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A FIXED POINT APPROACH TO THE STABILITY OF THE EQUATION $f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}$

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ABSTRACT. We will apply a fixed point method for proving the Hyers–Ulam stability of the functional equation $f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}$.

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1. INTRODUCTION

In 1940, S.M. Ulam [24] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among these was the following question concerning the stability of group homomorphisms:

> *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?*

The Ulam problem for the case of approximately additive functions was solved by D.H. Hyers [8] under the assumption that G_1 and G_2 are Banach spaces. Indeed, Hyers proved that each solution of the inequality $\|f(x+y) - f(x) - f(y)\| \le \varepsilon$, for all x and y, can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, is said to satisfy the Hyers–Ulam stability.

Th.M. Rassias [20] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

and derived Hyers' theorem for the stability of the additive mapping as a special case. Thus in [20], a proof of the generalized Hyers–Ulam stability for the linear mapping between Banach spaces was obtained. A particular case of Th.M. Rassias' theorem regarding the Hyers–Ulam stability of the additive mapping was proved by T. Aoki [1]. The stability concept that was introduced by Th.M. Rassias' theorem provided some influence to a number of mathematicians to develop the notion of what is known today with the term Hyers–Ulam–Rassias stability of the linear mapping. Since then, the stability of several functional equations has been extensively investigated by several mathematicians (see for example [1, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 18, 21, 22, 23] and the references therein).

The terminologies Hyers–Ulam–Rassias stability and Hyers–Ulam stability can also be applied to the case of other functional equations, differential equations, and to various integral equations.

The function

$$f(x) = \frac{c}{x} \ (x > 0)$$

is a solution of the functional equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}. \tag{1.1}$$

In this paper, we will adopt the ideas from [3, 15, 16, 19] and prove the Hyers–Ulam stability of the functional equation (1.1).

2. PRELIMINARIES

For a nonempty set X, we introduce the definition of the generalized metric on X. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies

(M_1) $d(x, y) = 0$ if and only if $x = y$;
(M_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(M_3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We remark that the only one difference of the generalized metric from the usual metric is that the range of the former is permitted to include infinity.

We now introduce one of fundamental results of fixed point theory. For the proof, we refer to [17]. This theorem will play an important role in proving our main theorem.

Theorem 2.1. *Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the following are true:*

(a) *The sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;*

(b) *x^* is the unique fixed point of Λ in*

$$X^* = \{y \in X \mid d(\Lambda^k x, y) < \infty\};$$

(c) *If $y \in X^*$, then*

$$d(y, x^*) \le \frac{1}{1-L} d(\Lambda y, y).$$

3. HYERS–ULAM STABILITY OF (1.1)

Recently, Cădariu and Radu [3] applied a fixed point method to the investigation of the Cauchy additive functional equation. Using such an idea they presented a proof for the Hyers–Ulam stability of that equation (ref. [2, 19]).

By using the idea of Cădariu and Radu, we prove our main theorem concerning the Hyers–Ulam stability of the functional equation (1.1).

In the following theorem, we will set $\frac{0^2}{0} = 0$.

Theorem 3.1. *If a function $f : (0, \infty) \to [0, \infty)$ satisfies the functional inequality*

$$\left| f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)} \right| \le \varepsilon \tag{3.1}$$

for all $x, y > 0$ and for some $\varepsilon > 0$, then there exists a unique solution function $F : (0, \infty) \to [0, \infty)$ of Eq. (1.1) such that

$$|f(x) - F(x)| \le 2\varepsilon \tag{3.2}$$

for any $x > 0$.

Proof. Let us define a set $\mathcal{X}$ by

$$\mathcal{X} = \{h : (0, \infty) \to [0, \infty) \mid h \text{ is a function}\}$$

and introduce a generalized metric d on $\mathcal{X}$ as follows:

$$d(g, h) = \inf\{C \in [0, \infty] \mid |g(x) - h(x)| \le C\varepsilon \text{ for all } x > 0\}. \tag{3.3}$$

(Here we will give a proof for the triangle inequality only. Assume that $d(g, h) > d(g, k) + d(k, h)$ would hold for some $g, h, k \in \mathcal{X}$. Then, by (3.3), there would exist an $x_0 > 0$ with

$$|g(x_0) - h(x_0)| > [d(g, k) + d(k, h)]\,\varepsilon \ge |g(x_0) - k(x_0)| + |k(x_0) - h(x_0)|,$$

a contradiction.)

We assert that $(\mathcal{X}, d)$ is complete. We will follow the idea from [16] to prove the completeness of $(\mathcal{X}, d)$. Let $\{h_n\}$ be a Cauchy sequence in $(\mathcal{X}, d)$. Then, for any $C > 0$, there exists an integer $N_c > 0$ such that $d(h_m, h_n) \le C$ for all $m, n \ge N_c$. It further follows from (3.3) that

$$\forall\, C > 0\; \exists\, N_c \in \mathbb{N}\; \forall\, m, n \ge N_c\; \forall\, x > 0 : \; |h_m(x) - h_n(x)| \le C\varepsilon. \tag{3.4}$$

If x is a fixed positive real number, (3.4) implies that $\{h_n(x)\}$ is a Cauchy sequence in $([0, \infty), |\cdot|)$. Since $([0, \infty), |\cdot|)$ is complete, $\{h_n(x)\}$ converges for each $x > 0$. Thus, we can define a function $h : (0, \infty) \to [0, \infty)$ by

$$h(x) = \lim_{n\to\infty} h_n(x),$$

and hence h belongs to $\mathcal{X}$.

If we let m increase to infinity, it then follows from (3.4) that

$$\forall\, C > 0 \;\exists\, N_c \in \mathbb{N} \;\forall\, n \geq N_c \;\forall\, x > 0 : \; |h(x) - h_n(x)| \leq C\varepsilon.$$

Further if we consider (3.3), then we conclude that

$$\forall\, C > 0 \;\exists\, N_c \in \mathbb{N} \;\forall\, n \geq N_c : \; d(h, h_n) \leq C,$$

that is, the Cauchy sequence $\{h_n\}$ converges to h in $(\mathcal{X}, d)$. Hence, $(\mathcal{X}, d)$ is complete.

Now, let us define an operator $\Lambda : \mathcal{X} \to \mathcal{X}$ by

$$(\Lambda h)(x) = \frac{1}{2} h\left(\frac{1}{2}x\right) \;\; (x > 0) \tag{3.5}$$

for all $h \in \mathcal{X}$. (It is obvious that $\Lambda h \in \mathcal{X}$.)

We assert that Λ is strictly contractive on $\mathcal{X}$. For any $g, h \in \mathcal{X}$, let us choose a $C_{gh} \in [0, \infty]$ satisfying $d(g, h) \leq C_{gh}$. Then, using (3.3), we have

$$|g(x) - h(x)| \leq C_{gh}\varepsilon \tag{3.6}$$

for all $x > 0$. Using (3.5) and (3.6), we get

$$|(\Lambda g)(x) - (\Lambda h)(x)| = \left| \frac{1}{2} g\left(\frac{1}{2}x\right) - \frac{1}{2} h\left(\frac{1}{2}x\right) \right| \leq \frac{1}{2} C_{gh}\varepsilon$$

for all $x > 0$, that is, $d(\Lambda g, \Lambda h) \leq \frac{1}{2} C_{gh}$. Hence, we may conclude that

$$d(\Lambda g, \Lambda h) \leq L d(g, h)$$

with $L = \frac{1}{2}$.

Moreover, it follows from (3.1) and (3.5) that

$$|(\Lambda f)(x) - f(x)| = \left| \frac{1}{2} f\left(\frac{1}{2}x\right) - f(x) \right| \leq \varepsilon$$

for every $x > 0$. Thus, (3.3) implies that

$$d(\Lambda f, f) \leq 1. \tag{3.7}$$

Therefore, it follows from Theorem 2.1 (a) that there exists a function $F : (0, \infty) \to [0, \infty)$ such that $\Lambda^n f \to F$ in $(\mathcal{X}, d)$ and $\Lambda F = F$. Indeed, it holds that

$$F(x) = \lim_{n \to \infty} (\Lambda^n f)(x) = \lim_{n \to \infty} \frac{1}{2^n} f\left(\frac{1}{2^n}x\right)$$

for any $x > 0$. Thus, it follows from (3.1) that

$$\begin{aligned} \left| F(x+y) - \frac{F(x)F(y)}{F(x)+F(y)} \right| &= \lim_{n \to \infty} \frac{1}{2^n} \left| f\left(\frac{1}{2^n}(x+y)\right) - \frac{f\left(\frac{1}{2^n}x\right) f\left(\frac{1}{2^n}y\right)}{f\left(\frac{1}{2^n}x\right) + f\left(\frac{1}{2^n}y\right)} \right| \\ &\leq \lim_{n \to \infty} \frac{1}{2^n}\varepsilon \\ &= 0 \end{aligned}$$

for all $x, y > 0$. Hence, F is a solution of the functional equation (1.1).

Moreover, Theorem 2.1 (c), together with (3.7), implies that

$$d(f, F) \leq \frac{1}{1-L} d(\Lambda f, f) \leq 2.$$

That is, in view of (3.3), the inequality (3.2) is true for each $x > 0$.

Finally, let $G : (0, \infty) \to [0, \infty)$ be another solution of the functional equation (1.1) with

$$|f(x) - G(x)| \leq 2\varepsilon$$

for all $x > 0$. (It easily follows from (1.1) that G is a fixed point of Λ, that is, $\Lambda G = G$.) Then, since $d(f, G) < \infty$, we have

$$G \in \mathcal{X}^* = \{g \in \mathcal{X} \mid d(f, g) < \infty\}.$$

Therefore, since both F and G are fixed points of Λ, Theorem 2.1 (b) implies that $F = G$, that is, F is unique. ∎

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