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INEQUALITIES FOR THE ČEBYŠEV FUNCTIONAL OF TWO FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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ABSTRACT. Some recent inequalities for the Čebyšev functional of two functions of selfadjoint linear operators in Hilbert spaces, under suitable assumptions for the involved functions and operators, are surveyed.

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CONTENTS

1. INTRODUCTION

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle\cdot,\cdot\rangle)$. The *Gelfand map* establishes a $*$-isometric isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted by $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [41, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have:

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
(iii) $\|\Phi(f)\| = \|f\| := \sup_{t\in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \quad \text{for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real-valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e., $f(A)$ is a *positive operator* on H. In addition, if both f and g are real-valued functions on $Sp(A)$, then the following important property holds:

$$\text{(P)} \qquad f(t) \geq g(t) \quad \text{for any } t \in Sp(A) \quad \text{implies that} \quad f(A) \geq g(A)$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [41] and the references therein.

For other results, see [45] and [59].

The main aim of the present paper is to survey a number of recent results due to the author (published in preprint form in the papers [29] and [30]) concerning some natural extensions of the celebrated *Čebyšev* and *Grüss* inequalities to two continuous functions of selfadjoint operators defined on a real or complex Hilbert space. Applications for a number of fundamental elementary functions such as the power, logarithmic and exponential functions of operators are also provided.

2. ČEBYŠEV'S INEQUALITY

2.1. Čebyšev's Inequality for Real Numbers.

First of all, let us recall a number of classical results for sequences of real numbers concerning the celebrated *Čebyšev inequality.*

Consider the real sequences ($n-$tuples) $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$ and the non-negative sequence $\mathbf{p} = (p_1, \dots, p_n)$ with $P_n := \sum_{i=1}^n p_i > 0$. Define the *weighted Čebyšev's functional as*

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) := \frac{1}{P_n}\sum_{i=1}^n p_i a_i b_i - \frac{1}{P_n}\sum_{i=1}^n p_i a_i \cdot \frac{1}{P_n}\sum_{i=1}^n p_i b_i. \tag{2.1}$$

In 1882 – 1883, Čebyšev [6] and [7] proved that if $\mathbf{a}$ and $\mathbf{b}$ are *monotonic* in the same (opposite) sense, then

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) \geq (\leq)\, 0. \tag{2.2}$$

For the special case $\mathbf{p} = \mathbf{a} \geq \mathbf{0}$, it appears that the inequality (2.2) had been obtained by Laplace long before Čebyšev (see for example [49, p. 240]).

The inequality (2.2) was mentioned by Hardy, Littlewood and Pólya in their 1934 book [43] in the more general setting of synchronous sequences, i.e., if $\mathbf{a}$, $\mathbf{b}$ are *synchronous* (*asynchronous*), then

$$(a_i - a_j)(b_i - b_j) \geq (\leq)\, 0 \quad \text{for any } i, j \in \{1, \dots, n\}, \tag{2.3}$$

and (2.2) also holds.

A relaxation of the synchronicity condition was provided by M. Biernacki in 1951, [4], who showed that, if $\mathbf{a}$, $\mathbf{b}$ are *monotonic in mean* in the same sense, i.e., for $P_k := \sum_{i=1}^{k} p_i$, $k = 1, \dots, n-1$,

$$\frac{1}{P_k}\sum_{i=1}^{k} p_i a_i \leq (\geq) \frac{1}{P_{k+1}}\sum_{i=1}^{k+1} p_i a_i, \quad k \in \{1, \dots, n-1\} \tag{2.4}$$

and

$$\frac{1}{P_k}\sum_{i=1}^{k} p_i b_i \leq (\geq) \frac{1}{P_{k+1}}\sum_{i=1}^{k+1} p_i b_i, \quad k \in \{1, \dots, n-1\}, \tag{2.5}$$

then (2.2) holds with "$\geq$". If $\mathbf{a}$, $\mathbf{b}$ are monotonic in mean in the opposite sense then (2.2) holds with "$\leq$".

If the assumption of nonnegativity for the components of $\mathbf{p}$ is dropped, then one may state the following inequality obtained by Mitrinović and Pečarić in 1991, [48]: If $0 \leq P_i \leq P_n$ for each $i \in \{1, \dots, n-1\}$, then

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) \geq 0, \tag{2.6}$$

provided that $\mathbf{a}$ and $\mathbf{b}$ are sequences with the same monotonicity.

If $\mathbf{a}$ and $\mathbf{b}$ are monotonic in the opposite sense, the sign of the inequality (2.6) reverses.

Similar integral inequalities may be stated, however we do not present them here.

For other recent results on the Čebyšev inequality in either discrete or integral form see [5], [14], [15], [25], [36], [37], [49], [47], [50], [54], [55], [57], and the references therein.

The main aim of the present section is to provide operator versions of the Čebyšev inequality in different settings. Related results and some particular cases of interest are also given.

2.2. **A Version of the Čebyšev Inequality for One Operator.** We say that the functions $f, g : [a, b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a, b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq)\, 0 \quad \text{for each } t, s \in [a, b].$$

It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a, b]$, then they are synchronous on $[a, b]$ while if they have opposite monotonicity, they are asynchronous.

For some extensions of the discrete *Čebyšev inequality* for *synchronous (asynchronous)* sequences of vectors in an inner product space, see [39] and [38].

The following result provides an inequality of Čebyšev type for functions of selfadjoint operators.

Theorem 2.1 (Dragomir, 2008, [29]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\langle f(A) g(A) x, x\rangle \geq (\leq) \langle f(A) x, x\rangle \cdot \langle g(A) x, x\rangle \tag{2.7}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. We consider only the case of synchronous functions. In this case we have then

$$f(t)\,g(t)+f(s)\,g(s)\geq f(t)\,g(s)+f(s)\,g(t) \tag{2.8}$$

for each $t,s\in[a,b]$.

If we fix $s\in[a,b]$ and apply the property (P) for the inequality (2.8) then we have for each $x\in H$ with $\|x\|=1$ that

$$\langle (f(A)\,g(A)+f(s)\,g(s)\,1_H)\,x,x\rangle \geq \langle (g(s)\,f(A)+f(s)\,g(A))\,x,x\rangle,$$

which is clearly equivalent with

$$\langle f(A)\,g(A)\,x,x\rangle+f(s)\,g(s)\geq g(s)\,\langle f(A)\,x,x\rangle+f(s)\,\langle g(A)\,x,x\rangle \tag{2.9}$$

for each $s\in[a,b]$.

Now, if we apply again the property (P) for the inequality (2.9), then we have for any $y\in H$ with $\|y\|=1$ that

$$\begin{aligned}\langle (\langle f(A)\,g(A)\,x,x\rangle\,1_H+f(A)\,g(A))\,y,y\rangle \\ \geq \langle (\langle f(A)\,x,x\rangle\,g(A)+\langle g(A)\,x,x\rangle\,f(A))\,y,y\rangle,\end{aligned}$$

which is clearly equivalent with

$$\begin{aligned}\langle f(A)\,g(A)\,x,x\rangle+\langle f(A)\,g(A)\,y,y\rangle \\ \geq \langle f(A)\,x,x\rangle\,\langle g(A)\,y,y\rangle+\langle f(A)\,y,y\rangle\,\langle g(A)\,x,x\rangle\end{aligned} \tag{2.10}$$

for each $x,y\in H$ with $\|x\|=\|y\|=1$. This is an inequality of interest in itself.

Finally, on making $y=x$ in (2.10) we deduce the desired result (2.7). ∎

Some particular cases are of interest for applications. In the first instance we consider the case of power functions.

Example 2.1. *Assume that A is a positive operator on the Hilbert space H and $p,q>0$. Then for each $x\in H$ with $\|x\|=1$, we have the inequality*

$$\left\langle A^{p+q}x,x\right\rangle \geq \langle A^p x,x\rangle\cdot\langle A^q x,x\rangle. \tag{2.11}$$

If A is positive definite, then the inequality (2.11) also holds for $p,q<0$.

If A is positive definite and either $p>0,q<0$ or $p<0,q>0$, then the reverse inequality holds in (2.11).

Another case of interest for applications is the exponential function.

Example 2.2. *Assume that A is a selfadjoint operator on H. If $\alpha,\beta>0$ or $\alpha,\beta<0$, then*

$$\langle \exp[(\alpha+\beta)\,A]\,x,x\rangle \geq \langle \exp(\alpha A)\,x,x\rangle\cdot\langle \exp(\beta A)\,x,x\rangle \tag{2.12}$$

for each $x\in H$ with $\|x\|=1$.

If either $\alpha>0,\beta<0$ or $\alpha<0,\beta>0$, then the reverse inequality holds in (2.12).

The following particular cases may be of interest as well:

Example 2.3.
a. *Assume that A is positive definite and $p>0$. Then*

$$\langle A^p\log Ax,x\rangle \geq \langle A^p x,x\rangle\cdot\langle \log Ax,x\rangle \tag{2.13}$$

for each $x\in H$ with $\|x\|=1$. If $p<0$, then the reverse inequality holds in (2.13).
b. *Assume that A is positive definite and $Sp(A)\subset(0,1)$. If $r,s>0$ or $r,s<0$ then*

$$\left\langle (1_H-A^r)^{-1}(1_H-A^s)^{-1}x,x\right\rangle \geq \left\langle (1_H-A^r)^{-1}x,x\right\rangle\cdot\left\langle (1_H-A^s)^{-1}x,x\right\rangle \tag{2.14}$$

for each $x \in H$ *with* $\|x\| = 1$.

If either $r > 0, s < 0$ *or* $r < 0, s > 0$, *then the reverse inequality holds in (2.14).*

Remark 2.1. We observe, from the proof of the above theorem that, if A and B are selfadjoint operators and $Sp(A), Sp(B) \subseteq [m, M]$, then for any continuous synchronous (asynchronous) functions $f, g : [m, M] \longrightarrow \mathbb{R}$ we have the more general result

$$\begin{aligned} (2.15)\quad & \langle f(A) g(A) x, x\rangle + \langle f(B) g(B) y, y\rangle \\ & \geq (\leq) \langle f(A) x, x\rangle \langle g(B) y, y\rangle + \langle f(B) y, y\rangle \langle g(A) x, x\rangle \end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

If $f : [m, M] \longrightarrow (0, \infty)$ is continuous, then the functions f^p, f^q are synchronous in the case when $p, q > 0$ or $p, q < 0$ and asynchronous when either $p > 0, q < 0$ or $p < 0, q > 0$. In this situation, if A and B are positive definite operators then we have the inequality

$$\begin{aligned} (2.16)\quad & \left\langle f^{p+q}(A) x, x\right\rangle + \left\langle f^{p+q}(B) y, y\right\rangle \\ & \geq \langle f^p(A) x, x\rangle \langle f^q(B) y, y\rangle + \langle f^p(B) y, y\rangle \langle f^q(A) x, x\rangle \end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$ where either $p, q > 0$ or $p, q < 0$. If $p > 0, q < 0$ or $p < 0, q > 0$ then the reverse inequality also holds in (2.16).

As particular cases, we should observe that for $p = q = 1$ and $f(t) = t$, we get from (2.16) the inequality

$$\left\langle A^2 x, x\right\rangle + \left\langle B^2 y, y\right\rangle \geq 2 \cdot \langle Ax, x\rangle \langle By, y\rangle \tag{2.17}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

For $p = 1$ and $q = -1$ we have from (2.16)

$$\langle Ax, x\rangle \left\langle B^{-1} y, y\right\rangle + \langle By, y\rangle \left\langle A^{-1} x, x\right\rangle \leq 2 \tag{2.18}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

2.3. **A Version of the Čebyšev Inequality for n Operators.** The following multiple operator version of Theorem 2.1 holds:

Theorem 2.2 (Dragomir, 2008, [29])**.** *Let* A_j *be selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous and synchronous (asynchronous) on* $[m, M]$, *then*

$$\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle \geq (\leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle, \tag{2.19}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Proof. As in [41, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix} \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = 1$,

$$\left\langle f\left(\widetilde{A}\right) g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle,$$

$$\left\langle f\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle \quad \text{and} \quad \left\langle g\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^{n}\left\langle g\left(A_j\right)x_j,x_j\right\rangle.$$

Applying Theorem 2.1 for $\widetilde{A}$ and $\widetilde{x}$, we deduce the desired result (2.19). ∎

The following particular cases may be of interest for applications.

Example 2.4. *Assume that $A_j, j\in\{1,\dots,n\}$ are positive operators on the Hilbert space H and $p,q>0$. Then for each $x_j\in H, j\in\{1,\dots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$ we have*

$$\left\langle\sum_{j=1}^{n}A_j^{p+q}x_j,x_j\right\rangle\geq\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle. \tag{2.20}$$

If A_j are positive definite, then the inequality (2.20) also holds for $p,q<0$.

If A_j are positive definite and either $p>0,q<0$ or $p<0,q>0$, then the reverse inequality holds in (2.20).

Another case of interest for applications is the exponential function.

Example 2.5. *Assume that $A_j, j\in\{1,\dots,n\}$ are selfadjoint operators on H. If $\alpha,\beta>0$ or $\alpha,\beta<0$, then*

$$\left\langle\sum_{j=1}^{n}\exp\left[\left(\alpha+\beta\right)A_j\right]x_j,x_j\right\rangle\geq\sum_{j=1}^{n}\left\langle\exp\left(\alpha A_j\right)x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle\exp\left(\beta A_j\right)x_j,x_j\right\rangle \tag{2.21}$$

for each $x_j\in H, j\in\{1,\dots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$.

If either $\alpha>0,\beta<0$ or $\alpha<0,\beta>0$, then the reverse inequality holds in (2.21).

The following particular cases may be of interest as well:

Example 2.6.

a. *Assume that $A_j, j\in\{1,\dots,n\}$ are positive definite operators and $p>0$. Then*

$$\left\langle\sum_{j=1}^{n}A_j^{p}\log A_jx_j,x_j\right\rangle\geq\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle\log A_jx_j,x_j\right\rangle \tag{2.22}$$

for each $x_j\in H, j\in\{1,\dots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$. If $p<0$, then the reverse inequality holds in (2.22).

b. *If A_j are positive definite and $Sp\left(A_j\right)\subset(0,1)$ for $j\in\{1,\dots,n\}$, then for $r,s>0$ or $r,s<0$ we have*

$$\begin{aligned}\left\langle\sum_{j=1}^{n}\left(1_H-A_j^{r}\right)^{-1}\left(1_H-A_j^{s}\right)^{-1}x_j,x_j\right\rangle\\ \geq\sum_{j=1}^{n}\left\langle\left(1_H-A_j^{r}\right)^{-1}x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle\left(1_H-A_j^{s}\right)^{-1}x_j,x_j\right\rangle\end{aligned} \tag{2.23}$$

for each $x_j\in H, j\in\{1,\dots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$.

If either $r>0,s<0$ or $r<0,s>0$, then the reverse inequality holds in (2.23).

2.4. **Another Version of the Čebyšev Inequality for n Operators.** The following different version of the Čebyšev inequality for a sequence of operators also holds:

Theorem 2.3 (Dragomir, 2008, [29])**.** *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle \geq (\leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle, \tag{2.24}$$

for any $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $x \in H$ with $\|x\| = 1$.

In particular,

$$\left\langle \frac{1}{n} \sum_{j=1}^{n} f(A_j) g(A_j) x, x \right\rangle \geq (\leq) \left\langle \frac{1}{n} \sum_{j=1}^{n} f(A_j) x, x \right\rangle \cdot \left\langle \frac{1}{n} \sum_{j=1}^{n} g(A_j) x, x \right\rangle, \tag{2.25}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. We provide here two proofs. The first is based on the inequality (2.15) and generates as a by-product a more general result. The second is derived from Theorem 2.2.

1. If we make use of the inequality (2.15), then we can write

$$\begin{aligned} &\langle f(A_j) g(A_j) x, x \rangle + \langle f(B_k) g(B_k) y, y \rangle \\ &\qquad \geq (\leq) \langle f(A_j) x, x \rangle \langle g(B_k) y, y \rangle + \langle f(B_k) y, y \rangle \langle g(A_j) x, x \rangle, \end{aligned} \tag{2.26}$$

which holds for any A_j and B_k selfadjoint operators with $Sp(A_j), Sp(B_k) \subseteq [m, M], j, k \in \{1, \ldots, n\}$ and for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, if $p_j \geq 0, q_k \geq 0, j, k \in \{1, \ldots, n\}$ and $\sum_{j=1}^{n} p_j = \sum_{k=1}^{n} q_k = 1$ then, by multiplying (2.26) with $p_j \geq 0, q_k \geq 0$ and summing over j and k from 1 to n we deduce the following inequality that is of interest in its own right:

$$\begin{aligned} &\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle + \left\langle \sum_{k=1}^{n} q_k f(B_k) g(B_k) y, y \right\rangle \\ &\qquad \geq (\leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \left\langle \sum_{k=1}^{n} q_k g(B_k) y, y \right\rangle \\ &\qquad\qquad + \left\langle \sum_{k=1}^{n} q_k f(B_k) y, y \right\rangle \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \end{aligned} \tag{2.27}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Finally, the choice $B_k = A_k, q_k = p_k$ and $y = x$ in (2.27) produces the desired result (2.24).

2. If we choose in Theorem 2.2 $x_j = \sqrt{p_j} \cdot x, j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$ with $\|x\| = 1$, then a simple calculation shows that the inequality (2.19) becomes (2.24). The details are omitted. ∎

Remark 2.2. We remark that the case $n = 1$ in (2.24) produces the inequality (2.7).

The following particular cases are of interest:

Example 2.7. *Assume that $A_j, j \in \{1,\dots,n\}$ are positive operators on the Hilbert space H, $p_j \geq 0, j \in \{1,\dots,n\}$ with $\sum_{j=1}^n p_j = 1$ and $p, q > 0$. Then for each $x \in H$ with $\|x\| = 1$ we have*

$$\left\langle \sum_{j=1}^n p_j A_j^{p+q} x, x \right\rangle \geq \left\langle \sum_{j=1}^n p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^n p_j A_j^{q} x, x \right\rangle . \tag{2.28}$$

If $A_j, j \in \{1,\dots,n\}$ are positive definite, then the inequality (2.28) also holds for $p, q < 0$.

If $A_j, j \in \{1,\dots,n\}$ are positive definite and either $p > 0, q < 0$ or $p < 0, q > 0$, then the reverse inequality holds in (2.28).

Another case of interest for applications is the exponential function.

Example 2.8. *Assume that $A_j, j \in \{1,\dots,n\}$ are selfadjoint operators on H and $p_j \geq 0, j \in \{1,\dots,n\}$ with $\sum_{j=1}^n p_j = 1$. If $\alpha, \beta > 0$ or $\alpha, \beta < 0$, then*

$$\left\langle \sum_{j=1}^n p_j \exp\left[(\alpha+\beta) A_j\right] x, x \right\rangle \geq \left\langle \sum_{j=1}^n p_j \exp\left(\alpha A_j\right) x, x \right\rangle \cdot \left\langle \sum_{j=1}^n p_j \exp\left(\beta A_j\right) x, x \right\rangle \tag{2.29}$$

for each $x \in H$ with $\|x\| = 1$.

If either $\alpha > 0, \beta < 0$ or $\alpha < 0, \beta > 0$, then the reverse inequality holds in (2.29).

The following particular cases may be of interest as well:

Example 2.9.

a. *Assume that $A_j, j \in \{1,\dots,n\}$ are positive definite operators on the Hilbert space H, $p_j \geq 0, j \in \{1,\dots,n\}$ with $\sum_{j=1}^n p_j = 1$ and $p > 0$. Then*

$$\left\langle \sum_{j=1}^n p_j A_j^{p} \log A_j x, x \right\rangle \geq \left\langle \sum_{j=1}^n p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^n p_j \log A_j x, x \right\rangle . \tag{2.30}$$

If $p < 0$, then the reverse inequality holds in (2.30).

b. *Assume that $A_j, j \in \{1,\dots,n\}$ are positive definite operators on the Hilbert space H, $Sp(A_j) \subset (0,1)$ and $p_j \geq 0, j \in \{1,\dots,n\}$ with $\sum_{j=1}^n p_j = 1$. If $r, s > 0$ or $r, s < 0$ then*

$$\left\langle \sum_{j=1}^n p_j \left(1_H - A_j^r\right)^{-1} \left(1_H - A_j^s\right)^{-1} x, x \right\rangle \geq \left\langle \sum_{j=1}^n p_j \left(1_H - A_j^r\right)^{-1} x, x \right\rangle \cdot \left\langle \sum_{j=1}^n p_j \left(1_H - A_j^s\right)^{-1} x, x \right\rangle \tag{2.31}$$

for each $x \in H$ with $\|x\| = 1$.

If either $r > 0, s < 0$ or $r < 0, s > 0$, then the reverse inequality holds in (2.31).

We remark that the following operator norm inequality can be stated as well:

Corollary 2.4. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1,\dots,n\}$ and some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous, asynchronous on $[m, M]$ and for*

$p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ the operator $\sum_{j=1}^{n} p_j f(A_j) g(A_j)$ is positive, then

$$\left\| \sum_{j=1}^{n} p_j f(A_j) g(A_j) \right\| \leq \left\| \sum_{j=1}^{n} p_j f(A_j) \right\| \cdot \left\| \sum_{j=1}^{n} p_j g(A_j) \right\|. \tag{2.32}$$

Proof. We have from (2.24) that

$$0 \leq \left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle \leq \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$. Taking the supremum in this inequality over $x \in H$ with $\|x\| = 1$, we deduce the desired result (2.32). ∎

Corollary 2.4 provides some interesting norm inequalities for sums of positive operators as follows:

Example 2.10.
a. *If $A_j, j \in \{1, \ldots, n\}$ are positive definite and either $p > 0, q < 0$ or $p < 0, q > 0$, then for $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ we have the norm inequality:*

$$\left\| \sum_{j=1}^{n} p_j A_j^{p+q} \right\| \leq \left\| \sum_{j=1}^{n} p_j A_j^{p} \right\| \cdot \left\| \sum_{j=1}^{n} p_j A_j^{q} \right\|. \tag{2.33}$$

In particular,

$$1 \leq \left\| \sum_{j=1}^{n} p_j A_j^{r} \right\| \cdot \left\| \sum_{j=1}^{n} p_j A_j^{-r} \right\| \tag{2.34}$$

for any $r > 0$.
b. *Assume that $A_j, j \in \{1, \ldots, n\}$ are selfadjoint operators on H and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$. If either $\alpha > 0, \beta < 0$ or $\alpha < 0, \beta > 0$, then*

$$\left\| \sum_{j=1}^{n} p_j \exp[(\alpha + \beta) A_j] \right\| \leq \left\| \sum_{j=1}^{n} p_j \exp(\alpha A_j) \right\| \cdot \left\| \sum_{j=1}^{n} p_j \exp(\beta A_j) \right\|. \tag{2.35}$$

In particular,

$$1 \leq \left\| \sum_{j=1}^{n} p_j \exp(\gamma A_j) \right\| \cdot \left\| \sum_{j=1}^{n} p_j \exp(-\gamma A_j) \right\|. \tag{2.36}$$

for any $\gamma > 0$.

2.5. **Related Results for One Operator.** The following result that is related to the Čebyšev inequality may be stated:

Theorem 2.5 (Dragomir, 2008, [29])**.** *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous on $[m, M]$, then*

$$\begin{aligned} &\langle f(A) g(A) x, x \rangle - \langle f(A) x, x \rangle \cdot \langle g(A) x, x \rangle \\ &\qquad \geq [\langle f(A) x, x \rangle - f(\langle Ax, x \rangle)] \cdot [g(\langle Ax, x \rangle) - \langle g(A) x, x \rangle] \end{aligned} \tag{2.37}$$

for any $x \in H$ with $\|x\| = 1$.

If f, g are asynchronous, then

$$\langle f(A)x,x\rangle \cdot \langle g(A)x,x\rangle - \langle f(A)g(A)x,x\rangle \geq [\langle f(A)x,x\rangle - f(\langle Ax,x\rangle)] \cdot [\langle g(A)x,x\rangle - g(\langle Ax,x\rangle)] \tag{2.38}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since f, g are synchronous and $m \leq \langle Ax,x\rangle \leq M$ for any $x \in H$ with $\|x\| = 1$, then we have

$$[f(t) - f(\langle Ax,x\rangle)][g(t) - g(\langle Ax,x\rangle)] \geq 0 \tag{2.39}$$

for any $t \in [a,b]$ and $x \in H$ with $\|x\| = 1$.

On utilising the property (P) for the inequality (2.39) we have that

$$\langle [f(B) - f(\langle Ax,x\rangle)][g(B) - g(\langle Ax,x\rangle)]y,y\rangle \geq 0 \tag{2.40}$$

for any B a bounded linear operator with $Sp(B) \subseteq [m,M]$ and $y \in H$ with $\|y\| = 1$.

Since

$$\begin{aligned} &\langle [f(B) - f(\langle Ax,x\rangle)][g(B) - g(\langle Ax,x\rangle)]y,y\rangle \\ &= \langle f(B)g(B)y,y\rangle + f(\langle Ax,x\rangle)g(\langle Ax,x\rangle) \\ &\quad - \langle f(B)y,y\rangle g(\langle Ax,x\rangle) - f(\langle Ax,x\rangle)\langle g(B)y,y\rangle, \end{aligned} \tag{2.41}$$

then from (2.40) we get

$$\langle f(B)g(B)y,y\rangle + f(\langle Ax,x\rangle)g(\langle Ax,x\rangle) \geq \langle f(B)y,y\rangle g(\langle Ax,x\rangle) + f(\langle Ax,x\rangle)\langle g(B)y,y\rangle$$

which is clearly equivalent with

$$\langle f(B)g(B)y,y\rangle - \langle f(A)y,y\rangle \cdot \langle g(A)y,y\rangle \geq [\langle f(B)y,y\rangle - f(\langle Ax,x\rangle)] \cdot [g(\langle Ax,x\rangle) - \langle g(B)y,y\rangle] \tag{2.42}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$. This inequality is of interest in its own right.

Now, if we choose $B = A$ and $y = x$ in (2.42), then we deduce the desired result (2.37). ∎

The following result which improves the Čebyšev inequality may be stated:

Corollary 2.6 (Dragomir, 2008, [29]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$. If $f, g : [m,M] \longrightarrow \mathbb{R}$ are continuous, synchronous and one is convex while the other is concave on $[m,M]$, then*

$$\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle \cdot \langle g(A)x,x\rangle \geq [\langle f(A)x,x\rangle - f(\langle Ax,x\rangle)] \cdot [g(\langle Ax,x\rangle) - \langle g(A)x,x\rangle] \geq 0 \tag{2.43}$$

for any $x \in H$ with $\|x\| = 1$.

If f, g are asynchronous and either both of them are convex or both of them areconcave on $[m,M]$, then

$$\langle f(A)x,x\rangle \cdot \langle g(A)x,x\rangle - \langle f(A)g(A)x,x\rangle \geq [\langle f(A)x,x\rangle - f(\langle Ax,x\rangle)] \cdot [\langle g(A)x,x\rangle - g(\langle Ax,x\rangle)] \geq 0 \tag{2.44}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. The second inequality follows by making use of the result due to Mond and Pečarić, see [51], [53] or [41, p. 5]:

$$\langle h(A)x, x\rangle \geq (\leq)\, h(\langle Ax, x\rangle) \tag{MP}$$

for any $x \in H$ with $\|x\| = 1$, provided that A is a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and h is convex (concave) on the given interval $[m, M]$. ∎

Corollary 2.6 offers the possibility of improving some of the results established before for power function as follows:

Example 2.11.
a. *Assume that A is a positive operator on the Hilbert space H. If $p \in (0,1)$ and $q \in (1, \infty)$, then for each $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\left\langle A^{p+q}x, x\right\rangle - \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle \geq [\langle A^q x, x\rangle - \langle Ax, x\rangle^q]\,[\langle Ax, x\rangle^p - \langle A^p x, x\rangle] \geq 0. \tag{2.45}$$

If A is positive definite and $p > 1, q < 0$, then

$$\langle A^p x, x\rangle \cdot \langle A^q x, x\rangle - \left\langle A^{p+q}x, x\right\rangle \geq [\langle A^q x, x\rangle - \langle Ax, x\rangle^q]\,[\langle A^p x, x\rangle - \langle Ax, x\rangle^p] \geq 0 \tag{2.46}$$

for each $x \in H$ with $\|x\| = 1$.
b. *Assume that A is positive definite and $p > 1$. Then*

$$\begin{aligned}\langle A^p \log Ax, x\rangle - \langle A^p x, x\rangle \cdot \langle \log Ax, x\rangle \\ \geq [\langle A^p x, x\rangle - \langle Ax, x\rangle^p]\,[\log\langle Ax, x\rangle - \langle \log Ax, x\rangle] \geq 0\end{aligned} \tag{2.47}$$

for each $x \in H$ with $\|x\| = 1$.

2.6. **Related Results for n Operators.** We can now state the following generalisation of Theorem 2.5 for n operators:

Theorem 2.7 (Dragomir, 2008, [29])**.** *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and some scalars $m < M$.*

(i) *If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous on $[m, M]$, then*

$$\begin{aligned}\sum_{j=1}^{n} \langle f(A_j)\, g(A_j)\, x_j, x_j\rangle - \sum_{j=1}^{n} \langle f(A_j)\, x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle g(A_j)\, x_j, x_j\rangle \\ \geq \left[\sum_{j=1}^{n} \langle f(A_j)\, x_j, x_j\rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right)\right] \\ \cdot \left[g\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right) - \sum_{j=1}^{n} \langle g(A_j)\, x_j, x_j\rangle\right]\end{aligned} \tag{2.48}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. Moreover, if one function is convex while the other is concave on $[m, M]$, then the right hand side of (2.48) is nonnegative.

(ii) *If f, g are asynchronous on $[m, M]$, then*

$$\sum_{j=1}^{n}\langle f(A_j)x_j, x_j\rangle \cdot \sum_{j=1}^{n}\langle g(A_j)x_j, x_j\rangle - \sum_{j=1}^{n}\langle f(A_j)g(A_j)x_j, x_j\rangle \geq \left[\sum_{j=1}^{n}\langle f(A_j)x_j, x_j\rangle - f\left(\sum_{j=1}^{n}\langle A_jx_j, x_j\rangle\right)\right] \cdot \left[\sum_{j=1}^{n}\langle g(A_j)x_j, x_j\rangle - g\left(\sum_{j=1}^{n}\langle A_jx_j, x_j\rangle\right)\right] \tag{2.49}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$. Observe that if either both of them are convex or both of them are concave on $[m, M]$, then the right hand side of (2.49) is nonnegative as well.

Proof. The argument is similar to the one from the proof of Theorem 2.2 on utilising the results from one operator case obtained in Theorem 2.5.

The nonnegativity of the right hand sides of the inequalities (2.48) and (2.49) follows by the use of the Jensen type result from [41, p. 5]

$$\sum_{j=1}^{n}\langle h(A_j)x_j, x_j\rangle \geq (\leq)\, h\left(\sum_{j=1}^{n}\langle A_jx_j, x_j\rangle\right) \tag{2.50}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$, which holds provided that A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$, some scalars $m < M$ and h is convex (concave) on $[m, M]$.

The details are omitted. ∎

Example 2.12.
a. *Assume that $A_j, j \in \{1, \ldots, n\}$ are positive operators on the Hilbert space H. If $p \in (0, 1)$ and $q \in (1, \infty)$, then for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$, we have*

$$\sum_{j=1}^{n}\left\langle A_j^{p+q}x_j, x_j\right\rangle - \sum_{j=1}^{n}\left\langle A_j^{p}x_j, x_j\right\rangle \cdot \sum_{j=1}^{n}\left\langle A_j^{q}x_j, x_j\right\rangle \geq \left[\sum_{j=1}^{n}\left\langle A_j^{q}x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\langle A_jx_j, x_j\rangle\right)^q\right] \cdot \left[\left(\sum_{j=1}^{n}\langle A_jx_j, x_j\rangle\right)^p - \sum_{j=1}^{n}\left\langle A_j^{p}x_j, x_j\right\rangle\right] \geq 0. \tag{2.51}$$

If A_j are positive definite and $p > 1, q < 0$, then

$$\sum_{j=1}^{n}\left\langle A_j^{p}x_j, x_j\right\rangle \cdot \sum_{j=1}^{n}\left\langle A_j^{q}x_j, x_j\right\rangle - \sum_{j=1}^{n}\left\langle A_j^{p+q}x_j, x_j\right\rangle \geq \left[\sum_{j=1}^{n}\left\langle A_j^{q}x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\langle A_jx_j, x_j\rangle\right)^q\right] \tag{2.52}$$

$$\cdot \left[\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)^p\right] \geq 0$$

for each $x_j \in H, j \in \{1,\ldots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.
***b.** Assume that A_j are positive definite and $p > 1$. Then*

$$\begin{aligned}(2.53)\quad \sum_{j=1}^{n}\left\langle A_j^p \log A x_j, x_j\right\rangle - \sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle \cdot \sum_{j=1}^{n}\left\langle \log A_j x_j, x_j\right\rangle \\ \geq \left[\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)^p\right] \\ \cdot \left[\sum_{j=1}^{n}\log\left\langle A_j x_j, x_j\right\rangle - \log\left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)\right] \geq 0\end{aligned}$$

for each $x_j \in H, j \in \{1,\ldots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

The following result may be stated as well:

Theorem 2.8 (Dragomir, 2008, [29])**.** *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1,\ldots,n\}$ and some scalars $m < M$.*

(i) *If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous on $[m, M]$, then*

$$\begin{aligned}(2.54)\quad \left\langle\sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle - \left\langle\sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle \\ \geq \left[f\left(\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle\right) - \left\langle\sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle\right] \\ \cdot \left[\left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle - g\left(\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle\right)\right]\end{aligned}$$

for any $p_j \geq 0, j \in \{1,\ldots,n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $x \in H$ with $\|x\| = 1$. Notice that if one is convex while the other is concave on $[m, M]$, then the right hand side of (2.54) is nonnegative.

(ii) *If f, g are asynchronous on $[m, M]$, then*

$$\begin{aligned}(2.55)\quad \left\langle\sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle - \left\langle\sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle \\ \geq \left[\left\langle\sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle - f\left(\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle\right)\right] \\ \cdot \left[\left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle - g\left(\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle\right)\right]\end{aligned}$$

for any $p_j \geq 0, j \in \{1,\ldots,n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $x \in H$ with $\|x\| = 1$. Note that, if either both of them are convex or both of them are concave on $[m, M]$, then the right hand side of (2.55) is nonnegative as well.

Proof. Follows from Theorem 2.7 on choosing $x_j = \sqrt{p_j} \cdot x, j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$.

Also, the positivity of the right hand term in (2.54) follows by the Jensen type inequality from the inequality (2.50) for the same choices, namely $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$. The details are omitted. ∎

Finally, we can list some particular inequalities that may be of interest for applications. They improve some of the results obtained above.

Example 2.13.
a. *Assume that $A_j, j \in \{1, \ldots, n\}$ are positive operators on the Hilbert space H and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$. If $p \in (0, 1)$ and $q \in (1, \infty)$, then for each $x \in H$ with $\|x\| = 1$ we have*

$$\left\langle \sum_{j=1}^{n} p_j A_j^{p+q} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^{q} x, x \right\rangle \geq \left[\left\langle \sum_{j=1}^{n} p_j A_j^{q} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{q} \right] \cdot \left[\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{p} - \left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \right] \geq 0. \tag{2.56}$$

If $A_j, j \in \{1, \ldots, n\}$ are positive definite and $p > 1, q < 0$, then

$$\left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^{q} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j^{p+q} x, x \right\rangle \geq \left[\left\langle \sum_{j=1}^{n} p_j A_j^{q} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{q} \right] \cdot \left[\left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{p} \right] \geq 0 \tag{2.57}$$

for each $x \in H$ with $\|x\| = 1$.
b. *Assume that A_j, $j \in \{1, \ldots, n\}$ are positive definite and $p > 1$. Then*

$$\left\langle \sum_{j=1}^{n} p_j A_j^{p} \log A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j \log A_j x, x \right\rangle \geq \left[\left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{p} \right] \cdot \left[\log \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j \log A_j x, x \right\rangle \right] \geq 0 \tag{2.58}$$

for each $x \in H$ with $\|x\| = 1$.

3. GRÜSS INEQUALITY

3.1. **Some Elementary Inequalities of Grüss Type.** In 1935, G. Grüss [42] proved the following integral inequality which gives an approximation of the integral of the product in terms of the product of the integrals as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \le \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma)\,, \tag{3.1}$$

where $f, g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi \le f(x) \le \Phi, \qquad \gamma \le g(x) \le \Gamma \tag{3.2}$$

for each $x \in [a,b]$, where $\phi, \Phi, \gamma, \Gamma$ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardjewski [49, Chapter X] established the following discrete version of Grüss' inequality:

Let $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ be two $n-$tuples of real numbers such that $r \le a_i \le R$ and $s \le b_i \le S$ for $i = 1, \dots, n$. Then one has

$$\left|\frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \cdot \frac{1}{n}\sum_{i=1}^{n} b_i\right| \le \frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s)\,, \tag{3.3}$$

where $[x]$ denotes the integer part of x, $x \in \mathbb{R}$.

For a simple proof of (3.1) as well as for some other integral inequalities of Grüss type, see Chapter X of the recent book [49]. For other related results see the papers [1] – [3], [8] – [10], [11] – [12], [18] – [35], [40], [56], [60] and the references therein.

3.2. **Operator Inequalities.** The following operator version of the Grüss inequality was obtained by Mond and Pečarić in [52]:

Theorem 3.1 (Mond-Pečarić, 1993, [52])**.** *Let C_j, $j \in \{1, \dots, n\}$ be selfadjoint operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and such that $m_j \cdot 1_H \le C_j \le M_j \cdot 1_H$ for $j \in \{1, \dots, n\}$, where 1_H is the identity operator on H. Further, let $g_j, h_j : [m_j, Mj] \to \mathbb{R}$, $j \in \{1, \dots, n\}$ be functions such that*

$$\varphi \cdot 1_H \le g_j(C_j) \le \Phi \cdot 1_H \quad \textit{and} \quad \gamma \cdot 1_H \le h_j(C_j) \le \Gamma \cdot 1_H \tag{3.4}$$

for each $j \in \{1, \dots, n\}$.

If $x_j \in H$, $j \in \{1, \dots, n\}$ are such that $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then

$$\left|\sum_{j=1}^{n} \langle g_j(C_j)\,h_j(C_j)\,x_j, x_j\rangle - \sum_{j=1}^{n} \langle g_j(C_j)\,x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle h_j(C_j)\,x_j, x_j\rangle\right| \le \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma)\,. \tag{3.5}$$

If $C_j, j \in \{1, \dots, n\}$ are selfadjoint operators such that $Sp(C_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$, some scalars $m < M$ and if $g, h : [m, M] \longrightarrow \mathbb{R}$ are continuous, then by the Mond-Pečarić inequality we deduce the following version of the Grüss inequality for operators

$$\left|\sum_{j=1}^{n}\langle g(C_j)h(C_j)x_j,x_j\rangle-\sum_{j=1}^{n}\langle g(C_j)x_j,x_j\rangle\cdot\sum_{j=1}^{n}\langle h(C_j)x_j,x_j\rangle\right| \leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{3.6}$$

where $x_j\in H$, $j\in\{1,\dots,n\}$ are such that $\sum_{j=1}^{n}\|x_j\|^2=1$ and $\varphi=\min_{t\in[m,M]}g(t)$, $\Phi=\max_{t\in[m,M]}g(t)$, $\gamma=\min_{t\in[m,M]}h(t)$ and $\Gamma=\max_{t\in[m,M]}h(t)$.

In particular, if the selfadjoint operator C satisfies the condition $Sp(C)\subseteq[m,M]$ for some scalars $m<M$, then

$$|\langle g(C)h(C)x,x\rangle-\langle g(C)x,x\rangle\cdot\langle h(C)x,x\rangle|\leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{3.7}$$

for any $x\in H$ with $\|x\|=1$.

Motivated by the above results we investigate in what follows other Grüss type inequalities for selfadjoint operators in Hilbert spaces. Some of the obtained results improve the inequalities (3.6) and (3.7) derived from the Mond-Pečarić inequality. Others provide different operator versions for the celebrated Grüss inequality mentioned above. Examples for power functions and the logarithmic function are given as well.

3.3. An Inequality of Grüss Type for One Operator.

The following result may be stated:

Theorem 3.2 (Dragomir, 2008, [30]). *Let A be a selfadjoint operator on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$ and assume that $Sp(A)\subseteq[m,M]$ for some scalars $m<M$. If f and g are continuous on $[m,M]$ and $\gamma:=\min_{t\in[m,M]}f(t)$ and $\Gamma:=\max_{t\in[m,M]}f(t)$, then*

$$\begin{aligned}\Big|\langle f(A)g(A)y,y\rangle&-\langle f(A)y,y\rangle\cdot\langle g(A)x,x\rangle\\&-\frac{\gamma+\Gamma}{2}[\langle g(A)y,y\rangle-\langle g(A)x,x\rangle]\Big|\\&\leq\frac{1}{2}\cdot(\Gamma-\gamma)\left[\|g(A)y\|^2+\langle g(A)x,x\rangle^2-2\langle g(A)x,x\rangle\langle g(A)y,y\rangle\right]^{1/2}\end{aligned} \tag{3.8}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

Proof. First of all, observe that, for each $\lambda\in\mathbb{R}$ and $x,y\in H$, $\|x\|=\|y\|=1$ we have the identity

$$\begin{aligned}&\langle(f(A)-\lambda\cdot 1_H)(g(A)-\langle g(A)x,x\rangle\cdot 1_H)y,y\rangle\\&=\langle f(A)g(A)y,y\rangle-\lambda\cdot[\langle g(A)y,y\rangle-\langle g(A)x,x\rangle]-\langle g(A)x,x\rangle\langle f(A)y,y\rangle.\end{aligned} \tag{3.9}$$

Taking the modulus in (3.9) we have

$$\begin{aligned}&|\langle f(A)g(A)y,y\rangle-\lambda\cdot[\langle g(A)y,y\rangle-\langle g(A)x,x\rangle]\\&\qquad-\langle g(A)x,x\rangle\langle f(A)y,y\rangle|\\&=|\langle(g(A)-\langle g(A)x,x\rangle\cdot 1_H)y,(f(A)-\lambda\cdot 1_H)y\rangle|\\&\leq\|g(A)y-\langle g(A)x,x\rangle y\|\,\|f(A)y-\lambda y\|\end{aligned} \tag{3.10}$$

$$= \left[\|g(A)y\|^2 + \langle g(A)x,x\rangle^2 - 2\langle g(A)x,x\rangle\langle g(A)y,y\rangle\right]^{1/2} \times \|f(A)y - \lambda y\|$$
$$\leq \left[\|g(A)y\|^2 + \langle g(A)x,x\rangle^2 - 2\langle g(A)x,x\rangle\langle g(A)y,y\rangle\right]^{1/2} \times \|f(A) - \lambda\cdot 1_H\|$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Now, since $\gamma = \min_{t\in[m,M]} f(t)$ and $\Gamma = \max_{t\in[m,M]} f(t)$, then by the property (P) we have that $\gamma \leq \langle f(A)y,y\rangle \leq \Gamma$ for each $y \in H$ with $\|y\| = 1$, which is clearly equivalent with

$$\left|\langle f(A)y,y\rangle - \frac{\gamma+\Gamma}{2}\|y\|^2\right| \leq \frac{1}{2}(\Gamma-\gamma)$$

or with

$$\left|\left\langle\left(f(A) - \frac{\gamma+\Gamma}{2}1_H\right)y,y\right\rangle\right| \leq \frac{1}{2}(\Gamma-\gamma)$$

for each $y \in H$ with $\|y\| = 1$.

Taking the supremum in this inequality we get

$$\left\|f(A) - \frac{\gamma+\Gamma}{2}\cdot 1_H\right\| \leq \frac{1}{2}(\Gamma-\gamma),$$

which together with the inequality (3.10) applied for $\lambda = \frac{\gamma+\Gamma}{2}$ produces the desired result (3.8). ∎

As a particular case of interest we can derive from the above theorem the following result of Grüss' type that improves (3.7):

Corollary 3.3 (Dragomir, 2008, [30])**.** *With the assumptions in Theorem 3.2 we have*

$$\text{(3.11)} \quad |\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle| \leq \frac{1}{2}\cdot(\Gamma-\gamma)\left[\|g(A)x\|^2 - \langle g(A)x,x\rangle^2\right]^{1/2}\left(\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right)$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

Proof. The first inequality follows from (3.8) by putting $y = x$.

Now, if we write the first inequality in (3.11) for $f = g$ we get

$$0 \leq \|g(A)x\|^2 - \langle g(A)x,x\rangle^2 = \left\langle g^2(A)x,x\right\rangle - \langle g(A)x,x\rangle^2 \leq \frac{1}{2}(\Delta-\delta)\left[\|g(A)x\|^2 - \langle g(A)x,x\rangle^2\right]^{1/2},$$

which implies that

$$\left[\|g(A)x\|^2 - \langle g(A)x,x\rangle^2\right]^{1/2} \leq \frac{1}{2}(\Delta-\delta)$$

for each $x \in H$ with $\|x\| = 1$.

This together with the first part of (3.11) proves the desired bound. ∎

The following particular cases that hold for power functions are of interest:

Example 3.1. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some scalars $m < M$.*

If A is positive $(m \geq 0)$ and $p, q > 0$, then

$$\text{(3.12)} \quad (0 \leq)\left\langle A^{p+q}x,x\right\rangle - \langle A^p x,x\rangle\cdot\langle A^q x,x\rangle \leq \frac{1}{2}\cdot(M^p - m^p)\left[\|A^q x\|^2 - \langle A^q x,x\rangle^2\right]^{1/2}$$

$$\left[\leq \frac{1}{4} \cdot (M^p - m^p)(M^q - m^q) \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

If A *is positive definite* $(m > 0)$ *and* $p, q < 0$, *then*

$$(3.13) \qquad (0 \leq) \left\langle A^{p+q}x, x \right\rangle - \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle \leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p}m^{-p}} \left[\|A^q x\|^2 - \langle A^q x, x\rangle^2 \right]^{1/2} \left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p}m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q}m^{-q}} \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

If A *is positive definite* $(m > 0)$ *and* $p < 0$, $q > 0$ *then*

$$(3.14) \qquad (0 \leq) \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle - \left\langle A^{p+q}x, x \right\rangle \leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p}m^{-p}} \left[\|A^q x\|^2 - \langle A^q x, x\rangle^2 \right]^{1/2} \left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p}m^{-p}} (M^q - m^q) \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

If A *is positive definite* $(m > 0)$ *and* $p > 0$, $q < 0$ *then*

$$(3.15) \qquad (0 \leq) \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle - \left\langle A^{p+q}x, x \right\rangle \leq \frac{1}{2} \cdot (M^p - m^p) \left[\|A^q x\|^2 - \langle A^q x, x\rangle^2 \right]^{1/2} \left[\leq \frac{1}{4} \cdot (M^p - m^p) \frac{M^{-q} - m^{-q}}{M^{-q}m^{-q}} \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (3.12) – (3.15) follows from Theorem 2.1.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

Example 3.2. *Let* A *be a positive definite operator with* $Sp(A) \subseteq [m, M]$ *for some scalars* $0 < m < M$.

If $p > 0$ *then*

$$(3.16) \qquad (0 \leq) \langle A^p \ln Ax, x\rangle - \langle A^p x, x\rangle \cdot \langle \ln Ax, x\rangle \leq \begin{cases} \frac{1}{2} \cdot (M^p - m^p) \left[\|\ln Ax\|^2 - \langle \ln Ax, x\rangle^2 \right]^{1/2} \\ \\ \ln\sqrt{\frac{M}{m}} \cdot \left[\|A^p x\|^2 - \langle A^p x, x\rangle^2 \right]^{1/2} \end{cases} \left[\leq \frac{1}{2} \cdot (M^p - m^p) \ln\sqrt{\frac{M}{m}} \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

If $p<0$, *then*

$$(0\leq)\left\langle A^{p}x,x\right\rangle \cdot\left\langle \ln Ax,x\right\rangle -\left\langle A^{p}\ln Ax,x\right\rangle \tag{3.17}$$
$$\leq\begin{cases}\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\left\Vert \ln Ax\right\Vert ^{2}-\left\langle \ln Ax,x\right\rangle ^{2}\right]^{1/2}\\ \\ \ln\sqrt{\frac{M}{m}}\cdot\left[\left\Vert A^{p}x\right\Vert ^{2}-\left\langle A^{p}x,x\right\rangle ^{2}\right]^{1/2}\end{cases}$$
$$\left[\leq\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\ln\sqrt{\frac{M}{m}}\right]$$

for each $x\in H$ *with* $\left\Vert x\right\Vert =1$.

3.4. An Inequality of Grüss Type for n Operators.

The following multiple operator version of Theorem 3.2 holds:

Theorem 3.4 (Dragomir, 2008, [30]). *Let* A_{j} *be selfadjoint operators with* $Sp\left(A_{j}\right)\subseteq\left[m,M\right]$ *for* $j\in\left\{ 1,\dots,n\right\}$ *and some scalars* $m<M$. *If* $f,g:\left[m,M\right]\longrightarrow\mathbb{R}$ *are continuous,* $\gamma:=\min_{t\in\left[m,M\right]}f\left(t\right)$ *and* $\Gamma:=\max_{t\in\left[m,M\right]}f\left(t\right)$, *then*

$$\left|\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)g\left(A_{j}\right)y_{j},y_{j}\right\rangle -\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)y_{j},y_{j}\right\rangle \cdot\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle \right.$$
$$\left.-\frac{\gamma+\Gamma}{2}\left[\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)y_{j},y_{j}\right\rangle -\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle \right]\right|$$
$$\leq\frac{1}{2}\left(\Gamma-\gamma\right)\left[\sum_{j=1}^{n}\left\Vert g\left(A_{j}\right)y_{j}\right\Vert ^{2}+\left(\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle \right)^{2}\right.$$
$$\left.-2\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle \sum_{j=1}^{n}\left\langle g\left(A_{j}\right)y_{j},y_{j}\right\rangle \right]^{\frac{1}{2}} \tag{3.18}$$

for each $x_{j},y_{j}\in H,j\in\left\{ 1,\dots,n\right\}$ *with* $\sum_{j=1}^{n}\left\Vert x_{j}\right\Vert ^{2}=\sum_{j=1}^{n}\left\Vert y_{j}\right\Vert ^{2}=1$.

Proof. As in [41, p. 6], if we put

$$\widetilde{A}:=\left(\begin{array}{ccc}A_{1} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & A_{n}\end{array}\right)\quad\text{and}\quad\widetilde{x}=\left(\begin{array}{c}x_{1}\\ \vdots\\ x_{n}\end{array}\right),\quad\widetilde{y}=\left(\begin{array}{c}y_{1}\\ \vdots\\ y_{n}\end{array}\right),$$

then we have $Sp\left(\widetilde{A}\right)\subseteq\left[m,M\right]$, $\left\Vert \widetilde{x}\right\Vert =\left\Vert \widetilde{y}\right\Vert =1$

$$\left\langle f\left(\widetilde{A}\right)g\left(\widetilde{A}\right)\widetilde{y},\widetilde{y}\right\rangle =\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)g\left(A_{j}\right)y_{j},y_{j}\right\rangle ,\qquad\left\langle g\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle =\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle ,$$

$$\left\langle f\left(\widetilde{A}\right)\widetilde{y},\widetilde{y}\right\rangle =\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)y_{j},y_{j}\right\rangle ,\quad\left\langle g\left(\widetilde{A}\right)\widetilde{y},\widetilde{y}\right\rangle =\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)y_{j},y_{j}\right\rangle ,$$

and

$$\left\Vert g\left(\widetilde{A}\right)\widetilde{y}\right\Vert ^{2}=\sum_{j=1}^{n}\left\Vert g\left(A_{j}\right)y_{j}\right\Vert ^{2},\quad\left\langle g\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle ^{2}=\left(\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle \right)^{2}.$$

Applying Theorem 3.2 for $\widetilde{A}$, $\widetilde{x}$ and $\widetilde{y}$ we deduce the desired result (3.18). ∎

The following particular case provides a refinement of the Mond- Pečarić result from (3.6).

Corollary 3.5 (Dragomir, 2008, [30]). *With the assumptions of Theorem 3.4 we have*

$$\begin{aligned}(3.19)\qquad &\left|\sum_{j=1}^{n}\langle f(A_j)\,g(A_j)\,x_j,x_j\rangle-\sum_{j=1}^{n}\langle f(A_j)\,x_j,x_j\rangle\cdot\sum_{j=1}^{n}\langle g(A_j)\,x_j,x_j\rangle\right|\\ &\le\frac{1}{2}\cdot(\Gamma-\gamma)\left[\sum_{j=1}^{n}\|g(A_j)\,x_j\|^2-\left(\sum_{j=1}^{n}\langle g(A_j)\,x_j,x_j\rangle\right)^2\right]^{1/2}\\ &\left(\le\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right)\end{aligned}$$

for each $x_j\in H, j\in\{1,\dots,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$, *where* $\delta:=\min_{t\in[m,M]}g(t)$ *and* $\Delta:=\max_{t\in[m,M]}g(t)$.

Example 3.3. *Let* A_j, $j\in\{1,\dots,n\}$ *be selfadjoint operators with* $Sp(A_j)\subseteq[m,M], j\in\{1,\dots,n\}$ *for some scalars* $m<M$.

If A_j *are positive* $(m\ge 0)$ *and* $p,q>0$, *then*

$$\begin{aligned}(3.20)\qquad &(0\le)\sum_{j=1}^{n}\left\langle A_j^{p+q}x_j,x_j\right\rangle-\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle\\ &\le\frac{1}{2}\cdot(M^p-m^p)\left[\sum_{j=1}^{n}\left\|A_j^{q}x_j\right\|^2-\left(\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle\right)^2\right]^{1/2}\\ &\left[\le\frac{1}{4}\cdot(M^p-m^p)(M^q-m^q)\right]\end{aligned}$$

for each $x_j\in H, j\in\{1,\dots,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$.

If A_j *are positive definite* $(m>0)$ *and* $p,q<0$, *then*

$$\begin{aligned}(3.21)\qquad &(0\le)\sum_{j=1}^{n}\left\langle A_j^{p+q}x_j,x_j\right\rangle-\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle\\ &\le\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\sum_{j=1}^{n}\left\|A_j^{q}x_j\right\|^2-\left(\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle\right)^2\right]^{1/2}\\ &\left[\le\frac{1}{4}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\frac{M^{-q}-m^{-q}}{M^{-q}m^{-q}}\right]\end{aligned}$$

for each $x_j\in H, j\in\{1,\dots,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$.

If A_j are positive definite $(m > 0)$ and $p < 0$, $q > 0$, then

$$(0 \le) \sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^q x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle \tag{3.22}$$
$$\le \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{j=1}^{n} \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^{n} \left\langle A_j^q x_j, x_j \right\rangle \right)^2 \right]^{1/2}$$
$$\left[\le \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left(M^q - m^q \right) \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If A_j are positive definite $(m > 0)$ and $p > 0$, $q < 0$, then

$$(0 \le) \sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^q x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle \tag{3.23}$$
$$\le \frac{1}{2} \cdot \left(M^p - m^p \right) \left[\sum_{j=1}^{n} \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^{n} \left\langle A_j^q x_j, x_j \right\rangle \right)^2 \right]^{1/2}$$
$$\left[\le \frac{1}{4} \cdot \left(M^p - m^p \right) \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

We notice that the positivity of the quantities in the left hand side of the inequalities (3.20) – (3.23) follows from Theorem 2.1.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

Example 3.4. *Let A_j be positive definite operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$ for some scalars $0 < m < M$.*

If $p > 0$, then

$$(0 \le) \sum_{j=1}^{n} \left\langle A_j^p \ln A_j x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle \tag{3.24}$$
$$\le \begin{cases} \frac{1}{2} \cdot \left(M^p - m^p \right) \left[\sum_{j=1}^{n} \left\| \ln A_j x_j \right\|^2 - \left(\sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle \right)^2 \right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\sum_{j=1}^{n} \left\| A_j^p x_j \right\|^2 - \left(\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \right)^2 \right]^{1/2} \end{cases}$$
$$\left[\le \frac{1}{2} \cdot \left(M^p - m^p \right) \ln \sqrt{\frac{M}{m}} \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If $p < 0$, *then*

$$(3.25)\qquad (0 \leq) \sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^p \ln A_j x_j, x_j \right\rangle$$

$$\leq \begin{cases} \frac{1}{2}\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}} \left[\sum_{j=1}^{n} \|\ln A_j x_j\|^2 - \left(\sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle\right)^2\right]^{1/2} \\ \\ \ln\sqrt{\frac{M}{m}} \cdot \left[\sum_{j=1}^{n} \left\|A_j^p x_j\right\|^2 - \left(\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle\right)^2\right]^{1/2} \end{cases}$$

$$\left[\leq \frac{1}{2} \cdot \frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}} \ln\sqrt{\frac{M}{m}}\right]$$

for each $x_j \in H, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

3.5. **Another Inequality of Grüss Type for n Operators.** The following different result for n operators can be stated as well:

Theorem 3.6 (Dragomir, 2008, [30]). *Let* A_j *be selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \dots, n\}$ *and some scalars* $m < M$. *If* f *and* g *are continuous on* $[m, M]$, $\gamma := \min_{t\in[m,M]} f(t)$ *and* $\Gamma := \max_{t\in[m,M]} f(t)$, *then for any* $p_j \geq 0, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$, *we have*

$$(3.26)\qquad \left|\left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) y, y \right\rangle - \frac{\gamma + \Gamma}{2} \cdot \left[\left\langle \sum_{k=1}^{n} p_k g(A_k) y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle\right] - \left\langle \sum_{k=1}^{n} p_k f(A_k) y, y \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle\right|$$

$$\leq \frac{\Gamma - \gamma}{2} \left[\sum_{k=1}^{n} p_k \|g(A_k) y\|^2 - 2\left\langle \sum_{k=1}^{n} p_k g(A_k) y, y \right\rangle \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle + \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle^2\right]^{1/2},$$

for each $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

Proof. Follows from Theorem 3.4 on choosing $x_j = \sqrt{p_j} \cdot x$, $y_j = \sqrt{p_j} \cdot y$, $j \in \{1, \dots, n\}$, where $p_j \geq 0, j \in \{1, \dots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x, y \in H$, with $\|x\| = \|y\| = 1$. The details are omitted. ∎

Remark 3.1. The case $n = 1$ (therefore $p = 1$) in (3.26) provides the result from Theorem 3.2.

As a particular case of interest we can derive from the above theorem the following Grüss type result:

Corollary 3.7 (Dragomir, 2008, [30])**.** *With the assumptions of Theorem 3.6, we have*

$$\left|\left\langle\sum_{k=1}^{n} p_k f\left(A_k\right) g\left(A_k\right) x, x\right\rangle-\left\langle\sum_{k=1}^{n} p_k f\left(A_k\right) x, x\right\rangle \cdot\left\langle\sum_{k=1}^{n} p_k g\left(A_k\right) x, x\right\rangle\right| \tag{3.27}$$
$$\leq \frac{\Gamma-\gamma}{2}\left(\sum_{k=1}^{n} p_k\left\|g\left(A_k\right) x\right\|^2-\left\langle\sum_{k=1}^{n} p_k g\left(A_k\right) x, x\right\rangle^2\right)^{1/2}$$
$$\left[\leq \frac{1}{4} \cdot(\Gamma-\gamma)(\Delta-\delta)\right]$$

for each $x \in H$ *with* $\|x\|=1$, *where* $\delta:=\min_{t \in[m, M]} g(t)$ *and* $\Delta:=\max_{t \in[m, M]} g(t)$.

Proof. It is similar with the proof from Corollary 3.3 and the details are omitted. ∎

The following particular cases that hold for power functions are of interest:

Example 3.5. *Let* A_j, $j \in\{1, \ldots, n\}$ *be selfadjoint operators with* $Sp\left(A_j\right) \subseteq[m, M]$, $j \in \{1, \ldots, n\}$ *for some scalars* $m<M$ *and* $p_j \geq 0, j \in\{1, \ldots, n\}$ *with* $\sum_{j=1}^n p_j=1$.
If A_j, $j \in\{1, \ldots, n\}$ *are positive* $(m \geq 0)$ *and* $p, q>0$, *then*

$$(0 \leq)\left\langle\sum_{k=1}^{n} p_k A_k^{p+q} x, x\right\rangle-\left\langle\sum_{k=1}^{n} p_k A_k^p x, x\right\rangle \cdot\left\langle\sum_{k=1}^{n} p_k A_k^q x, x\right\rangle \tag{3.28}$$
$$\leq \frac{1}{2} \cdot\left(M^p-m^p\right)\left[\sum_{k=1}^{n} p_k\left\|A_k^q x\right\|^2-\left\langle\sum_{k=1}^{n} p_k A_k^q x, x\right\rangle^2\right]^{1/2}$$
$$\left[\leq \frac{1}{4} \cdot\left(M^p-m^p\right)\left(M^q-m^q\right)\right]$$

for each $x \in H$ *with* $\|x\|=1$.
If A_j, $j \in\{1, \ldots, n\}$ *are positive definite* $(m>0)$ *and* $p, q<0$, *then*

$$(0 \leq)\left\langle\sum_{k=1}^{n} p_k A_k^{p+q} x, x\right\rangle-\left\langle\sum_{k=1}^{n} p_k A_k^p x, x\right\rangle \cdot\left\langle\sum_{k=1}^{n} p_k A_k^q x, x\right\rangle \tag{3.29}$$
$$\leq \frac{1}{2} \cdot \frac{M^{-p}-m^{-p}}{M^{-p} m^{-p}}\left[\sum_{k=1}^{n} p_k\left\|A_k^q x\right\|^2-\left\langle\sum_{k=1}^{n} p_k A_k^q x, x\right\rangle^2\right]^{1/2}$$
$$\left[\leq \frac{1}{4} \cdot \frac{M^{-p}-m^{-p}}{M^{-p} m^{-p}} \frac{M^{-q}-m^{-q}}{M^{-q} m^{-q}}\right]$$

for each $x \in H$ *with* $\|x\|=1$.

If A_j, $j \in \{1, \dots, n\}$ are positive definite $(m > 0)$ and $p < 0$, $q > 0$, then

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle \leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{k=1}^{n} p_k \|A_k^q x\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle^2 \right]^{1/2} \left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} (M^q - m^q) \right] \tag{3.30}$$

for each $x \in H$ with $\|x\| = 1$.

If A_j, $j \in \{1, \dots, n\}$ are positive definite $(m > 0)$ and $p > 0$, $q < 0$, then

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle \leq \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{k=1}^{n} p_k \|A_k^q x\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle^2 \right]^{1/2} \left[\leq \frac{1}{4} \cdot (M^p - m^p) \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right] \tag{3.31}$$

for each $x \in H$ with $\|x\| = 1$.

We notice that the positivity of the quantities in the left hand side of the inequalities (3.28) – (3.31) follows from Theorem 2.1.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

Example 3.6. *Let A_j, $j \in \{1, \dots, n\}$ be positive definite operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \dots, n\}$ for some scalars $0 < m < M$ and $p_j \geq 0$, $j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} p_j = 1$.*

If $p > 0$, then

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^p \ln A_k x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k \ln A_k x, x \right\rangle \leq \begin{cases} \frac{1}{2} \cdot (M^p - m^p) \cdot \left[\sum_{k=1}^{n} p_k \|\ln A_k x\|^2 - \langle \sum_{k=1}^{n} p_k \ln A_k x, x \rangle^2 \right]^{1/2} \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\sum_{k=1}^{n} p_k \|A_k^p x\|^2 - \langle \sum_{k=1}^{n} p_k A_k^p x, x \rangle^2 \right]^{1/2} \end{cases} \left[\leq \frac{1}{2} \cdot (M^p - m^p) \ln \sqrt{\frac{M}{m}} \right] \tag{3.32}$$

for each $x \in H$ with $\|x\| = 1$.

If $p < 0$, then

$$(0 \le) \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k \ln A_k x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^p \ln A_k x, x \right\rangle \tag{3.33}$$
$$\le \begin{cases} \frac{1}{2} \cdot \frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}} \left[\sum_{k=1}^{n} p_k \left\|\ln A_k x\right\|^2 - \left\langle \sum_{k=1}^{n} p_k \ln A_k x, x\right\rangle^2\right]^{1/2} \\ \ln\sqrt{\frac{M}{m}} \cdot \left[\sum_{k=1}^{n} p_k \left\|A_k^p x\right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^p x, x\right\rangle^2\right]^{1/2} \end{cases}$$
$$\left[\le \frac{1}{2} \cdot \frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}} \ln\sqrt{\frac{M}{m}}\right]$$

for each $x \in H$ with $\|x\| = 1$.

The following norm inequalities may be stated as well:

Corollary 3.8 (Dragomir, 2008, [30])**.** *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous, then for each $p_j \ge 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} p_j = 1$, we have the norm inequality:*

$$\left\|\sum_{j=1}^{n} p_j f(A_j) g(A_j)\right\| \le \left\|\sum_{j=1}^{n} p_j f(A_j)\right\| \cdot \left\|\sum_{j=1}^{n} p_j g(A_j)\right\| + \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{3.34}$$

where $\gamma := \min_{t\in[m,M]} f(t)$, $\Gamma := \max_{t\in[m,M]} f(t)$, $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

Proof. Utilising the inequality (3.27) we obtain

$$\left|\left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) x, x\right\rangle\right| \le \left|\left\langle \sum_{k=1}^{n} p_k f(A_k) x, x\right\rangle\right| \cdot \left|\left\langle \sum_{k=1}^{n} p_k g(A_k) x, x\right\rangle\right| + \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta)$$

for each $x \in H$ with $\|x\| = 1$. Taking the supremum over $\|x\| = 1$ we deduce the desired inequality (3.34). ∎

Example 3.7.
a. *Let $A_j, j \in \{1, \dots, n\}$ be selfadjoint operators with $Sp(A_j) \subseteq [m, M], j \in \{1, \dots, n\}$ for some scalars $m < M$ and $p_j \ge 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} p_j = 1$.*

If $A_j, j \in \{1, \dots, n\}$ are positive $(m \ge 0)$ and $p, q > 0$, then

$$\left\|\sum_{k=1}^{n} p_k A_k^{p+q}\right\| \le \left\|\sum_{k=1}^{n} p_k A_k^{p}\right\| \cdot \left\|\sum_{k=1}^{n} p_k A_k^{q}\right\| + \frac{1}{4}(M^p - m^p)(M^q - m^q). \tag{3.35}$$

If $A_j, j \in \{1, \dots, n\}$ are positive definite $(m > 0)$ and $p, q < 0$, then

$$\left\|\sum_{k=1}^{n} p_k A_k^{p+q}\right\| \le \left\|\sum_{k=1}^{n} p_k A_k^{p}\right\| \cdot \left\|\sum_{k=1}^{n} p_k A_k^{q}\right\| + \frac{1}{4} \cdot \frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}} \cdot \frac{M^{-q}-m^{-q}}{M^{-q}m^{-q}}. \tag{3.36}$$

b. *Let $A_j, j \in \{1, \dots, n\}$ be positive definite operators with $Sp(A_j) \subseteq [m, M], j \in \{1, \dots, n\}$ for some scalars $0 < m < M$ and $p_j \ge 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} p_j = 1$.*

If $p > 0$, *then*

$$\left\| \sum_{k=1}^{n} p_k A_k^p \ln A_k \right\| \le \left\| \sum_{k=1}^{n} p_k A_k^p \right\| \cdot \left\| \sum_{k=1}^{n} p_k \ln A_k \right\| + \frac{1}{2}\left(M^p - m^p\right) \ln \sqrt{\frac{M}{m}}. \tag{3.37}$$

4. MORE INEQUALITIES OF GRÜSS TYPE

4.1. **Some Vectorial Grüss Type Inequalities.** The following lemmas, that are of interest in their own right, collect some Grüss type inequalities for vectors in inner product spaces obtained earlier by the author:

Lemma 4.1 (Dragomir, 2003 & 2004, [22], [26])**.** *Let* $(H, \langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, *and* $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ *such that*

$$\operatorname{Re}\langle \beta e - u, u - \alpha e\rangle \ge 0, \qquad \operatorname{Re}\langle \delta e - v, v - \gamma e\rangle \ge 0 \tag{4.1}$$

or equivalently,

$$\left\| u - \frac{\alpha+\beta}{2} e \right\| \le \frac{1}{2}\left|\beta - \alpha\right|, \qquad \left\| v - \frac{\gamma+\delta}{2} e \right\| \le \frac{1}{2}\left|\delta - \gamma\right|. \tag{4.2}$$

Then

$$\begin{aligned} &\left|\langle u, v\rangle - \langle u, e\rangle \langle e, v\rangle\right| \\ &\le \frac{1}{4}\cdot\left|\beta-\alpha\right|\left|\delta-\gamma\right| - \begin{cases} \left[\operatorname{Re}\langle \beta e - u, u - \alpha e\rangle \operatorname{Re}\langle \delta e - v, v - \gamma e\rangle\right]^{\frac{1}{2}}, \\ \left|\langle u, e\rangle - \frac{\alpha+\beta}{2}\right| \left|\langle v, e\rangle - \frac{\gamma+\delta}{2}\right|. \end{cases} \end{aligned} \tag{4.3}$$

The first inequality has been obtained in [22] (see also [27, p. 44]) while the second result was established in [26] (see also [27, p. 90]). They provide refinements of the earlier result from [16] where only the first part of the bound, i.e., $\frac{1}{4}\left|\beta-\alpha\right|\left|\delta-\gamma\right|$, has been given. Notice that, as pointed out in [26], the upper bounds for the Grüss functional incorporated in (4.3) cannot be compared in general, meaning that one is better than the other depending on appropriate choices of the vectors and scalars involved.

Another result of this type is the following one:

Lemma 4.2 (Dragomir, 2004 & 2006, [23], [28])**.** *With the assumptions in Lemma 4.1 and if* $\operatorname{Re}(\beta\overline{\alpha}) > 0, \operatorname{Re}(\delta\overline{\gamma}) > 0$ *then*

$$\begin{aligned} &\left|\langle u, v\rangle - \langle u, e\rangle \langle e, v\rangle\right| \\ &\le \begin{cases} \frac{1}{4}\cdot \frac{\left|\beta-\alpha\right|\left|\delta-\gamma\right|}{\left[\operatorname{Re}(\beta\overline{\alpha}) \operatorname{Re}(\delta\overline{\gamma})\right]^{\frac{1}{2}}} \left|\langle u, e\rangle \langle e, v\rangle\right|, \\ \left[\left(\left|\alpha+\beta\right| - 2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{\frac{1}{2}}\right)\left(\left|\delta+\gamma\right| - 2\left[\operatorname{Re}(\delta\overline{\gamma})\right]^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \left[\left|\langle u, e\rangle \langle e, v\rangle\right|\right]^{\frac{1}{2}}. \end{cases} \end{aligned} \tag{4.4}$$

The first inequality has been established in [23] (see [27, p. 62]) while the second one can be obtained in a canonical manner from the reverse of the Schwarz inequality given in [28]. The details are omitted.

Finally, another inequality of Grüss type that has been obtained in [24] (see also [27, p. 65]) can be stated as:

Lemma 4.3 (Dragomir, 2004, [24]). *With the assumptions in Lemma 4.1 and if $\beta \neq -\alpha$, $\delta \neq -\gamma$ then*

$$(4.5)\qquad |\langle u,v\rangle - \langle u,e\rangle\langle e,v\rangle| \leq \frac{1}{4}\cdot\frac{|\beta-\alpha|\,|\delta-\gamma|}{[|\beta+\alpha|\,|\delta+\gamma|]^{\frac{1}{2}}}\left[(\|u\|+|\langle u,e\rangle|)(\|v\|+|\langle v,e\rangle|)\right]^{\frac{1}{2}}.$$

4.2. **Some Inequalities of Grüss Type for One Operator.** The following results incorporate some new inequalities of Grüss type for two functions of a selfadjoint operator.

Theorem 4.4 (Dragomir, 2008, [31]). *Let A be a selfadjoint operator on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$ and assume that $Sp(A)\subseteq[m,M]$ for some scalars $m<M$. If f and g are continuous on $[m,M]$ and $\gamma:=\min_{t\in[m,M]}f(t)$, $\Gamma:=\max_{t\in[m,M]}f(t)$, $\delta:=\min_{t\in[m,M]}g(t)$ and $\Delta:=\max_{t\in[m,M]}g(t)$, then*

$$(4.6)\qquad |\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle\langle g(A)x,x\rangle|$$
$$\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta) - \begin{cases} [\langle \Gamma x - f(A)x, f(A)x-\gamma x\rangle\langle \Delta x - g(A)x, g(A)x-\delta x\rangle]^{\frac{1}{2}}, \\ \left|\langle f(A)x,x\rangle - \frac{\Gamma+\gamma}{2}\right|\left|\langle g(A)x,x\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases}$$

for each $x\in H$ with $\|x\|=1$.

Observe that if γ and δ are positive, then we also have

$$(4.7)\qquad |\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle\langle g(A)x,x\rangle|$$
$$\leq \begin{cases} \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}}\langle f(A)x,x\rangle\langle g(A)x,x\rangle, \\ \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)[\langle f(A)x,x\rangle\langle g(A)x,x\rangle]^{\frac{1}{2}}, \end{cases}$$

while for $\Gamma+\gamma, \Delta+\delta\neq 0$ we have

$$(4.8)\qquad |\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle\langle g(A)x,x\rangle|$$
$$\leq \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{[|\Gamma+\gamma|\,|\Delta+\delta|]^{\frac{1}{2}}}\left[(\|f(A)x\|+|\langle f(A)x,x\rangle|)(\|g(A)x\|+|\langle g(A)x,x\rangle|)\right]^{\frac{1}{2}}$$

for each $x\in H$ with $\|x\|=1$.

Proof. Since $\gamma:=\min_{t\in[m,M]}f(t)$, $\Gamma:=\max_{t\in[m,M]}f(t)$, $\delta:=\min_{t\in[m,M]}g(t)$ and $\Delta:=\max_{t\in[m,M]}g(t)$, then by the property (P) we have

$$\gamma\cdot 1_H \leq f(A) \leq \Gamma\cdot 1_H \quad\text{and}\quad \delta\cdot 1_H\leq g(A)\leq \Delta\cdot 1_H$$

in the operator order, which implies that

$$(4.9)\qquad [f(A)-\gamma\cdot 1][\Gamma\cdot 1_H - f(A)]\geq 0 \text{ and } [\Delta\cdot 1_H - g(A)][g(A)-\delta\cdot 1_H]\geq 0$$

are in the operator order.

We then have from (4.9).

$$\langle [f(A)-\gamma\cdot 1][\Gamma\cdot 1_H - f(A)]x,x\rangle \geq 0$$

and

$$\langle [\Delta\cdot 1_H - g(A)][g(A)-\delta\cdot 1_H]x,x\rangle\geq 0,$$

for each $x\in H$ with $\|x\|=1$, which, by the fact that the involved operators are selfadjoint, are equivalent with the inequalities

$$(4.10)\qquad \langle \Gamma x - f(A)x, f(A)x-\gamma x\rangle \geq 0 \text{ and } \langle \Delta x - g(A)x, g(A)x - \delta x\rangle\geq 0,$$

for each $x \in H$ with $\|x\| = 1$.

Now, if we apply Lemma 4.1 for $u = f(A)x$, $v = g(A)x$, $e = x$, and the real scalars Γ, γ, Δ and δ defined in the statement of the theorem, then we can state that

$$|\langle f(A)x, g(A)x\rangle - \langle f(A)x, x\rangle\langle x, g(A)x\rangle| \le \frac{1}{4}\cdot(\Gamma-\gamma)(\Delta-\delta) - \begin{cases} [\operatorname{Re}\langle \Gamma x - f(A)x, f(A)x - \gamma x\rangle \operatorname{Re}\langle \Delta x - g(A)x, g(A)x - \delta x\rangle]^{\frac{1}{2}}, \\ \left|\langle f(A)x, x\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\langle g(A)x, x\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases} \tag{4.11}$$

for each $x \in H$ with $\|x\| = 1$, which is clearly equivalent with the inequality (4.6).

The inequalities (4.7) and (4.8) follow by Lemma 4.2 and Lemma 4.3 respectively and the details are omitted. ∎

Remark 4.1. The first inequality in (4.7) can be written in a more convenient way as

$$\left|\frac{\langle f(A)g(A)x, x\rangle}{\langle f(A)x, x\rangle\langle g(A)x, x\rangle} - 1\right| \le \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \tag{4.12}$$

for each $x \in H$ with $\|x\| = 1$, while the second inequality has the following equivalent form

$$\left|\frac{\langle f(A)g(A)x, x\rangle}{[\langle f(A)x, x\rangle\langle g(A)x, x\rangle]^{1/2}} - [\langle f(A)x, x\rangle\langle g(A)x, x\rangle]^{1/2}\right| \le \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right) \tag{4.13}$$

for each $x \in H$ with $\|x\| = 1$.

We know, from [29] that if f, g are synchronous (asynchronous) functions on the interval $[m, M]$, i.e., we recall that

$$[f(t) - f(s)][g(t) - g(s)] (\ge) \le 0 \quad \text{for each } t, s \in [m, M],$$

then we have the inequality

$$\langle f(A)g(A)x, x\rangle \ge (\le) \langle f(A)x, x\rangle\langle g(A)x, x\rangle \tag{4.14}$$

for each $x \in H$ with $\|x\| = 1$, provided f, g are continuous on $[m, M]$ and A is a selfadjoint operator with $Sp(A) \subseteq [m, M]$.

Therefore, if f, g are synchronous, then we have from (4.12) and (4.13) the following results:

$$0 \le \frac{\langle f(A)g(A)x, x\rangle}{\langle f(A)x, x\rangle\langle g(A)x, x\rangle} - 1 \le \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \tag{4.15}$$

and

$$0 \le \frac{\langle f(A)g(A)x, x\rangle}{[\langle f(A)x, x\rangle\langle g(A)x, x\rangle]^{1/2}} - [\langle f(A)x, x\rangle\langle g(A)x, x\rangle]^{1/2} \le \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right) \tag{4.16}$$

for each $x \in H$ with $\|x\| = 1$, respectively.

If f, g are asynchronous then

$$0 \le 1 - \frac{\langle f(A)g(A)x, x\rangle}{\langle f(A)x, x\rangle\langle g(A)x, x\rangle} \le \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \tag{4.17}$$

and

$$0 \le [\langle f(A)x,x\rangle \langle g(A)x,x\rangle]^{1/2} - \frac{\langle f(A)g(A)x,x\rangle}{[\langle f(A)x,x\rangle \langle g(A)x,x\rangle]^{1/2}} \le \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right) \tag{4.18}$$

for each $x \in H$ with $\|x\| = 1$, respectively.

It is obvious that all the inequalities from Theorem 4.4 can be used to obtain reverse inequalities of Grüss type for various particular instances of operator functions, see for instance [30]. However, we give here only a few provided by the inequalities (4.15) and (4.16) above.

Example 4.1. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$.*
If A is positive $(m \ge 0)$ and $p, q > 0$, then

$$0 \le \frac{\langle A^{p+q}x,x\rangle}{\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle} - 1 \le \frac{1}{4} \cdot \frac{(M^p - m^p)(M^q - m^q)}{M^{\frac{p+q}{2}} m^{\frac{p+q}{2}}} \tag{4.19}$$

and

$$0 \le \frac{\langle A^{p+q}x,x\rangle}{[\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle]^{1/2}} - [\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle]^{1/2} \le \left(M^{\frac{p}{2}} - m^{\frac{p}{2}}\right)\left(M^{\frac{q}{2}} - m^{\frac{q}{2}}\right) \tag{4.20}$$

for each $x \in H$ with $\|x\| = 1$.
If A is positive definite $(m > 0)$ and $p, q < 0$, then

$$0 \le \frac{\langle A^{p+q}x,x\rangle}{\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle} - 1 \le \frac{1}{4} \cdot \frac{(M^{-p} - m^{-p})(M^{-q} - m^{-q})}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}} \tag{4.21}$$

and

$$0 \le \frac{\langle A^{p+q}x,x\rangle}{[\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle]^{1/2}} - [\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle]^{1/2} \le \frac{\left(M^{-\frac{p}{2}} - m^{-\frac{p}{2}}\right)\left(M^{-\frac{q}{2}} - m^{-\frac{q}{2}}\right)}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}} \tag{4.22}$$

for each $x \in H$ with $\|x\| = 1$.

Similar inequalities may be stated for either $p > 0, q < 0$ or $p < 0, q > 0$. The details are omitted.

Example 4.2. *Let A be a positive definite operator with $Sp(A) \subseteq [m, M]$ for some scalars $1 < m < M$. If $p > 0$ then*

$$0 \le \frac{\langle A^p \ln Ax,x\rangle}{\langle A^p x,x\rangle \cdot \langle \ln Ax,x\rangle} - 1 \le \frac{1}{4} \cdot \frac{(M^p - m^p)\ln \frac{M}{m}}{M^{\frac{p}{2}} m^{\frac{p}{2}} \sqrt{\ln M \cdot \ln m}} \tag{4.23}$$

and

$$0 \le \frac{\langle A^p \ln Ax,x\rangle}{[\langle A^p x,x\rangle \cdot \langle \ln Ax,x\rangle]^{1/2}} - [\langle A^p x,x\rangle \cdot \langle \ln Ax,x\rangle]^{1/2} \le \left(M^{\frac{p}{2}} - m^{\frac{p}{2}}\right)\left[\sqrt{\ln M} - \sqrt{\ln m}\right], \tag{4.24}$$

for each $x \in H$ with $\|x\| = 1$.

4.3. **Some Inequalities of Grüss Type for n Operators.** The following extension for sequences of operators can be stated:

Theorem 4.5 (Dragomir, 2008, [31]). *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and some scalars $m < M$. If f and g are continuous on $[m, M]$ and $\gamma := \min_{t\in[m,M]} f(t)$, $\Gamma := \max_{t\in[m,M]} f(t)$, $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then*

$$\left|\sum_{j=1}^{n}\langle f(A_j) g(A_j) x_j, x_j\rangle - \sum_{j=1}^{n}\langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n}\langle g(A_j) x_j, x_j\rangle\right| \tag{4.25}$$
$$\leq \frac{1}{4}\cdot(\Gamma-\gamma)(\Delta-\delta) - \begin{cases} \left[\sum_{j=1}^{n}\langle \Gamma x_j - f(A_j) x_j, f(A_j) x_j - \gamma x_j\rangle \right. \\ \qquad \left. \times \sum_{j=1}^{n}\langle \Delta x_j - g(A_j) x_j, g(A_j) x - \delta x_j\rangle\right]^{\frac{1}{2}}, \\ \left|\sum_{j=1}^{n}\langle f(A_j) x_j, x_j\rangle - \frac{\Gamma+\gamma}{2}\right|\left|\sum_{j=1}^{n}\langle g(A_j) x_j, x_j\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

If γ and δ are positive, then we also have

$$\left|\sum_{j=1}^{n}\langle f(A_j) g(A_j) x_j, x_j\rangle - \sum_{j=1}^{n}\langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n}\langle g(A_j) x_j, x_j\rangle\right| \tag{4.26}$$
$$\leq \begin{cases} \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \sum_{j=1}^{n}\langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n}\langle g(A_j) x_j, x_j\rangle, \\ \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)\left[\sum_{j=1}^{n}\langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n}\langle g(A_j) x_j, x_j\rangle\right]^{\frac{1}{2}}, \end{cases}$$

while for $\Gamma+\gamma, \Delta+\delta \neq 0$ we have

$$\left|\sum_{j=1}^{n}\langle f(A_j) g(A_j) x_j, x_j\rangle - \sum_{j=1}^{n}\langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n}\langle g(A_j) x_j, x_j\rangle\right| \tag{4.27}$$
$$\leq \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\left[|\Gamma+\gamma||\Delta+\delta|\right]^{\frac{1}{2}}}\cdot\left[\left(\left(\sum_{j=1}^{n}\|f(A_j) x_j\|^2\right)^{1/2} + \left|\sum_{j=1}^{n}\langle f(A_j) x_j, x_j\rangle\right|\right)\right.$$
$$\left.\cdot\left(\left(\sum_{j=1}^{n}\|g(A_j) x_j\|^2\right)^{1/2} + \left|\sum_{j=1}^{n}\langle g(A_j) x_j, x_j\rangle\right|\right)\right]^{1/2},$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

Proof. As in [41, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix} \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = 1$

$$\left\langle f\left(\widetilde{A}\right) g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle, \left\langle g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle,$$

$$\left\langle f\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle, \ \left\| f\left(\widetilde{A}\right) \widetilde{x}\right\|^2 = \sum_{j=1}^{n} \|f(A_j) x_j\|^2$$

and

$$\left\| g\left(\widetilde{A}\right) \widetilde{x}\right\|^2 = \sum_{j=1}^{n} \|g(A_j) x_j\|^2.$$

Applying Theorem 4.4 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired results. The details are omitted. ∎

Remark 4.2. The first inequality in (4.26) can be written in a more convenient way as

$$\left| \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle}{\sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle} - 1 \right| \le \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \tag{4.28}$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, while the second inequality has the following equivalent form

$$\left| \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle}{\left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle\right]^{1/2}} - \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle\right]^{1/2} \right| \le \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \tag{4.29}$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

We know, from [29] that if f, g are synchronous (asynchronous) functions on the interval $[m, M]$, then we have the inequality

$$\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle \ge (\le) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle \tag{4.30}$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, provided that f, g are continuous on $[m, M]$ and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \dots, n\}$.

Therefore, if f, g are synchronous, then we have from (4.28) and (4.29) the following results:

$$0 \le \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle}{\sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle} - 1 \le \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \tag{4.31}$$

and

$$0 \leq \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2}} - \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2} \leq \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \tag{4.32}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, respectively.

If f, g are asynchronous then

$$0 \leq 1 - \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle} \leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \tag{4.33}$$

and

$$0 \leq \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2} - \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2}} \leq \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \tag{4.34}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, respectively.

It is obvious that all the inequalities from Theorem 4.5 can be used to obtain reverse inequalities of Grüss type for various particular instances of operator functions, see for instance [30]. However we give here only a few provided by the inequalities (4.31) and (4.32) above.

Example 4.3. *Let A_j, $j \in \{1, \ldots, n\}$ be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$ for some scalars $m < M$.*

If A_j are positive $(m \geq 0)$ and $p, q > 0$, then

$$0 \leq \frac{\sum_{j=1}^{n} \langle A_j^{p+q} x_j, x_j \rangle}{\sum_{j=1}^{n} \langle A_j^{p} x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^{q} x_j, x_j \rangle} - 1 \leq \frac{1}{4} \cdot \frac{(M^p - m^p)(M^q - m^q)}{M^{\frac{p+q}{2}} m^{\frac{p+q}{2}}} \tag{4.35}$$

and

$$0 \le \frac{\sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle}{\left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle\right]^{1/2}} - \left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle\right]^{1/2} \le \left(M^{\frac{p}{2}} - m^{\frac{p}{2}}\right)\left(M^{\frac{q}{2}} - m^{\frac{q}{2}}\right) \tag{4.36}$$

for each $x_j \in H, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1.$

If A *is positive definite* $(m > 0)$ *and* $p, q < 0$, *then*

$$0 \le \frac{\sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle} - 1 \le \frac{1}{4} \cdot \frac{\left(M^{-p} - m^{-p}\right)\left(M^{-q} - m^{-q}\right)}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}} \tag{4.37}$$

and

$$0 \le \left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle\right]^{1/2} - \frac{\sum_{j=1}^{n} \left\langle A_j^{p+q} x, x \right\rangle}{\left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle\right]^{1/2}} \le \frac{\left(M^{-\frac{p}{2}} - m^{-\frac{p}{2}}\right)\left(M^{-\frac{q}{2}} - m^{-\frac{q}{2}}\right)}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}} \tag{4.38}$$

for each $x_j \in H, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1.$

Similar inequalities may be stated for either $p > 0, q < 0$ or $p < 0, q > 0$. The details are omitted.

Example 4.4. *Let* A *be a positive definite operator with* $Sp(A) \subseteq [m, M]$ *for some scalars* $1 < m < M$. *If* $p > 0$, *then*

$$0 \le \frac{\sum_{j=1}^{n} \left\langle A_j^{p} \ln A_j x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle} - 1 \le \frac{1}{4} \cdot \frac{\left(M^p - m^p\right) \ln \frac{M}{m}}{M^{\frac{p}{2}} m^{\frac{p}{2}} \sqrt{\ln M \cdot \ln m}} \tag{4.39}$$

and

$$0 \le \frac{\sum_{j=1}^{n} \left\langle A_j^{p} \ln A_j x_j, x_j \right\rangle}{\left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle\right]^{1/2}} - \left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle\right]^{1/2} \le \left(M^{\frac{p}{2}} - m^{\frac{p}{2}}\right)\left[\sqrt{\ln M} - \sqrt{\ln m}\right], \tag{4.40}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Similar inequalities may be stated for $p < 0$. The details are omitted.

The following result for n operators can be stated as well:

Corollary 4.6. *Let* A_j *be selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and some scalars* $m < M$. *If* f *and* g *are continuous on* $[m, M]$ *and* $\gamma := \min_{t\in[m,M]} f(t)$, $\Gamma := \max_{t\in[m,M]} f(t)$, $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$, *then for any* $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *we have*

$$\left|\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle\right| \tag{4.41}$$

$$\leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta) - \begin{cases} \left[\sum_{j=1}^{n} p_j \langle \Gamma x - f(A_j) x, f(A_j) x - \gamma x \rangle \right. \\ \qquad \left. \times \sum_{j=1}^{n} p_j \langle \Delta x - g(A_j) x, g(A_j) x - \delta x \rangle \right]^{\frac{1}{2}}, \\ \left|\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases}$$

for each $x \in H$, *with* $\|x\|^2 = 1$.

Moreover, if γ *and* δ *are positive, then*

$$\left|\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle\right| \tag{4.42}$$

$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle, \\ \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle\right]^{\frac{1}{2}}, \end{cases}$$

while for $\Gamma + \gamma, \Delta + \delta \neq 0$ *we have*

$$\left|\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle\right| \tag{4.43}$$

$$\leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{[|\Gamma + \gamma| |\Delta + \delta|]^{\frac{1}{2}}} \left[\left(\left(\sum_{j=1}^{n} p_j \|f(A_j) x\|^2\right)^{1/2} + \left|\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle\right|\right)\right.$$

$$\left. \cdot \left(\left(\sum_{j=1}^{n} p_j \|g(A_j) x\|^2\right)^{1/2} + \left|\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle\right|\right)\right]^{1/2}$$

for each $x \in H$, *with* $\|x\|^2 = 1$.

Proof. Follows from Theorem 4.5 on choosing $x_j = \sqrt{p_j} \cdot x, j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$. The details are omitted. ∎

Remark 4.3. The first inequality in (4.42) can be written in a more convenient way as

$$\left|\frac{\left\langle\sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle} - 1\right| \leq \frac{1}{4} \cdot \frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \tag{4.44}$$

for each $x \in H$, with $\|x\|^2 = 1$, while the second inequality has the following equivalent form

$$\begin{aligned} &\left|\frac{\left\langle\sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle}{\left[\left\langle\sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle\right]^{1/2}}\right. \\ &\qquad \left. - \left[\left\langle\sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle\right]^{1/2}\right| \\ &\qquad\qquad \leq \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right) \end{aligned} \tag{4.45}$$

for each $x \in H$, with $\|x\|^2 = 1$.

We know, from [29] that if f, g are synchronous (asynchronous) functions on the interval $[m, M]$, then we have the inequality

$$\left\langle\sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle \geq (\leq) \left\langle\sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle \tag{4.46}$$

for each $x \in H$, with $\|x\|^2 = 1$, provided that f, g are continuous on $[m, M]$ and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$.

Therefore, if f, g are synchronous then we have from (4.44) and (4.45) the following results:

$$\begin{aligned} 0 &\leq \frac{\left\langle\sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle} - 1 \\ &\leq \frac{1}{4} \cdot \frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \end{aligned} \tag{4.47}$$

and

$$\begin{aligned} 0 &\leq \frac{\left\langle\sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle}{\left[\left\langle\sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle\right]^{1/2}} \\ &\qquad - \left[\left\langle\sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle\right]^{1/2} \\ &\leq \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right) \end{aligned} \tag{4.48}$$

for each $x \in H$, with $\|x\|^2 = 1$, respectively.

If f, g are asynchronous then

$$0 \le 1 - \frac{\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle} \le \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \tag{4.49}$$

and

$$0 \le \left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right]^{1/2} - \frac{\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle}{\left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right]^{1/2}} \le \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right) \tag{4.50}$$

for each $x \in H$, with $\|x\|^2 = 1$, respectively.

The above inequalities (4.47) – (4.50) can be used to state various particular inequalities as in the previous examples, however the details are left to the interested reader.

5. More Inequalities for the Čebyšev Functional

5.1. A Refinement and Some Related Results. The following result can be stated:

Theorem 5.1 (Dragomir, 2008, [32]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous with $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$, then*

$$|C(f, g; A; x)| \le \frac{1}{2} (\Delta - \delta) \langle |f(A) - \langle f(A) x, x \rangle \cdot 1_H| x, x \rangle \le \frac{1}{2} (\Delta - \delta) C^{1/2}(f, f; A; x), \tag{5.1}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$, we have

$$\left| g(t) - \frac{\Delta + \delta}{2} \right| \le \frac{1}{2} (\Delta - \delta), \tag{5.2}$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

If we multiply the inequality (5.2) with $|f(t) - \langle f(A) x, x \rangle|$ we get

$$\left| f(t) g(t) - \langle f(A) x, x \rangle g(t) - \frac{\Delta + \delta}{2} f(t) + \frac{\Delta + \delta}{2} \langle f(A) x, x \rangle \right| \le \frac{1}{2} (\Delta - \delta) |f(t) - \langle f(A) x, x \rangle|, \tag{5.3}$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

Now, if we apply the property (P) for the inequality (5.3) and a selfadjoint operator B with $Sp(B) \subset [m, M]$, then we get the following inequality which is of interest in itself:

$$\begin{aligned}(5.4)\quad & \left|\langle f(B)\, g(B)\, y, y\rangle - \langle f(A)\, x, x\rangle \langle g(B)\, y, y\rangle \right.\\ & \left. -\frac{\Delta+\delta}{2}\langle f(B)\, y, y\rangle + \frac{\Delta+\delta}{2}\langle f(A)\, x, x\rangle\right| \\ & \leq \frac{1}{2}(\Delta-\delta)\langle |f(B) - \langle f(A)\, x, x\rangle \cdot 1_H|\, y, y\rangle,\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

If we choose in (5.4) $y = x$ and $B = A$, then we deduce the first inequality in (5.1).

Now, by the Schwarz inequality in H we have

$$\begin{aligned}\langle |f(A) - \langle f(A)\, x, x\rangle \cdot 1_H|\, x, x\rangle &\leq \| |f(A) - \langle f(A)\, x, x\rangle \cdot 1_H|\, x\| \\ &= \| f(A)\, x - \langle f(A)\, x, x\rangle \cdot x\| \\ &= \left[\| f(A)\, x\|^2 - \langle f(A)\, x, x\rangle^2\right]^{1/2} \\ &= C^{1/2}(f, f; A; x),\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, and the second part of (5.1) is also proved. ∎

Let U be a selfadjoint operator on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{R}$, it is well known that we have the following representation in terms of the Riemann-Stieltjes integral:

$$(5.5)\qquad \langle f(U)\, x, x\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x\rangle),$$

for any $x \in H$ with $\|x\| = 1$. The function $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* on the interval $[m, M]$ and

$$(5.6)\qquad g_x(m-0) = 0 \quad \text{and} \quad g_x(M) = 1$$

for any $x \in H$ with $\|x\| = 1$.

The following result is of interest:

Theorem 5.2 (Dragomir, 2008, [32]). *Let A and B be selfadjoint operators with $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type, i.e., for a given $r \in (0, 1]$ and $L > 0$ we have*

$$|f(s) - f(t)| \leq L |s - t|^r \quad \textit{for any } s, t \in [m, M],$$

then we have the Ostrowski type inequality for selfadjoint operators:

$$(5.7)\qquad |f(s) - \langle f(A)\, x, x\rangle| \leq L\left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^r,$$

for any $s \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

Moreover, we have

$$\begin{aligned}(5.8)\quad |\langle f(B)\, y, y\rangle - \langle f(A)\, x, x\rangle| &\leq \langle |f(B) - \langle f(A)\, x, x\rangle \cdot 1_H|\, y, y\rangle \\ &\leq L\left[\frac{1}{2}(M-m) + \left\langle\left|B - \frac{m+M}{2}\cdot 1_H\right| y, y\right\rangle\right]^r,\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. We use the following Ostrowski type inequality for the Riemann-Stieltjes integral obtained by the author in [21]:

$$\left|f(s)[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right| \leq L\left[\frac{1}{2}(b-a)+\left|s-\frac{a+b}{2}\right|\right]^r \bigvee_a^b(u) \tag{5.9}$$

for any $s\in[a,b]$, provided that f is of $r-L-$Hölder type on $[a,b]$, u is of bounded variation on $[a,b]$ and $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$.

Now, applying this inequality for $u(\lambda)=g_x(\lambda):=\langle E_\lambda x,x\rangle$, where $x\in H$ with $\|x\|=1$ we get

$$\left|f(s)-\int_{m-0}^M f(\lambda)\,d(\langle E_\lambda x,x\rangle)\right| \leq L\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^r \bigvee_{m-0}^M(g_x) \tag{5.10}$$

which, by (5.5) and (5.6) is equivalent with (5.7).

By applying the property (P) for the inequality (5.7) and the operator B we have

$$\begin{aligned}\langle|f(B)-\langle f(A)x,x\rangle\cdot 1_H|\,y,y\rangle &\leq L\left\langle\left[\frac{1}{2}(M-m)+\left|B-\frac{m+M}{2}\cdot 1_H\right|\right]^r y,y\right\rangle\\ &\leq L\left\langle\left[\frac{1}{2}(M-m)+\left|B-\frac{m+M}{2}\right|\cdot 1_H\right]y,y\right\rangle^r\\ &= L\left[\frac{1}{2}(M-m)+\left\langle\left|B-\frac{m+M}{2}\cdot 1_H\right|y,y\right\rangle\right]^r\end{aligned}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$, which proves the second inequality in (5.8).

Further, by the Jensen inequality for convex functions of selfadjoint operators (see for instance [41, p. 5]) applied for the modulus, we can state that

$$|\langle h(A)x,x\rangle|\leq\langle|h(A)|\,x,x\rangle \tag{M}$$

for any $x\in H$ with $\|x\|=1$, where h is a continuous function on $[m,M]$.

Now, if we apply the inequality (M), then we have

$$|\langle[f(B)-\langle f(A)x,x\rangle\cdot 1_H]\,y,y\rangle|\leq\langle|f(B)-\langle f(A)x,x\rangle\cdot 1_H|\,y,y\rangle,$$

which shows the first part of (5.8), and the proof is complete. ∎

Remark 5.1. With the above assumptions for f,A and B, we have the following particular inequalities of interest:

$$\left|f\left(\frac{m+M}{2}\right)-\langle f(A)x,x\rangle\right|\leq\frac{1}{2^r}L(M-m)^r \tag{5.11}$$

and

$$|f(\langle Ax,x\rangle)-\langle f(A)x,x\rangle|\leq L\left[\frac{1}{2}(M-m)+\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right]^r, \tag{5.12}$$

for any $x\in H$ with $\|x\|=1$.

We also have the inequalities:

$$
\begin{aligned}
(5.13)\qquad & |\langle f(A)y,y\rangle - \langle f(A)x,x\rangle| \\
&\le \langle |f(A) - \langle f(A)x,x\rangle \cdot 1_H|\, y,y\rangle \\
&\le L\left[\frac{1}{2}(M-m) + \left\langle \left|A - \frac{m+M}{2}\cdot 1_H\right| y,y\right\rangle\right]^r,
\end{aligned}
$$

for any $x,y \in H$ with $\|x\| = \|y\| = 1,$

$$
\begin{aligned}
(5.14)\qquad |\langle [f(B) - f(A)]x,x\rangle| &\le \langle |f(B) - \langle f(A)x,x\rangle \cdot 1_H|\, x,x\rangle \\
&\le L\left[\frac{1}{2}(M-m) + \left\langle \left|B - \frac{m+M}{2}\cdot 1_H\right| x,x\right\rangle\right]^r
\end{aligned}
$$

and, more particularly,

$$
\begin{aligned}
(5.15)\qquad & \langle |f(A) - \langle f(A)x,x\rangle \cdot 1_H|\, x,x\rangle \\
&\le L\left[\frac{1}{2}(M-m) + \left\langle \left|A - \frac{m+M}{2}\cdot 1_H\right| x,x\right\rangle\right]^r,
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1.$

We also have the norm inequality

$$
(5.16)\qquad \|f(B) - f(A)\| \le L\left[\frac{1}{2}(M-m) + \left\|B - \frac{m+M}{2}\cdot 1_H\right\|\right]^r.
$$

The following corollary of Theorem 5.2 can be useful for applications:

Corollary 5.3 (Dragomir, 2008, [32]). *Let A and B be selfadjoint operators with $Sp(A), Sp(B) \subseteq [m,M]$ for some real numbers $m < M$. If $f : [m,M] \longrightarrow \mathbb{R}$ is absolutely continuous, then we have the Ostrowski type inequality for selfadjoint operators:*

$$
(5.17)\qquad |f(s) - \langle f(A)x,x\rangle| \le \begin{cases} \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]\|f'\|_{\infty,[m,M]} & \text{if } f' \in L_\infty[m,M]; \\[2ex] \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^{1/q}\|f'\|_{p,[m,M]} & \text{if } f' \in L_p[m,M], \\ & p,q>1, \frac{1}{p}+\frac{1}{q}=1, \end{cases}
$$

for any $s \in [m,M]$ and any $x \in H$ with $\|x\| = 1$, where $\|\cdot\|_{p,[m,M]}$ are the Lebesgue norms, i.e.,

$$
\|h\|_{\infty,[m,M]} := ess \sup_{t\in[m,M]} \|h(t)\|
$$

and

$$
\|h\|_{p,[m,M]} := \left(\int_m^M |h(t)|^p\right)^{1/p}, \quad p \ge 1.
$$

Additionally, we have

$$
\begin{aligned}
(5.18)\qquad & |\langle f(B)y,y\rangle - \langle f(A)x,x\rangle| \\
&\le \langle |f(B) - \langle f(A)x,x\rangle \cdot 1_H|\, y,y\rangle \\
&\le \begin{cases} \left[\frac{M-m}{2} + \left\langle \left|B - \frac{m+M}{2}\cdot 1_H\right| y,y\right\rangle\right]\|f'\|_{\infty,[m,M]} & \text{if } f' \in L_\infty[m,M]; \\[2ex] \left[\frac{M-m}{2} + \left\langle \left|B - \frac{m+M}{2}\cdot 1_H\right| y,y\right\rangle\right]^{\frac{1}{q}}\|f'\|_{p,[m,M]} & \text{if } f' \in L_p[m,M], \\ & p,q>1, \frac{1}{p}+\frac{1}{q}=1, \end{cases}
\end{aligned}
$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

Now, on utilising Theorem 5.1 we can provide the following upper bound for the Čebyšev functional that may be more useful in applications:

Corollary 5.4 (Dragomir, 2008, [32])**.** *Let* A *be a selfadjoint operator with* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$*. If* $g : [m, M] \longrightarrow \mathbb{R}$ *is continuous with* $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$ *, then for any* $f : [m, M] \longrightarrow \mathbb{R}$ *of* $r - L-$*Hölder type we have:*

$$|C(f, g; A; x)| \leq \frac{1}{2}(\Delta - \delta) L \left[\frac{1}{2}(M - m) + \left\langle \left| A - \frac{m + M}{2} \cdot 1_H \right| x, x \right\rangle \right]^r, \tag{5.19}$$

for any $x \in H$ *with* $\|x\| = 1$.

Remark 5.2. With the assumptions from Corollary 5.4 for g and A and if f is absolutely continuous on $[m, M]$, then we have the inequalities:

$$|C(f, g; A; x)| \leq \frac{1}{2}(\Delta - \delta) \times \begin{cases} \left[\frac{1}{2}(M - m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle \right] \|f'\|_{\infty,[m,M]} & \text{if } f' \in L_\infty [m, M]; \\ \left[\frac{1}{2}(M - m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle \right]^{1/q} \|f'\|_{p,[m,M]} & \text{if } f' \in L_\infty [m, M], \\ & p, q > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \tag{5.20}$$

for any $x \in H$ with $\|x\| = 1$.

5.2. **Some Inequalities for Sequences of Operators.** Consider the sequence of selfadjoint operators $\mathbf{A} = (A_1, \ldots, A_n)$ with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ are such that $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then we can consider the following Čebyšev type functional

$$C(f, g; \mathbf{A}, \mathbf{x}) := \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle .$$

As a particular case of the above functional and for a probability sequence $\mathbf{p} = (p_1, \ldots, p_n)$, i.e., $p_j \geq 0$ for $j \in \{1, \ldots, n\}$ and $\sum_{j=1}^{n} p_j = 1$, we can also consider the functional

$$C(f, g; \mathbf{A}, \mathbf{p}, x) := \left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle$$

where $x \in H$, $\|x\| = 1$.

We know, from [29] that for the sequence of selfadjoint operators $\mathbf{A} = (A_1, \ldots, A_n)$ with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for the synchronous (asynchronous) functions $f, g : [m, M] \longrightarrow \mathbb{R}$ we have the inequality

$$C(f, g; \mathbf{A}, \mathbf{x}) \geq (\leq) 0 \tag{5.21}$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. Also, for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$ and any $x \in H$, $\|x\| = 1$ we have

$$C(f, g; \mathbf{A}, \mathbf{p}, x) \geq (\leq) 0. \tag{5.22}$$

On the other hand, the following Grüss type inequality is valid as well [30]:

$$|C(f,g;\mathbf{A},\mathbf{x})| \leq \frac{1}{2}\cdot(\Gamma-\gamma)\left[C(g,g;\mathbf{A},\mathbf{x})\right]^{1/2}\left(\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right) \tag{5.23}$$

for any $\mathbf{x}=(x_1,\dots,x_n)\in H^n$ with $\sum_{j=1}^n \|x_j\|^2=1$, where f and g are continuous on $[m,M]$ and $\gamma := \min_{t\in[m,M]} f(t)$, $\Gamma := \max_{t\in[m,M]} f(t)$, $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

Similarly, for any probability distribution $\mathbf{p}=(p_1,\dots,p_n)$ and any $x\in H$, $\|x\|=1$ we also have the inequality:

$$|C(f,g;\mathbf{A},\mathbf{p},x)| \leq \frac{1}{2}\cdot(\Gamma-\gamma)\left[C(g,g;\mathbf{A},\mathbf{p},x)\right]^{1/2}\left(\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right). \tag{5.24}$$

We can state now the following new result:

Theorem 5.5 (Dragomir, 2008, [32]). *Consider the sequence of selfadjoint operators* $\mathbf{A}=(A_1,\dots,A_n)$ *with* $Sp(A_j)\subseteq[m,M]$ *for* $j\in\{1,\dots,n\}$ *and some scalars* $m<M$. *If* $f,g:[m,M]\longrightarrow\mathbb{R}$ *are continuous with* $\delta:=\min_{t\in[m,M]} g(t)$ *and* $\Delta:=\max_{t\in[m,M]} g(t)$, *then*

$$\begin{aligned} |C(f,g;\mathbf{A};\mathbf{x})| &\leq \frac{1}{2}(\Delta-\delta)\sum_{j=1}^n\left\langle\left|f(A_j)-\sum_{k=1}^n\langle f(A_k)x_k,x_k\rangle\cdot 1_H\right|x_j,x_j\right\rangle \\ &\leq \frac{1}{2}(\Delta-\delta)C^{1/2}(f,f;\mathbf{A};\mathbf{x}), \end{aligned} \tag{5.25}$$

for any $\mathbf{x}=(x_1,\dots,x_n)\in H^n$ *such that* $\sum_{j=1}^n\|x_j\|^2=1$.

Proof. As in [41, p. 6], if we put

$$\widetilde{A}:=\begin{pmatrix} A_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & A_n\end{pmatrix} \quad \text{and} \quad \widetilde{x}=\begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix},$$

then we have $Sp\left(\widetilde{A}\right)\subseteq[m,M]$, $\|\widetilde{x}\|=1$,

$$\left\langle f\left(\widetilde{A}\right)g\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^n\langle f(A_j)g(A_j)x_j,x_j\rangle,$$

$$\left\langle f\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^n\langle f(A_j)x_j,x_j\rangle, \qquad \left\langle g\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^n\langle g(A_j)x_j,x_j\rangle,$$

and so on.

Applying Theorem 5.1 for $\widetilde{A}$ and $\widetilde{x}$, we deduce the desired result (5.25). ∎

The following particular result is of interest for applications:

Corollary 5.6 (Dragomir, 2008, [32]). *Consider the sequence of selfadjoint operators* $\mathbf{A}=(A_1,\dots,A_n)$ *with* $Sp(A_j)\subseteq[m,M]$ *for* $j\in\{1,\dots,n\}$ *and some scalars* $m<M$. *If* $f,g:[m,M]\longrightarrow\mathbb{R}$ *are continuous with* $\delta:=\min_{t\in[m,M]} g(t)$ *and* $\Delta:=\max_{t\in[m,M]} g(t)$, *then for*

any $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$, *we have*

$$|C(f, g; \mathbf{A}, \mathbf{p}, x)| \leq \frac{1}{2}(\Delta - \delta)\left\langle \sum_{j=1}^{n} p_j \left| f(A_j) - \left\langle \sum_{k=1}^{n} p_k f(A_k) x, x \right\rangle \cdot 1_H \right| x, x \right\rangle \leq \frac{1}{2}(\Delta - \delta) C^{1/2}(f, f; \mathbf{A}, \mathbf{p}, x). \tag{5.26}$$

Proof. In we choose in Theorem 5.5, $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$ then a simple calculation shows that the inequality (5.25) becomes (5.26). The details are omitted. ∎

In a similar manner we can prove the following result as well:

Theorem 5.7 (Dragomir, 2008, [32]). *Consider the sequences of selfadjoint operators* $\mathbf{A} = (A_1, \ldots, A_n)$, $\mathbf{B} = (B_1, \ldots, B_n)$ *with* $Sp(A_j), Sp(B_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and some scalars* $m < M$. *If* $f : [m, M] \longrightarrow \mathbb{R}$ *is of* $r - L-$*Hölder type, then we have the Ostrowski type inequality for sequences of selfadjoint operators:*

$$\left| f(s) - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \right| \leq L \left[\frac{1}{2}(M - m) + \left| s - \frac{m + M}{2} \right| \right]^r, \tag{5.27}$$

for any $s \in [m, M]$ *and any* $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ *such that* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Moreover,

$$\left| \sum_{j=1}^{n} \langle f(B_j) y_j, y_j \rangle - \sum_{k=1}^{n} \langle f(A_k) x_k, x_k \rangle \right| \leq \sum_{j=1}^{n} \left\langle \left| f(B_j) - \sum_{k=1}^{n} \langle f(A_k) x_k, x_k \rangle \cdot 1_H \right| y_j, y_j \right\rangle \leq L \left[\frac{1}{2}(M - m) + \sum_{j=1}^{n} \left\langle \left| B_j - \frac{m + M}{2} \cdot 1_H \right| y_j, y_j \right\rangle \right]^r, \tag{5.28}$$

for any $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in H^n$ *such that* $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$.

Corollary 5.8 (Dragomir, 2008, [32]). *Consider the sequences of selfadjoint operators* $\mathbf{A} = (A_1, \ldots, A_n)$, $\mathbf{B} = (B_1, \ldots, B_n)$ *with* $Sp(A_j), Sp(B_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and some scalars* $m < M$. *If* $f : [m, M] \longrightarrow \mathbb{R}$ *is of* $r - L-$*Hölder type, then for any* $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$ *we have the weighted Ostrowski type inequality for sequences of selfadjoint operators:*

$$\left| f(s) - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \right| \leq L \left[\frac{1}{2}(M - m) + \left| s - \frac{m + M}{2} \right| \right]^r, \tag{5.29}$$

for any $s \in [m, M]$.

In addition, we have

$$\begin{aligned}(5.30)\qquad &\left|\left\langle \sum_{j=1}^{n} q_j f(B_j)\, y, y\right\rangle - \left\langle \sum_{k=1}^{n} p_k f(A_k)\, x, x\right\rangle\right| \\ &\leq \left\langle \sum_{j=1}^{n} q_j \left| f(B_j) - \left\langle \sum_{k=1}^{n} p_k f(A_k)\, x, x\right\rangle \cdot 1_H \right| y, y\right\rangle \\ &\leq L\left[\frac{1}{2}(M-m) + \left\langle \sum_{j=1}^{n} q_j \left| B_j - \frac{m+M}{2}\cdot 1_H\right| y, y\right\rangle\right]^r,\end{aligned}$$

for any $q_k \geq 0, k \in \{1,\ldots,n\}$ *with* $\sum_{k=1}^{n} q_k = 1$ *and* $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

5.3. **Some Reverses of Jensen's Inequality.** It is clear that all of the above inequalities can be applied for various particular instances of functions f and g. However, in the following we only consider the inequalities

$$(5.31)\qquad |f(\langle Ax, x\rangle) - \langle f(A)\, x, x\rangle| \leq L\left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right]^r$$

for any $x \in H$ with$\|x\| = 1$, where the function $f : [m, M] \to \mathbb{R}$ is of $r - L-$Hölder type, and

$$(5.32)\qquad |f(\langle Ax, x\rangle) - \langle f(A)\, x, x\rangle| \leq \begin{cases} \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right] \|f'\|_{\infty,[m,M]}, & \text{if } f' \in L_\infty[m, M] \\ \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right]^q \|f'\|_{p,[m,M]}, & \text{if } f' \in L_p[m, M];\ p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases}$$

for any $x \in H$ with $\|x\| = 1$, where the function $f : [m, M] \to \mathbb{R}$ is absolutely continuous on $[m, M]$, which are related to the *Jensen inequality* for convex functions.

1. Now, if we consider the concave function $f : [m, M] \subset [0, \infty) \to \mathbb{R}$, $f(t) = t^r$ with $r \in (0, 1)$ and take into account that it is of $r - L-$Hölder type with the constant $L = 1$, then from (5.31) we derive the following reverse of the *Hölder-McCarthy inequality* [46]

$$(5.33)\qquad 0 \leq \langle A^r x, x\rangle - \langle Ax, x\rangle^r \leq \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right]^r$$

for any $x \in H$ with $\|x\| = 1$.

2. Now, if we consider the functions $f : [m, M] \subset (0, \infty) \to \mathbb{R}$ with $f(t) = t^s$ and $s \in (-\infty, 0) \cup (0, \infty)$, then they are absolutely continuous and

$$\|f'\|_{\infty,[m,M]} = \begin{cases} sM^{s-1} & \text{for } s \in [1, \infty), \\ |s|\, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1). \end{cases}$$

If $p \geq 1$, then

$$\begin{aligned}\|f'\|_{p,[m,M]} &= |s|\left(\int_m^M t^{p(s-1)} dt\right)^{1/p} \\ &= |s| \times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s \neq 1 - \frac{1}{p}; \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s = 1 - \frac{1}{p}. \end{cases}\end{aligned}$$

On making use of the first inequality from (5.32), we deduce for a given $s \in (-\infty, 0) \cup (0, \infty)$ that

$$|\langle Ax, x\rangle^s - \langle A^s x, x\rangle| \le \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right] \times \begin{cases} sM^{s-1} & \text{for } s \in [1, \infty), \\ |s|\, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1), \end{cases} \tag{5.34}$$

for any $x \in H$ with $\|x\| = 1$.

The second part of (5.32) will produce the following reverse of the *Hölder-McCarthy inequality* as well:

$$|\langle Ax, x\rangle^s - \langle A^s x, x\rangle| \le |s| \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right]^q \times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s \ne 1 - \frac{1}{p}; \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s = 1 - \frac{1}{p}, \end{cases} \tag{5.35}$$

for any $x \in H$ with $\|x\| = 1$, where $s \in (-\infty, 0) \cup (0, \infty)$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

3. Now, if we consider the function $f(t) = \ln t$ defined on the interval $[m, M] \subset (0, \infty)$, then f is also absolutely continuous and

$$\|f'\|_{p,[m,M]} = \begin{cases} m^{-1} & \text{for } p = \infty, \\ \left(\frac{M^{p-1} - m^{p-1}}{(p-1)M^{p-1}m^{p-1}}\right)^{1/p} & \text{for } p > 1, \\ \ln\left(\frac{M}{m}\right) & \text{for } p = 1. \end{cases}$$

Making use of the first inequality in (5.32), we deduce

$$0 \le \ln(\langle Ax, x\rangle) - \langle \ln(A)\, x, x\rangle \le \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right] m^{-1} \tag{5.36}$$

and

$$\begin{aligned} 0 &\le \ln(\langle Ax, x\rangle) - \langle \ln(A)\, x, x\rangle \\ &\le \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right]^q \left(\frac{M^{p-1} - m^{p-1}}{(p-1)\, M^{p-1} m^{p-1}}\right)^{1/p} \end{aligned} \tag{5.37}$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Similar results can be stated for sequences of operators, however the details are left to the interested reader.

5.4. Some Particular Grüss Type Inequalities.

In what follows, we provide some particular cases that can be obtained via the Grüss type inequalities established before. For this purpose, we select only two examples as follows.

Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$, then for any $f:[m,M]\longrightarrow\mathbb{R}$ of $r-L-$Hölder type we have the inequality:

$$|\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle| \le \frac{1}{2}(\Delta-\delta)L\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r, \tag{5.38}$$

for any $x\in H$ with $\|x\|=1$.

Also, if f is absolutely continuous on $[m,M]$, then we have the inequalities:

$$|\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle|\le\frac{1}{2}(\Delta-\delta) \tag{5.39}$$
$$\times\begin{cases}\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]\|f'\|_{\infty,[m,M]} & \text{if } f'\in L_\infty[m,M];\\ \left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^{1/q}\|f'\|_{p,[m,M]} & \text{if } f'\in L_p[m,M],\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1\end{cases}$$

for any $x\in H$ with $\|x\|=1$.

1. If we consider the concave function $f:[m,M]\subset[0,\infty)\to\mathbb{R}$, $f(t)=t^r$ with $r\in(0,1)$ and take into account that it is of $r-L-$Hölder type with the constant $L=1$, then from (5.38) we derive the following result:

$$|\langle A^r g(A)x,x\rangle-\langle A^r x,x\rangle\cdot\langle g(A)x,x\rangle| \le\frac{1}{2}(\Delta-\delta)\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r, \tag{5.40}$$

for any $x\in H$ with $\|x\|=1$, where $g:[m,M]\longrightarrow\mathbb{R}$ is continuous with $\delta:=\min_{t\in[m,M]}g(t)$ and $\Delta:=\max_{t\in[m,M]}g(t)$.

Now, consider the function $g:[m,M]\subset(0,\infty)\to\mathbb{R}$, $g(t)=t^p$ with $p\in(-\infty,0)\cup(0,\infty)$. Obviously,

$$\Delta-\delta=\begin{cases}M^p-m^p & \text{if } p>0,\\ \frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}} & \text{if } p<0,\end{cases}$$

and by (5.40) we get for any $x\in H$ with $\|x\|=1$ that

$$0\le\left\langle A^{r+p}x,x\right\rangle-\langle A^r x,x\rangle\cdot\langle A^p x,x\rangle \le\frac{1}{2}(M^p-m^p)\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r, \tag{5.41}$$

when $p>0$ and

$$0\le\langle A^r x,x\rangle\cdot\langle A^p x,x\rangle-\left\langle A^{r+p}x,x\right\rangle \le\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r, \tag{5.42}$$

when $p<0$.

If $g:[m,M]\subset(0,\infty)\to\mathbb{R}$, $g(t)=\ln t$, then by (5.40) we also obtain the inequality for the logarithm:

$$0\le\langle A^r\ln Ax,x\rangle-\langle A^r x,x\rangle\cdot\langle\ln Ax,x\rangle \le\ln\sqrt{\frac{M}{m}}\cdot\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r, \tag{5.43}$$

for any $x\in H$ with $\|x\|=1$.

2. Now consider the functions $f, g : [m, M] \subset (0, \infty) \to \mathbb{R}$, with $f(t) = t^s$ and $g(t) = t^w$ with $s, w \in (-\infty, 0) \cup (0, \infty)$. We have

$$\|f'\|_{\infty,[m,M]} = \begin{cases} sM^{s-1} & \text{for } s \in [1, \infty), \\ |s|\, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1). \end{cases}$$

and, for $p \geq 1$,

$$\|f'\|_{p,[m,M]} = |s| \times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s \neq 1 - \frac{1}{p}, \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s = 1 - \frac{1}{p}. \end{cases}$$

If $w > 0$, then by the first inequality in (5.39) we have

$$(5.44)\quad \left|\left\langle A^{s+w}x, x\right\rangle - \langle A^s x, x\rangle \cdot \langle A^w x, x\rangle\right| \leq \frac{1}{2}(M^w - m^w)\left[\frac{1}{2}(M - m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle\right] \times \begin{cases} sM^{s-1} & \text{for } s \in [1, \infty), \\ |s|\, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1), \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

If $w < 0$, then by the same inequality we also have

$$(5.45)\quad \left|\left\langle A^{s+w}x, x\right\rangle - \langle A^s x, x\rangle \cdot \langle A^w x, x\rangle\right| \leq \frac{1}{2} \cdot \frac{M^{-w} - m^{-w}}{M^{-w}m^{-w}}\left[\frac{1}{2}(M - m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle\right] \times \begin{cases} sM^{s-1} & \text{for } s \in [1, \infty), \\ |s|\, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1), \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

Finally, if we assume that $p > 1$ and $w > 0$, then by the second inequality in (5.39) we have

$$(5.46)\quad \left|\left\langle A^{s+w}x, x\right\rangle - \langle A^s x, x\rangle \cdot \langle A^w x, x\rangle\right| \leq \frac{1}{2}|s|(M^w - m^w)\left[\frac{1}{2}(M - m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle\right]^{1/q} \times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s \neq 1 - \frac{1}{p}, \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s = 1 - \frac{1}{p}, \end{cases}$$

while for $w < 0$, we also have

$$(5.47)\quad \left|\left\langle A^{s+w}x, x\right\rangle - \langle A^s x, x\rangle \cdot \langle A^w x, x\rangle\right| \leq \frac{1}{2}|s| \cdot \frac{M^{-w} - m^{-w}}{M^{-w}m^{-w}}\left[\frac{1}{2}(M - m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle\right]^{1/q} \times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s \neq 1 - \frac{1}{p}, \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s = 1 - \frac{1}{p}, \end{cases}$$

where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in H$ with $\|x\| = 1$.

6. Bounds for the Čebyšev Functional of Lipschitzian Functions

6.1. The Case of Lipschitzian Functions. The following result can be stated:

Theorem 6.1 (Dragomir, 2008, [33]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is Lipschitzian with the constant $L > 0$ and $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then*

$$|C(f, g; A; x)| \leq \frac{1}{2}(\Delta - \delta) L \langle \ell_{A,x}(A) x, x\rangle \leq (\Delta - \delta) LC(e, e; A; x) \tag{6.1}$$

for any $x \in H$ with $\|x\| = 1$, where

$$\ell_{A,x}(t) := \langle |t \cdot 1_H - A| x, x\rangle$$

is a continuous function on $[m, M]$, $e(t) = t$ and

$$C(e, e; A; x) = \|Ax\|^2 - \langle Ax, x\rangle^2 \, (\geq 0). \tag{6.2}$$

Proof. First of all, by the Jensen inequality for convex functions of selfadjoint operators (see for instance [41, p. 5]) applied for the modulus, we can state that

$$|\langle h(A) x, x\rangle| \leq \langle |h(A)| x, x\rangle \tag{M}$$

for any $x \in H$ with $\|x\| = 1$, where h is a continuous function on $[m, M]$.

Since f is Lipschitzian with the constant $L > 0$, then for any $t, s \in [m, M]$ we have

$$|f(t) - f(s)| \leq L|t - s|. \tag{6.3}$$

Now, if we fix $t \in [m, M]$ and apply the property (P) for the inequality (6.3) and the operator A, we obtain

$$\langle |f(t) \cdot 1_H - f(A)| x, x\rangle \leq L \langle |t \cdot 1_H - A| x, x\rangle, \tag{6.4}$$

for any $x \in H$ with $\|x\| = 1$.

Utilising the property (M) we get

$$|f(t) - \langle f(A) x, x\rangle| = |\langle f(t) \cdot 1_H - f(A) x, x\rangle| \leq \langle |f(t) \cdot 1_H - f(A)| x, x\rangle$$

which together with (6.4) gives

$$|f(t) - \langle f(A) x, x\rangle| \leq L\ell_{A,x}(t) \tag{6.5}$$

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

Since $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, we also have

$$\left| g(t) - \frac{\Delta + \delta}{2} \right| \leq \frac{1}{2}(\Delta - \delta) \tag{6.6}$$

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

If we multiply the inequality (6.5) with (6.6) we get

$$\left|f(t)\,g(t)-\langle f(A)\,x,x\rangle\,g(t)-\frac{\Delta+\delta}{2}f(t)+\frac{\Delta+\delta}{2}\langle f(A)\,x,x\rangle\right| \tag{6.7}$$
$$\le\frac{1}{2}(\Delta-\delta)\,L\ell_{A,x}(t)=\frac{1}{2}(\Delta-\delta)\,L\,\langle|t\cdot 1_H-A|\,x,x\rangle$$
$$\le\frac{1}{2}(\Delta-\delta)\,L\left\langle|t\cdot 1_H-A|^2\,x,x\right\rangle^{1/2}$$
$$=\frac{1}{2}(\Delta-\delta)\,L\left(\left\langle A^2x,x\right\rangle-2\,\langle Ax,x\rangle\,t+t^2\right)^{1/2},$$

for any $t\in[m,M]$ and for any $x\in H$ with $\|x\|=1$.

Now, if we apply the property (P) for the inequality (6.7) and a selfadjoint operator B with $Sp(B)\subset[m,M]$, then we obtain the following inequality which is of interest in itself:

$$\Big|\langle f(B)\,g(B)\,y,y\rangle-\langle f(A)\,x,x\rangle\,\langle g(B)\,y,y\rangle \tag{6.8}$$
$$-\frac{\Delta+\delta}{2}\langle f(B)\,y,y\rangle+\frac{\Delta+\delta}{2}\langle f(A)\,x,x\rangle\Big|$$
$$\le\frac{1}{2}(\Delta-\delta)\,L\,\langle\ell_{A,x}(B)\,y,y\rangle$$
$$\le\frac{1}{2}(\Delta-\delta)\,L\left\langle\left(\left\langle A^2x,x\right\rangle 1_H-2\,\langle Ax,x\rangle\,B+B^2\right)^{1/2}y,y\right\rangle$$
$$\le\frac{1}{2}(\Delta-\delta)\,L\left(\left\langle A^2x,x\right\rangle-2\,\langle Ax,x\rangle\,\langle By,y\rangle+\left\langle B^2y,y\right\rangle\right),$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

Finally, if we choose in (6.8) $y=x$ and $B=A$, then we deduce the desired result (6.1). ∎

In the case of two Lipschitzian functions, the following result may be stated as well:

Theorem 6.2 (Dragomir, 2008, [33]). *Let A be a selfadjoint operator with $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. If $f,g:[m,M]\longrightarrow\mathbb{R}$ are Lipschitzian with the constants $L,K>0$, then*

$$|C(f,g;A;x)|\le LKC(e,e;A;x), \tag{6.9}$$

for any $x\in H$ with $\|x\|=1$.

Proof. Since $f,g:[m,M]\longrightarrow\mathbb{R}$ are Lipschitzian, then

$$|f(t)-f(s)|\le L\,|t-s|\quad\text{and}\quad|g(t)-g(s)|\le K\,|t-s|$$

for any $t,s\in[m,M]$, which gives the inequality

$$|f(t)\,g(t)-f(t)\,g(s)-f(s)\,g(t)+f(s)\,g(s)|\le KL\left(t^2-2ts+s^2\right)$$

for any $t,s\in[m,M]$.

Now, fix $t\in[m,M]$ and if we apply the properties (P) and (M) for the operator A we get successively

$$|f(t)\,g(t)-\langle g(A)\,x,x\rangle\,f(t)-\langle f(A)\,x,x\rangle\,g(t)+\langle f(A)\,g(A)\,x,x\rangle| \tag{6.10}$$
$$=|\langle[f(t)\,g(t)\cdot 1_H-f(t)\,g(A)-f(A)\,g(t)+f(A)\,g(A)]\,x,x\rangle|$$
$$\le\langle|f(t)\,g(t)\cdot 1_H-f(t)\,g(A)-f(A)\,g(t)+f(A)\,g(A)|\,x,x\rangle$$
$$\le KL\left\langle\left(t^2\cdot 1_H-2tA+A^2\right)x,x\right\rangle=KL\left(t^2-2t\,\langle Ax,x\rangle+\left\langle A^2x,x\right\rangle\right)$$

for any $x \in H$ with $\|x\| = 1$.

Further, fix $x \in H$ with $\|x\| = 1$. On applying the same properties for the inequality (6.10) and another selfadjoint operator B with $Sp(B) \subset [m, M]$, we have

$$\begin{aligned}(6.11)\quad & |\langle f(B) g(B) y, y\rangle - \langle g(A) x, x\rangle \langle f(B) y, y\rangle \\ &\qquad - \langle f(A) x, x\rangle \langle g(B) y, y\rangle + \langle f(A) g(A) x, x\rangle| \\ &= |\langle [f(B) g(B) - \langle g(A) x, x\rangle f(B) - \langle f(A) x, x\rangle g(B) + \langle f(A) g(A) x, x\rangle 1_H] y, y\rangle| \\ &\le \langle |f(B) g(B) - \langle g(A) x, x\rangle f(B) - \langle f(A) x, x\rangle g(B) + \langle f(A) g(A) x, x\rangle 1_H| y, y\rangle \\ &\le KL \left\langle \left(B^2 - 2\langle Ax, x\rangle B + \left\langle A^2 x, x\right\rangle 1_H\right) y, y\right\rangle \\ &= KL \left(\left\langle B^2 y, y\right\rangle - 2\langle Ax, x\rangle \langle By, y\rangle + \left\langle A^2 x, x\right\rangle\right)\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, which is an inequality of interest in its own right.

Finally, on making $B = A$ and $y = x$ in (6.11) we deduce the desired result (6.9). ∎

6.2. **Some Inequalities for Sequences of Operators.** Consider the sequence of selfadjoint operators $\mathbf{A} = (A_1, \dots, A_n)$ with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$. If $\mathbf{x} = (x_1, \dots, x_n) \in H^n$ are such that $\sum_{j=1}^n \|x_j\|^2 = 1$, then we can consider the following Čebyšev type functional

$$C(f, g; \mathbf{A}, \mathbf{x}) := \sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j\rangle - \sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j\rangle .$$

As a particular case of the above functional and for a probability sequence $\mathbf{p} = (p_1, \dots, p_n)$, i.e., $p_j \ge 0$ for $j \in \{1, \dots, n\}$ and $\sum_{j=1}^n p_j = 1$, we can also consider the functional

$$\begin{aligned}C(f, g; \mathbf{A}, \mathbf{p}, x) := & \left\langle \sum_{j=1}^n p_j f(A_j) g(A_j) x, x \right\rangle \\ & - \left\langle \sum_{j=1}^n p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^n p_j g(A_j) x, x \right\rangle,\end{aligned}$$

where $x \in H$, $\|x\| = 1$.

We know, from [29] that for the sequence of selfadjoint operators $\mathbf{A} = (A_1, \dots, A_n)$ with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for the synchronous (asynchronous) functions $f, g : [m, M] \longrightarrow \mathbb{R}$ we have the inequality

$$C(f, g; \mathbf{A}, \mathbf{x}) \ge (\le)\, 0 \tag{6.12}$$

for any $\mathbf{x} = (x_1, \dots, x_n) \in H^n$ with $\sum_{j=1}^n \|x_j\|^2 = 1$. Also, for any probability distribution $\mathbf{p} = (p_1, \dots, p_n)$ and any $x \in H$, $\|x\| = 1$ we have

$$C(f, g; \mathbf{A}, \mathbf{p}, x) \ge (\le)\, 0. \tag{6.13}$$

On the other hand, the following Grüss type inequality is valid as well [29]:

$$|C(f, g; \mathbf{A}, \mathbf{x})| \le \frac{1}{2} \cdot (\Gamma - \gamma) [C(g, g; \mathbf{A}, \mathbf{x})]^{1/2} \left(\le \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta)\right) \tag{6.14}$$

for any $\mathbf{x} = (x_1, \dots, x_n) \in H^n$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, where f and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m,M]} f(t)$, $\Gamma := \max_{t \in [m,M]} f(t)$, $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$.

Similarly, for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$ and any $x \in H$, $\|x\| = 1$ we also have the inequality:

$$|C(f, g; \mathbf{A}, \mathbf{p}, x)| \le \frac{1}{2} \cdot (\Gamma - \gamma) [C(g, g; \mathbf{A}, \mathbf{p}, x)]^{1/2} \left(\le \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta) \right). \tag{6.15}$$

We can now state the following new result:

Theorem 6.3 (Dragomir, 2008, [33]). *Let* $\mathbf{A} = (A_1, \ldots, A_n)$ *be a sequence of selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and some scalars* $m < M$. *If* $f : [m, M] \longrightarrow \mathbb{R}$ *is Lipschitzian with the constant* $L > 0$ *and* $g : [m, M] \longrightarrow \mathbb{R}$ *is continuous with* $\delta := \min_{t \in [m,M]} g(t)$ *and* $\Delta := \max_{t \in [m,M]} g(t)$, *then*

$$\begin{aligned} |C(f, g; \mathbf{A}, \mathbf{x})| &\le \frac{1}{2} (\Delta - \delta) L \sum_{k=1}^{n} \langle \ell_{\mathbf{A},\mathbf{x}}(A_k) x_k, x_k \rangle \\ &\le (\Delta - \delta) LC(e, e; \mathbf{A}; \mathbf{x}) \end{aligned} \tag{6.16}$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$, *where*

$$\ell_{\mathbf{A},\mathbf{x}}(t) := \sum_{j=1}^{n} \langle |t \cdot 1_H - A_j| x_j, x_j \rangle$$

is a continuous function on $[m, M]$, $e(t) = t$ *and*

$$C(e, e; \mathbf{A}; \mathbf{x}) = \sum_{j=1}^{n} \|Ax_j\|^2 - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)^2 (\ge 0).$$

Proof. As in [41, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix} \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = 1$,

$$\left\langle f\left(\widetilde{A}\right) g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle,$$

$$\left\langle f\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle, \qquad \left\langle g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle,$$

and so on.

Applying Theorem 6.1 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired result (6.16). ∎

As a particular case we have:

Corollary 6.4 (Dragomir, 2008, [33]). *Let* $\mathbf{A} = (A_1, \ldots, A_n)$ *be a sequence of selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and some scalars* $m < M$. *If* $f : [m, M] \longrightarrow \mathbb{R}$ *is Lipschitzian with the constant* $L > 0$ *and* $g : [m, M] \longrightarrow \mathbb{R}$ *is continuous with* $\delta := \min_{t \in [m,M]} g(t)$ *and* $\Delta := \max_{t \in [m,M]} g(t)$, *then for any* $p_j \ge 0, j \in \{1, \ldots, n\}$

with $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$, *we have*

$$\text{(6.17)} \qquad |C(f,g;\mathbf{A},\mathbf{p},x)| \leq \frac{1}{2}(\Delta - \delta) L \left\langle \sum_{k=1}^{n} p_k \ell_{\mathbf{A},\mathbf{p},x}(A_k)\, x, x \right\rangle \leq (\Delta - \delta) LC(e,e;\mathbf{A},\mathbf{p},x)$$

where

$$\ell_{\mathbf{A},\mathbf{p},x}(t) := \left\langle \sum_{j=1}^{n} p_j |t \cdot 1_H - A_j|\, x, x \right\rangle$$

is a continuous function on $[m, M]$ *and*

$$C(e,e;\mathbf{A},\mathbf{p},x) = \sum_{j=1}^{n} p_j \|Ax_j\|^2 - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^2 (\geq 0).$$

Proof. In we choose in Theorem 6.3, $x_j = \sqrt{p_j} \cdot x$, $j \in \{1,\dots,n\}$, where $p_j \geq 0, j \in \{1,\dots,n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$, then a simple calculation shows that the inequality (6.16) becomes (6.17). The details are omitted. ∎

In a similar manner we obtain the following results as well:

Theorem 6.5 (Dragomir, 2008, [33])**.** *Let* $\mathbf{A} = (A_1,\dots,A_n)$ *be a sequence of selfadjoint operators with* $Sp(A_j) \subseteq [m,M]$ *for* $j \in \{1,\dots,n\}$ *and some scalars* $m < M$. *If* $f, g : [m,M] \longrightarrow \mathbb{R}$ *are Lipschitzian with the constants* $L, K > 0$, *then*

$$\text{(6.18)} \qquad |C(f,g;\mathbf{A},\mathbf{x})| \leq LKC(e,e;\mathbf{A},\mathbf{x}),$$

for any $\mathbf{x} = (x_1,\dots,x_n) \in H^n$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Corollary 6.6. *Let* $\mathbf{A} = (A_1,\dots,A_n)$ *be a sequence of selfadjoint operators with* $Sp(A_j) \subseteq [m,M]$ *for* $j \in \{1,\dots,n\}$ *and some scalars* $m < M$. *If* $f, g : [m,M] \longrightarrow \mathbb{R}$ *are Lipschitzian with the constants* $L, K > 0$, *then for any* $p_j \geq 0, j \in \{1,\dots,n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *we have*

$$\text{(6.19)} \qquad |C(f,g;\mathbf{A},\mathbf{p},x)| \leq LKC(e,e;\mathbf{A},\mathbf{p},x),$$

for any $x \in H$ *with* $\|x\| = 1$.

6.3. **The Case of** (φ, Φ) **−Lipschitzian Functions.** The following lemma may be stated.

Lemma 6.7. *Let* $u : [a,b] \to \mathbb{R}$ *and* $\varphi, \Phi \in \mathbb{R}$ *with* $\Phi > \varphi$. *The following statements are equivalent:*

(i) *The function* $u - \frac{\varphi+\Phi}{2} \cdot e$, *where* $e(t) = t, t \in [a,b]$, *is* $\frac{1}{2}(\Phi - \varphi)$ *−Lipschitzian;*
(ii) *We have the inequality:*

$$\text{(6.20)} \qquad \varphi \leq \frac{u(t) - u(s)}{t - s} \leq \Phi \quad \textit{for each} \quad t, s \in [a,b] \quad \textit{with } t \neq s;$$

(iii) *We have the inequality:*

$$\text{(6.21)} \qquad \varphi(t-s) \leq u(t) - u(s) \leq \Phi(t-s) \quad \textit{for each} \quad t, s \in [a,b] \quad \textit{with } t > s.$$

Following [44], we can introduce the concept:

Definition 6.1. The function $u : [a,b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be (φ, Φ) −Lipschitzian on $[a,b]$.

Notice that in [44], the definition was introduced on utilising the statement (iii) and only the equivalence (i) $\Leftrightarrow$ (iii) was considered.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides practical examples of (φ, Φ) $-$Lipschitzian functions.

Proposition 6.8. *Let $u : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If*

$$-\infty < \gamma := \inf_{t\in(a,b)} u'(t), \qquad \sup_{t\in(a,b)} u'(t) =: \Gamma < \infty, \tag{6.22}$$

then u is (γ, Γ) $-$Lipschitzian on $[a, b]$.

The following result can be stated:

Theorem 6.9 (Dragomir, 2008, [33])**.** *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is (φ, Φ) $-$Lipschitzian on $[a, b]$ and $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then*

$$\begin{aligned}\left|C(f, g; A; x) - \frac{\varphi + \Phi}{2} C(e, g; A; x)\right| &\le \frac{1}{4}(\Delta - \delta)(\Phi - \varphi)\langle \ell_{A,x}(A)x, x\rangle \\ &\le \frac{1}{2}(\Delta - \delta)(\Phi - \varphi) C(e, e; A; x)\end{aligned} \tag{6.23}$$

for any $x \in H$ with $\|x\| = 1$.

The proof follows by Theorem 6.1 applied for the $\frac{1}{2}(\Phi - \varphi)$ $-$Lipschitzian function $f - \frac{\varphi+\Phi}{2} \cdot e$ (see Lemma 6.7) and the details are omitted.

Theorem 6.10 (Dragomir, 2008, [33])**.** *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $f, g : [m, M] \longrightarrow \mathbb{R}$. If f is (φ, Φ) $-$Lipschitzian and g is (ψ, Ψ) $-$Lipschitzian on $[a, b]$, then*

$$\begin{aligned}&\Bigg|C(f, g; A; x) - \frac{\Phi + \varphi}{2} C(e, g; A; x) \\ &\qquad - \frac{\Psi + \psi}{2} C(f, e; A; x) + \frac{\Phi + \varphi}{2} \cdot \frac{\Psi + \psi}{2} C(e, e; A; x)\Bigg| \\ &\qquad\qquad \le \frac{1}{4}(\Phi - \varphi)(\Psi - \psi) C(e, e; A; x),\end{aligned} \tag{6.24}$$

for any $x \in H$ with $\|x\| = 1$.

The proof follows by Theorem 6.2 applied for the $\frac{1}{2}(\Phi - \varphi)$ $-$Lipschitzian function $f - \frac{\varphi+\Phi}{2} \cdot e$ and the $\frac{1}{2}(\Psi - \psi)$ $-$Lipschitzian function $g - \frac{\Psi+\psi}{2} \cdot e$. The details are omitted.

Similar results can be derived for sequences of operators, however they will not be presented here.

6.4. **Some Applications.** It is clear that all the inequalities obtained in the previous sections can be applied to obtain particular inequalities of interest for different selections of the functions f and g involved. However we will present here only some particular results that can be derived from the inequality

$$|C(f, g; A; x)| \le LKC(e, e; A; x), \tag{6.25}$$

that holds for the Lipschitzian functions f and g, the first with the constant $L > 0$ and the second with the constant $K > 0$.

1. Now, if we consider the functions $f,g:[m,M]\subset(0,\infty)\to\mathbb{R}$ with $f(t)=t^p, g(t)=t^q$ and $p,q\in(-\infty,0)\cup(0,\infty)$, then they are Lipschitzian with the constants $L=\|f'\|_\infty$ and $K=\|g'\|_\infty$. Since $f'(t)=pt^{p-1}, g(t)=qt^{q-1}$, then

$$\|f'\|_\infty=\begin{cases} pM^{p-1} & \text{for } p\in[1,\infty),\\ |p|\,m^{p-1} & \text{for } p\in(-\infty,0)\cup(0,1)\end{cases}$$

and

$$\|g'\|_\infty=\begin{cases} qM^{q-1} & \text{for } q\in[1,\infty),\\ |q|\,m^{q-1} & \text{for } q\in(-\infty,0)\cup(0,1).\end{cases}$$

Therefore we can state the following inequalities for the powers of a positive definite operator A with $Sp(A)\subset[m,M]\subset(0,\infty)$.

If $p,q\geq 1$, then

$$(0\leq)\left\langle A^{p+q}x,x\right\rangle-\langle A^px,x\rangle\cdot\langle A^qx,x\rangle\leq pqM^{p+q-2}\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)\tag{6.26}$$

for each $x\in H$ with $\|x\|=1$.

If $p\geq 1$ and $q\in(-\infty,0)\cup(0,1)$, then

$$\left|\left\langle A^{p+q}x,x\right\rangle-\langle A^px,x\rangle\cdot\langle A^qx,x\rangle\right|\leq p\,|q|\,M^{p-1}m^{q-1}\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)\tag{6.27}$$

for each $x\in H$ with $\|x\|=1$.

If $p\in(-\infty,0)\cup(0,1)$ and $q\geq 1$, then

$$\left|\left\langle A^{p+q}x,x\right\rangle-\langle A^px,x\rangle\cdot\langle A^qx,x\rangle\right|\leq |p|\,qM^{q-1}m^{p-1}\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)\tag{6.28}$$

for each $x\in H$ with $\|x\|=1$.

If $p,q\in(-\infty,0)\cup(0,1)$, then

$$\left|\left\langle A^{p+q}x,x\right\rangle-\langle A^px,x\rangle\cdot\langle A^qx,x\rangle\right|\leq |pq|\,m^{p+q-2}\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)\tag{6.29}$$

for each $x\in H$ with $\|x\|=1$.

Moreover, if we take $p=1$ and $q=-1$ in (6.27), then we get the following result

$$(0\leq)\langle Ax,x\rangle\cdot\left\langle A^{-1}x,x\right\rangle-1\leq m^{-2}\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)\tag{6.30}$$

for each $x\in H$ with $\|x\|=1$.

2. Consider now the functions $f,g:[m,M]\subset(0,\infty)\to\mathbb{R}$ with $f(t)=t^p, p\in(-\infty,0)\cup(0,\infty)$ and $g(t)=\ln t$. Then g is also Lipschitzian with the constant $K=\|g'\|_\infty=m^{-1}$. Applying the inequality (6.25) we then have for any $x\in H$ with $\|x\|=1$ that

$$(0\leq)\langle A^p\ln Ax,x\rangle-\langle A^px,x\rangle\cdot\langle\ln Ax,x\rangle\leq pM^{p-1}m^{-1}\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)\tag{6.31}$$

if $p\geq 1$,

$$(0\leq)\langle A^p\ln Ax,x\rangle-\langle A^px,x\rangle\cdot\langle\ln Ax,x\rangle\leq pm^{p-2}\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)\tag{6.32}$$

if $p\in(0,1)$ and

$$(0\leq)\langle A^px,x\rangle\cdot\langle\ln Ax,x\rangle-\langle A^p\ln Ax,x\rangle\leq(-p)\,m^{p-2}\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)\tag{6.33}$$

if $p\in(-\infty,0)$.

3. Now consider the functions $f,g:[m,M]\subset\mathbb{R}\to\mathbb{R}$ given by $f(t)=\exp(\alpha t)$ and $g(t)=\exp(\beta t)$ with α,β nonzero real numbers. It is obvious that

$$\|f'\|_\infty=|\alpha|\times\begin{cases}\exp(\alpha M) & \text{for } \alpha>0,\\ \exp(\alpha m) & \text{for } \alpha<0\end{cases}$$

and

$$\|g'\|_\infty = |\beta| \times \begin{cases} \exp(\beta M) & \text{for } \beta > 0, \\ \exp(\beta m) & \text{for } \beta < 0 \end{cases}.$$

Finally, on applying the inequality (6.25) we get

$$(0 \leq) \langle \exp[(\alpha+\beta)A]x, x\rangle - \langle \exp(\alpha A)x, x\rangle \cdot \langle \exp(\beta A)x, x\rangle \leq |\alpha\beta| \left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \times \begin{cases} \exp[(\alpha+\beta)M] & \text{for } \alpha,\beta > 0, \\ \exp[(\alpha+\beta)m] & \text{for } \alpha,\beta < 0 \end{cases} \tag{6.34}$$

and

$$(0 \leq) \langle \exp(\alpha A)x, x\rangle \cdot \langle \exp(\beta A)x, x\rangle - \langle \exp[(\alpha+\beta)A]x, x\rangle \leq |\alpha\beta| \left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \times \begin{cases} \exp(\alpha M + \beta m) & \text{for } \alpha > 0, \beta < 0, \\ \exp(\alpha m + \beta M) & \text{for } \alpha < 0, \beta > 0 \end{cases} \tag{6.35}$$

for each $x \in H$ with $\|x\| = 1$.

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