$. The mapping Q is defined by

$$Q(x)=\lim_{n\to\infty}4^n f\left(\frac{x}{2^n}\right)$$

for all $x\in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Proof. We note that $\varepsilon(0,\ldots,0)=0$ implies $f(0)=0$ and

$$\varepsilon\left(\frac{x_1}{2^i},\ldots,\frac{x_{d+1}}{2^i}\right)\le\left(\frac{L}{4}\right)^i\varepsilon(x_1,\ldots,x_{d+1})$$

for all $x\in X$. It follows from (2.7) that for all integers with $n>l\ge 0$

$$\begin{aligned}
\left\|4^l f\left(\frac{x}{2^l}\right)-4^n f\left(\frac{x}{2^n}\right)\right\| &\le \sum_{i=l}^{n-1}\left\|4^i f\left(\frac{x}{2^i}\right)-4^{i+1}f\left(\frac{x}{2^{i+1}}\right)\right\| \\
&\le \frac{2}{d(d+1)}\sum_{i=l}^{n-1}4^i\varepsilon\left(\frac{x}{2^{i+1}},\ldots,\frac{x}{2^{i+1}}\right) \\
&= \frac{1}{2d(d+1)}\sum_{i=l}^{n-1}L^{i+1}\,\varepsilon(x,\ldots,x)
\end{aligned}$$

for all $x\in X$.

The rest of the proof is similar to that of Theorem 2.2. ∎

If we apply Theorem 2.2 and Theorem 2.3 to each case $L := \frac{2^p}{4}$, $p < 2$ and $L := \frac{4}{2^p}$, $p > 2$, we have the following corollary.

Corollary 2.4. *Suppose that there exist a real number $\varepsilon \geq 0$ and a real $p \neq 2$ such that a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \ldots, x_{d+1})\| \leq \varepsilon(\|x_1\|^p + \cdots + \|x_{d+1}\|^p)$$

for all $(d+1)$-variables $x_1, \ldots, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\left\| f(x) - \frac{f(0)}{3} - Q(x) \right\| \leq \frac{2\varepsilon\|x\|^p}{d|4 - 2^p|}$$

for all $x \in X$, where $x \neq 0$ if $p \leq 0$ and $f(0) = 0$ if $p > 0$. The mapping Q is defined by

$$Q(x) = \begin{cases} \lim_{n\to\infty} \frac{f(2^n x)}{4^n}, & \text{if } p < 2, \\ \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right), & \text{if } p > 2, \end{cases}$$

for all $x \in X$.

Further, we establish other theorems about the Ulam stability problem of the equation (1.4) as follows.

Theorem 2.5. *Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \ldots, x_{d+1})\| \leq \varepsilon(x_1, \ldots, x_{d+1}) \tag{2.11}$$

for all $(d+1)$-variables $x_1, \ldots, x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=0}^{\infty} \frac{\varepsilon(2^i x_1, \ldots, 2^i x_{d+1})}{2^{2i}}$$

converges for all $x_1, \ldots, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\left\| f(x) - \frac{f(0)}{3} - Q(x) \right\| \leq \frac{1}{2d(d+1)} \sum_{i=0}^{\infty} \frac{\varepsilon(2^i x, \ldots, 2^i x)}{2^{2i}} \tag{2.12}$$

for all $x \in X$. The mapping Q is defined by

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}} \tag{2.13}$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Proof. Now by (2.7), we have that for any integers m, n with $n > m \geq 0$,

$$\begin{aligned} \left\| \frac{q(2^m x)}{4^m} - \frac{q(2^n x)}{4^n} \right\| &\leq \sum_{i=m}^{n-1} \left\| \frac{q(2^{i+1} x)}{4^{i+1}} - \frac{q(2^i x)}{4^i} \right\| \\ &\leq \frac{1}{2d(d+1)} \sum_{i=m}^{n-1} \frac{\varepsilon(2^i x, \ldots, 2^i x)}{4^i} \end{aligned} \tag{2.14}$$

for all $x \in X$. Since the right hand side of the above inequality tends to 0 as $m \to \infty$, the sequence $\left\{\frac{q(2^n x)}{4^n}\right\}$ is Cauchy in Y and thus converges in Y. Therefore, we may define a mapping $Q : X \to Y$ as

$$Q(x) = \lim_{n\to\infty} \frac{q(2^n x)}{4^n} = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

for all $x \in X$, and then by letting $n \to \infty$ in (2.14) with $m = 0$ we arrive at the formula (2.12).

We claim that Q satisfies the equation (1.4). For this purpose, we calculate the following inequality from (2.11)

$$\begin{aligned}\|DQ(x_1, \ldots, x_{d+1})\| &= \lim_{n\to\infty} \frac{1}{4^n} \|Df(2^n x_1, \ldots, 2^n x_{d+1})\| \\ &\leq \lim_{n\to\infty} \frac{1}{4^n} \varepsilon(2^n x_1, \ldots, 2^n x_{d+1}),\end{aligned}$$

which implies that $DQ(x_1, \ldots, x_{d+1}) = 0$ for all $(x_1, \ldots, x_{d+1}) \in X$. Hence the mapping Q is quadratic by Lemma 2.1.

The rest of the proof is similar to that of Theorem 2.2. ∎

Theorem 2.6. *Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \ldots, x_{d+1})\| \leq \varepsilon(x_1, \ldots, x_{d+1})$$

for all $(d+1)$-variables $x_1, \ldots, x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=1}^{\infty} 4^i \varepsilon\left(\frac{x_1}{2^i}, \ldots, \frac{x_{d+1}}{2^i}\right)$$

converges for all $x_1, \ldots, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\|f(x) - Q(x)\| \leq \frac{1}{2d(d+1)} \sum_{i=1}^{\infty} 4^i \varepsilon\left(\frac{x}{2^i}, \ldots, \frac{x}{2^i}\right)$$

for all $x \in X$. The mapping Q is defined by

$$Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Note that one has $f(0) = 0$ in the above theorem because $\varepsilon(0, \ldots, 0) = 0$ by the convergence of the series.

Corollary 2.7. *Suppose that there exists a nonnegative real number ε for which a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \ldots, x_{d+1})\| \leq \varepsilon$$

for all $(d+1)$-variables $x_1, \ldots, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\left\|f(x) - \frac{f(0)}{3} - Q(x)\right\| \leq \frac{2\varepsilon}{3d(d+1)}$$

for all $x \in X$.

Remark 2.1. If $d = 1$ in Theorem 2.5 and Corollaries 2.4 and 2.7, then the results exactly coincide with the classical results [2, 17, 22].

Corollary 2.8. *A mapping $f : X \to Y$ with $f(0) = 0$ is quadratic if and only if*

$$\sup_{x_1,\ldots,x_{d+1}} \|Df(x_1, x_2, \ldots, x_{d+1})\|$$

is bounded for all $d \geq 1$.

Proof. Let $\sup_{x_1,\ldots,x_{d+1}} \|Df(x_1, x_2, \ldots, x_{d+1})\| \leq M < \infty$ for all $d \geq 1$. Then for each $d \geq 1$, there exists a unique quadratic mapping $Q_d : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\|f(x) - Q_d(x)\| \leq \frac{2M}{3d(d+1)}$$

for all $x \in X$ by Corollary 2.7. Let m be a positive integer with $m > d$. Then, we obtain

$$\|f(x) - Q_m(x)\| \leq \frac{2M}{3m(m+1)} \leq \frac{2M}{3d(d+1)}$$

for all $x \in X$. The uniqueness of Q_d implies that $Q_m = Q_d$ for all m with $m > d$, and so

$$\|f(x) - Q_d(x)\| \leq \frac{2M}{3m(m+1)}$$

for all $x \in X$. By letting $m \to \infty$, we conclude that f is itself quadratic.

The reverse assertion is trivial. ∎

Remark 2.2. If we apply Theorem 2.5 and Theorem 2.6 to Corollary 2.4, then we have exactly the same conclusion, and if we apply Theorem 2.2 to Corollary 2.7 with $L := \frac{1}{4}$, then we have exactly the same conclusion.

3. APPROXIMATELY QUADRATIC MAPPINGS ON RESTRICTED DOMAINS

In this section we investigate the Hyers–Ulam stability problem for the equation (1.4) on an unbounded restricted domain. As results, we have corollaries concerning an asymptotic property of the equation (1.4).

Theorem 3.1. *Let $r > 0$ be fixed. Suppose that there exists a nonnegative real number ε for which a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \ldots, x_{d+1})\| \leq \varepsilon \tag{3.1}$$

for all $(d+1)$-variables $x_1, \ldots, x_{d+1} \in X$ with $\sum_{i=1}^{d+1} \|x_i\| \geq r$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\left\| f(x) + \frac{(d^2+d-2)f(0)}{2} - Q(x) \right\| \leq \frac{3\varepsilon}{2} \tag{3.2}$$

for all $x \in X$.

Proof. Taking $(x_1, \ldots, x_{d+1})$ as $(x, y, 0, \ldots, 0)$ in (3.1) with $\|x\| + \|y\| \geq r$, we obtain

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y) - (d^2+d-2)f(0)\| \leq \varepsilon,$$

which yields

$$\|q(x+y) + q(x-y) - 2q(x) - 2q(y)\| \leq \varepsilon, \tag{3.3}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq r$, where

$$q(x) := f(x) + \frac{(d^2+d-2)f(0)}{2}.$$

In particular, we have $\|q(0)\| \leq \frac{\varepsilon}{2}$ by setting $y := 0$ and $x := t$ with $\|t\| \geq r$ in (3.3). Now, assume $\|x\| + \|y\| < r$ and choose a $t \in X$ with $\|t\| \geq 2r$. Then it clearly holds that

$$\|x \pm t\| \geq r, \quad \|y \pm t\| \geq r, \quad \text{and} \quad \|2t\| + \|x+y\| \geq r.$$

Therefore from (3.3) and the following functional identity

$$\begin{aligned}
&2\big[q(x+y) + q(x-y) - 2q(x) - 2q(y) - q(0)\big] \\
&= \big[q(x+y+2t) + q(x-y) - 2q(x+t) - 2q(y+t)\big] \\
&\qquad + \big[q(x+y-2t) + q(x-y) - 2q(x-t) - 2q(y-t)\big] \\
&\qquad + \big[-q(x+y+2t) - q(x+y-2t) + 2q(x+y) + 2q(2t)\big] \\
&\qquad + \big[2q(x+t) + 2q(x-t) - 4q(x) - 4q(t)\big] \\
&\qquad + \big[2q(y+t) + 2q(y-t) - 4q(y) - 4q(t)\big] \\
&\qquad + \big[-2q(2t) - 2q(0) + 4q(t) + 4q(t)\big],
\end{aligned}$$

we get

$$\|q(x+y) + q(x-y) - 2q(x) - 2q(y) - q(0)\| \leq \frac{9\varepsilon}{2} \tag{3.4}$$

for all $x, y \in X$ with $\|x\| + \|y\| < r$. Consequently, the last functional inequality holds for all $x, y \in X$ in view of (3.3) and (3.4). Now letting $y := x$ in (3.4), we obtain

$$\|q(2x) - 4q(x)\| \leq \frac{9\varepsilon}{2}.$$

Applying a standard procedure of direct method [9] to the last inequality, we see that there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\|q(x) - Q(x)\| \leq \frac{3\varepsilon}{2}$$

for all $x \in X$. ∎

Obviously our inequality (3.2) is sharper than the corresponding inequalities of Jung [13] and J.M. Rassias [15], where the approximate estimations were equal to $\frac{7}{2}\varepsilon$ and $\frac{5}{2}\varepsilon$, respectively.

We note that if we define $S_{d+1} = \{(x_1, \ldots, x_{d+1}) \in X^{d+1} : \|x_i\| < r, \forall i = 1, \ldots, d+1\}$ for some fixed $r > 0$, then we have

$$\left\{(x_1, \ldots, x_{d+1}) \in X^{d+1} : \sum_{i=1}^{d+1} \|x_i\| \geq (d+1)r\right\} \subset X^{d+1} \setminus S_{d+1}.$$

Thus the following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2. *If a mapping $f : X \to Y$ satisfies the functional inequality (3.1) for all $(x_1, \ldots, x_{d+1}) \in X^{d+1} \setminus S_{d+1}$, then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality (3.2)*

From Theorem 3.1, we have the following corollary concerning an asymptotic property of quadratic mappings.

Corollary 3.3. *A mapping $f : X \to Y$ with $f(0) = 0$ is quadratic if and only if*

$$\|Df(x_1, \ldots, x_{d+1})\| \to 0$$

as $\sum_{i=1}^{d+1} \|x_i\| \to \infty$.

Proof. According to our asymptotic condition, there is a sequence (ε_m) decreasing to zero such that $\|Df(x_1, \ldots, x_{d+1})\| \le \varepsilon_m$ for all $(d+1)$-variables $x_1, \ldots, x_{d+1} \in X$ with $\sum_{i=1}^{d+1} \|x_i\| \ge m$. Hence, it follows from Theorem 3.1 that there exists a unique quadratic mapping $Q_m : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\|f(x) - Q_m(x)\| \le \frac{3\varepsilon_m}{2}$$

for all $x \in X$. Let m and l be positive integers with $m > l$. Then, we obtain

$$\|f(x) - Q_m(x)\| \le \frac{3\varepsilon_m}{2} \le \frac{3\varepsilon_l}{2}$$

for all $x \in X$. The uniqueness of Q_l implies that $Q_m = Q_l$ for all m, l, and so

$$\|f(x) - Q_l(x)\| \le \frac{3\varepsilon_m}{2}$$

for all $x \in X$. By letting $m \to \infty$, we conclude that f is itself quadratic.

The reverse assertion is trivial. ∎

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