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POSITIVE PERIODIC SOLUTIONS FOR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH GENERALIZED NEUTRAL OPERATOR

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ABSTRACT. By some analysis of the neutral operator $(Ax)(t) = x(t) - \sum_{i=1}^{n} c_i x(t - \delta_i)$ and an application of the fixed-point index theorem, we obtain sufficient conditions for the existence, multiplicity and nonexistence of periodic solutions to a second-order differential equation with the prescribed neutral operator, which improve and extend some recent results of Lu-Ge, Wu-Wang, and Zhang. An example is given to illustrate our results. Moreover, the analysis of the generalized neutral operator will be helpful for other types of differential equations.

Key words and phrases: Generalized neutral operator, Positive periodic solutions, Fixed-point.

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1. INTRODUCTION

Consider the following second-order neutral differential equation

$$\left(x(t) - \sum_{i=1}^{n} c_i x(t-\delta_i)\right)'' + a(t)x(t) = \lambda b(t) f(x(t-\tau(t))), \tag{1.1}$$

where λ is a positive parameter; c_i and δ_i, $i = 1, 2, \ldots, n$, $n \in \mathbb{Z}^+$, are constants with $\sum_{i=1}^{n} |c_i| \neq 1$; $f(x) \in C(\mathbb{R}, [0, \infty))$, and $f(x) > 0$ for $x > 0$; $a(t) \in C(\mathbb{R}, (0, \infty))$ with $\max\{a(t) : t \in [0, \omega]\} < (\frac{\pi}{\omega})^2$, $b(t) \in C(\mathbb{R}, (0, \infty))$, $\tau \in C(\mathbb{R}, \mathbb{R})$, $a(t)$, $b(t)$ and $\tau(t)$ are ω-periodic functions.

In recent years, the existence of positive periodic solutions for differential delay equations has attracted much attention, see [1, 3, 6, 8] and the references therein. However, compared with the ample results on the existence of positive periodic solutions for various types of first-order or second-order ordinary delay differential equations, studies on positive periodic solutions for neutral differential equations are relatively less. This is because the latter is much more intricate than the former. In [11], Zhang presents some results for the neutral operator $(\tilde{A}x)(t) = x(t) - cx(t-\delta)$ which become effective tools for research on differential equations with the prescribed neutral operator, see [4, 7, 9, 10], etc. Lu and Ge in [5] obtain some results on the generalized neutral operator $Ax(t) = x(t) - \sum_{i=1}^{n} c_i x(t-\delta_i)$, and using Mawhin's continuation theorem they proved the existence of periodic solutions for a differential delay equation. However, the results in [5] do not apply to the study of positive periodic solutions to the generalized neutral differential equations.

Motivated by this problem, we first analyze properties of the generalized neutral operator A which will be helpful for further study on differential equations with a generalized neutral operator, and then by an application of the fixed-point index theorem, we obtain sufficient conditions for the existence, multiplicity and nonexistence of positive periodic solutions to (1.1). An example is also given to illustrate our results. Our results improve and extend the works in [5, 9, 11].

2. ANALYSIS OF THE GENERALIZED NEUTRAL OPERATOR

Let $X = \{x(t) \in C(\mathbb{R}, \mathbb{R}) : x(t+\omega) = x(t), t \in \mathbb{R}\}$ with norm $\|x\| = \sup_{t\in[0,\omega]} |x(t)|$. Then $(X, \|\cdot\|)$ is a Banach space. A cone K in X is defined by $K = \{x \in X : x(t) \geq \alpha\|x\|\}$, where α is a fixed positive number. Moreover, define operators $A,\ A_1 : X \to X$ by

$$(Ax)(t) = x(t) - \sum_{i=1}^{n} c_i x(t-\delta_i), \qquad (A_1 x)(t) = \sum_{i=1}^{n} c_i x(t-\delta_i),$$

respectively, here c_i and δ_i are defined as in the previous section. We have

Lemma 2.1. *If $\sum_{i=1}^{n} |c_i| < 1$, then A has a continuous bounded inverse A^{-1} on X and*

$$\begin{aligned}[A^{-1}y](t) \\ = y(t) + \sum_{j\geq 1} \sum_{r_1=1}^{n} \sum_{r_2=1}^{n} \cdots \sum_{r_j=1}^{n} c_{r_1} c_{r_2} \cdots c_{r_j} y(t - \delta_{r_1} - \delta_{r_2} - \cdots - \delta_{r_j}), \text{ for all } y \in X.\end{aligned} \tag{2.1}$$

Proof. From the definition of $A_1 y$, we obtain

$$\begin{aligned}\|A_1\| &= \sup_{\|y\|=1} \|A_1 y\| \\ &= \sup_{\|y\|=1} \max_{t\in[0,\omega]} \left| \sum_{i=1}^{n} c_i y(t-\delta_i) \right| \\ &\le \sup_{\|y\|=1} \max_{t\in[0,\omega]} \sum_{i=1}^{n} |c_i||y(t-\delta_i)| \\ &\le \sum_{i=1}^{n} |c_i| < 1.\end{aligned}$$

Then, by the Neumann expansion of A^{-1}, i.e., $A^{-1} = (I - A_1)^{-1} = I + \sum_{j\ge 1} A_1^j$, we have

$$[A^{-1}y](t) = y(t) + \sum_{j\ge1} \sum_{r_1=1}^{n} \sum_{r_2=1}^{n} \cdots \sum_{r_j=1}^{n} c_{r_1} c_{r_2} \cdots c_{r_j} y(t - \delta_{r_1} - \delta_{r_2} - \cdots - \delta_{r_j}).$$

∎

We now define an operator $H : X \to X$ by

$$H(y(t)) = -\sum_{i=1}^{n} c_i (A^{-1}y)(t-\delta_i).$$

Lemma 2.2. *If $c_i < 0$ for all $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} |c_i| < \min\{1, \alpha\}$, we have for $y \in K$ that:*

(a) $$\frac{\alpha - \sum_{i=1}^{n} |c_i|}{1 - \left(\sum_{i=1}^{n} c_i\right)^2} \|y\| \le (A^{-1}y)(t) \le \frac{1}{1 - \sum_{i=1}^{n} |c_i|} \|y\|;$$

(b) $$\frac{\sum_{i=1}^{n} |c_i| \left(\alpha - \sum_{i=1}^{n} |c_i|\right)}{1 - \left(\sum_{i=1}^{n} c_i\right)^2} \|y\| \le H(y(t)) \le \frac{\sum_{i=1}^{n} |c_i|}{1 - \sum_{i=1}^{n} |c_i|} \|y\|.$$

Proof. (a). Since $c_i < 0$ for all $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} |c_i| < \min\{1, \alpha\}$, by Lemma 2.1, we have for $y \in K$ that

$$\begin{aligned}(A^{-1}y)(t) &= y(t) + \sum_{j\ge1} \sum_{r_1=1}^{n} \sum_{r_2=1}^{n} \cdots \sum_{r_j=1}^{n} c_{r_1} c_{r_2} \cdots c_{r_j} y(t - \delta_{r_1} - \delta_{r_2} - \cdots - \delta_{r_j}) \\ &= y(t) + \sum_{j=2(m+1)} \sum_{r_1=1}^{n} \sum_{r_2=1}^{n} \cdots \sum_{r_j=1}^{n} c_{r_1} c_{r_2} \cdots c_{r_j} y(t - \delta_{r_1} - \delta_{r_2} - \cdots - \delta_{r_j}) \\ &\quad - \sum_{j=2m+1} \left| \sum_{r_1=1}^{n} \sum_{r_2=1}^{n} \cdots \sum_{r_j=1}^{n} c_{r_1} c_{r_2} \cdots c_{r_j} y(t - \delta_{r_1} - \delta_{r_2} - \cdots - \delta_{r_j}) \right| \\ &= y(t) + \sum_{r_1=1}^{n} \sum_{r_2=1}^{n} c_{r_1} c_{r_2} y(t - \delta_{r_1} - \delta_{r_2}) + \cdots - \left| \sum_{r_1=1}^{n} c_{r_1} y(t - \delta_{r_1}) \right| \\ &\quad - \left| \sum_{r_1=1}^{n} \sum_{r_2=1}^{n} \sum_{r_3=1}^{n} c_{r_1} c_{r_2} c_{r_3} y(t - \delta_{r_1} - \delta_{r_2} - \delta_{r_3}) \right| - \cdots\end{aligned}$$

$$
\begin{aligned}
&\geq \alpha\|y\| + \sum_{r_1=1}^{n}\sum_{r_2=1}^{n} c_{r_1}c_{r_2}\alpha\|y\| + \cdots - \left|\sum_{r_1=1}^{n} c_{r_1}\right| \|y\| \\
&\quad - \left|\sum_{r_1=1}^{n}\sum_{r_2=1}^{n}\sum_{r_3=1}^{n} c_{r_1}c_{r_2}c_{r_3}\right| \|y\| - \cdots \\
&= \alpha\|y\| + \left(\sum_{i=1}^{n} c_i\right)^2 \alpha\|y\| + \cdots - \left|\sum_{i=1}^{n} c_i\right| \|y\| - \left|\sum_{i=1}^{n} c_i\right|^3 \|y\| - \cdots \\
&= \frac{\alpha - |\sum_{i=1}^{n} c_i|}{1-(\sum_{i=1}^{n} c_i)^2}\|y\| \\
&= \frac{\alpha - \sum_{i=1}^{n} |c_i|}{1-(\sum_{i=1}^{n} c_i)^2}\|y\|,
\end{aligned}
$$

where $m \in \mathbb{N}$. On the other hand, from [5], we can obtain the other inequality.

(b). It can be directly derived from the definition of $H(y(t))$ and part (a). ∎

Lemma 2.3. *If $c_i > 0$ for all $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} c_i < 1$, $\alpha < 1$, then for $y \in K$ we have:*

(a)
$$\frac{\alpha}{1-\sum_{i=1}^{n} c_i}\|y\| \leq (A^{-1}y)(t) \leq \frac{1}{1-\sum_{i=1}^{n} c_i}\|y\|;$$

(b)
$$\frac{\alpha\sum_{i=1}^{n} c_i}{1-\sum_{i=1}^{n} c_i}\|y\| \leq H(y(t)) \leq \frac{\sum_{i=1}^{n} c_i}{1-\sum_{i=1}^{n} c_i}\|y\|.$$

Proof. (a). Since $c_i > 0$ for all $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} c_i < 1$. By Lemma 2.1, we have for $y \in K$ that

$$
\begin{aligned}
(A^{-1}y)(t) &= y(t) + \sum_{j\geq 1}\sum_{r_1=1}^{n}\sum_{r_2=1}^{n}\cdots\sum_{r_j=1}^{n} c_{r_1}c_{r_2}\cdots c_{r_j} y(t-\delta_{r_1}-\delta_{r_2}-\cdots-\delta_{r_j}) \\
&= y(t) + \sum_{r_1=1}^{n} c_{r_1} y(t-\delta_{r_1}) + \sum_{r_1=1}^{n}\sum_{r_2=1}^{n} c_{r_1}c_{r_2} y(t-\delta_{r_1}-\delta_{r_2}) \\
&\quad + \sum_{r_1=1}^{n}\sum_{r_2=1}^{n}\sum_{r_3=1}^{n} c_{r_1}c_{r_2}c_{r_3} y(t-\delta_{r_1}-\delta_{r_2}-\delta_{r_3}) + \cdots \\
&\geq \alpha\|y\| + \sum_{r_1=1}^{n} c_{r_1}\alpha\|y\| + \sum_{r_1=1}^{n}\sum_{r_2=1}^{n} c_{r_1}c_{r_2}\alpha\|y\| \\
&\quad + \sum_{r_1=1}^{n}\sum_{r_2=1}^{n}\sum_{r_3=1}^{n} c_{r_1}c_{r_2}c_{r_3}\alpha\|y\| + \cdots \\
&= \alpha\|y\| + \sum_{i=1}^{n} c_i\alpha\|y\| + \left(\sum_{i=1}^{n} c_i\right)^2 \alpha\|y\| + \left(\sum_{i=1}^{n} c_i\right)^3 \alpha\|y\| + \cdots \\
&= \frac{\alpha}{1-\sum_{i=1}^{n} c_i}\|y\|.
\end{aligned}
$$

The proof of the remaining parts is similar to that for Lemma 2.2, and will be omitted. ∎

3. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR (1.1)

Define the Banach space X as in Section 2, and let $C_\omega^+ = \{x(t) \in C(\mathbb{R}, (0, +\infty)) : x(t + \omega) = x(t)\}$. Denote

$$M = \max\{a(t) : t \in [0, \omega]\},\ m = \min\{a(t) : t \in [0, \omega]\},\ \beta = \sqrt{M},$$

$$L = \frac{1}{2\beta \sin\frac{\beta\omega}{2}},\ l = \frac{\cos\frac{\beta\omega}{2}}{2\beta \sin\frac{\beta\omega}{2}},\ k = l(M + m) + LM,$$

$$k_1 = \frac{k - \sqrt{k^2 - 4LlMm}}{2LM},\ \alpha = \frac{l\left[m - (M + m)\sum_{i=1}^n |c_i|\right]}{LM\left(1 - \sum_{i=1}^n |c_i|\right)}.$$

It is easy to see that $M, m, \beta, L, l, k, k_1 > 0$.

Here, the cone K in X is defined by $K = \{x \in X : x(t) \geq \alpha\|x\|\}$ as in Section 2, where α is as defined above. Note that $K_r = \{x \in K : \|x\| < r\}$ and $\partial K_r = \{x \in K : \|x\| = r\}$.

Now we consider (1.1). First let

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{x},\ f_\infty = \lim_{x \to \infty} \frac{f(x)}{x},$$

and denote

$$i_0 = \text{number of zeros in the set } \{f_0, f_\infty\},\ i_\infty = \text{number of infinities in the set } \{f_0, f_\infty\}.$$

It is clear that $i_0, i_\infty = 0, 1, 2$. We will show that (1.1) has i_0 or i_∞ positive w-periodic solution(s) for sufficiently large or small λ, respectively.

In the following we discuss (1.1) in two cases, namely, the case where $c_i < 0$ for all $i = 1, 2, \ldots, n$, and $\sum_{i=1}^n c_i > -\min\left\{k_1, \frac{m}{M+m}\right\}$ $\left[\sum_{i=1}^n c_i > -\frac{m}{M+m}\right.$ assures $\alpha > 0$; when $\sum_{i=1}^n c_i > -k_1$, then $\left.\sum_{i=1}^n |c_i| < \alpha\right]$; and the case where $c_i < 0$ for all $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n c_i < \min\left\{\frac{m}{M+m}, \frac{LM-lm}{(L-l)M-lm}\right\}$ $\left[\text{when } \sum_{i=1}^n c_i < \frac{m}{M+m}\right.$, then $\alpha > 0$; when $\sum_{i=1}^n c_i < \frac{LM-lm}{(L-l)M-lm}$, then $\left.\alpha < 1\right]$. Obviously, we have $\sum_{i=1}^n |c_i| < 1$ which makes Lemma 2.1 hold for both cases, and Lemma 2.2 or 2.3 also hold, respectively.

Let $y(t) = (Ax)(t)$, then from Lemma 2.1 we have $x(t) = (A^{-1}y)(t)$. Hence (1.1) can be transformed into

$$y''(t) + a(t)(A^{-1}y)(t) = \lambda b(t) f((A^{-1}y)(t - \tau(t))), \tag{3.1}$$

which can be further rewritten as

$$y''(t) + a(t)y(t) - a(t)H(y(t)) = \lambda b(t) f((A^{-1}y)(t - \tau(t))), \tag{3.2}$$

where

$$H(y(t)) = y(t) - (A^{-1}y)(t) = -\sum_{i=1}^n c_i (A^{-1}y)(t - \delta_i)$$

is defined as Section 2.

Now we discuss the two cases separately.

3.1. **Case I.** $c_i < 0$ for all $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} c_i > -\min\left\{k_1, \frac{m}{M+m}\right\}$.
Denote

$$F(r) = \max\left\{ f(t) : 0 \le t \le \frac{r}{1 - \sum\limits_{i=1}^{n} |c_i|} \right\},$$

$$f_1(r) = \min\left\{ f(t) : \frac{\alpha - \sum\limits_{i=1}^{n} |c_i|}{1 - (\sum\limits_{i=1}^{n} c_i)^2} r \le t \le \frac{r}{1 - \sum\limits_{i=1}^{n} |c_i|} \right\}.$$

Lemma 3.1. *The equation*

$$y''(t) + My(t) = h(t),\ h \in C_\omega^+, \tag{3.3}$$

has a unique positive ω-periodic solution

$$y(t) = \int_t^{t+\omega} G(t,s)h(s)\mathrm{d}s, \tag{3.4}$$

where

$$G(t,s) = \frac{\cos\beta\left(\frac{\omega}{2} + t - s\right)}{2\beta\sin\frac{\beta\omega}{2}}, \quad s \in [t, t+\omega]. \tag{3.5}$$

Remark 3.1. The conclusion has been presented in [9] without a proof. For the convenience of readers, we give the details here.

Proof. First it is easy to see that the associate homogeneous equation of (3.3) has the solution $y(t) = c_1 \cos\beta t + c_2 \sin\beta t$. Applying the method of variation of parameters, we get

$$c_1'(t) = \frac{-\sin\beta t}{2\beta} h(t),\ c_2'(t) = \frac{\cos\beta t}{2\beta} h(t).$$

Noticing that $y(t),\ y'(t)$ are periodic functions, we have

$$c_1(t) = \int_t^{t+\omega} \frac{h(s)\cos\left(s - \frac{\omega}{2}\right)}{2\beta\sin\frac{\beta\omega}{2}}\mathrm{d}s,\ \ c_2(t) = \int_t^{t+\omega} \frac{h(s)\sin\left(s - \frac{\omega}{2}\right)}{2\beta\sin\frac{\beta\omega}{2}}\mathrm{d}s.$$

Therefore

$$\begin{aligned} y(t) &= c_1(t)\cos\beta t + c_2(t)\sin\beta t \\ &= \int_t^{t+\omega} G(t,s)h(s)\mathrm{d}s, \end{aligned}$$

where $G(t,s)$ is as defined in (3.5). ∎

Lemma 3.2 ([9])**.** *We have $\int_t^{t+\omega} G(t,s)\mathrm{d}s = \frac{1}{M}$. Furthermore, if $\max\{a(t) : t \in [0,\omega]\} < (\frac{\pi}{\omega})^2$, then $0 < l \le G(t,s) \le L$ for all $t \in [0,\omega]$ and $s \in [t, t+\omega]$.*

Proof. For the proof, readers are referred to [9]. ∎

Now we study the following equation corresponding to (3.2),

$$y''(t) + a(t)y(t) - a(t)H(y(t)) = h(t),\ h \in C_\omega^+. \tag{3.6}$$

We define the operators $T,\ B : X \to X$ by

$$(Th)(t) = \int_t^{t+\omega} G(t,s)h(s)\mathrm{d}s,\ (By)(t) = (M - a(t))y(t) + a(t)H(y(t)). \tag{3.7}$$

Clearly T, B are completely continuous, $(Th)(t) > 0$ for $h(t) > 0$ and

$$\|B\| \le \left(M - m + M \frac{\sum_{i=1}^n |c_i|}{1 - \sum_{i=1}^n |c_i|} \right).$$

By Lemma 3.1, the solution of (3.6) can be written in the form

$$y(t) = (Th)(t) + (TBy)(t). \tag{3.8}$$

In view of $c_i < 0$ for all $i = 1, 2, \dots, n$, and $\sum_{i=1}^n c_i > -\min\left\{k_1, \frac{m}{M+m}\right\}$, we have

$$\|TB\| \le \|T\|\|B\| \le \frac{M - m + m\sum_{i=1}^n |c_i|}{M\left(1 - \sum_{i=1}^n |c_i|\right)} < 1,$$

and so

$$y(t) = (I - TB)^{-1}(Th)(t). \tag{3.9}$$

We define an operator $P : X \to X$ by

$$(Ph)(t) = (I - TB)^{-1}(Th)(t). \tag{3.10}$$

Obviously, for any $h \in C_\omega^+$, if $\max\{a(t) : t \in [0, \omega]\} < (\frac{\pi}{\omega})^2$, $y(t) = (Ph)(t)$ is the unique positive ω-periodic solution of (3.6).

Lemma 3.3. *P is completely continuous and*

$$(Th)(t) \le (Ph)(t) \le \frac{M\left(1 - \sum_{i=1}^n |c_i|\right)}{m - (M+m)\sum_{i=1}^n |c_i|}\|Th\|, \text{ for all } h \in C_\omega^+. \tag{3.11}$$

Proof. By Neumann expansions of P, we have

$$\begin{aligned} P &= (I - TB)^{-1}T \\ &= (I + TB + (TB)^2 + \cdots + (TB)^n + \cdots)T \\ &= T + TBT + (TB)^2T + \cdots + (TB)^nT + \cdots . \end{aligned} \tag{3.12}$$

Since T and B are completely continuous, so is P. Moreover, by (3.12), and recalling that

$$\|TB\| \le \frac{M - m + m\sum_{i=1}^n |c_i|}{M\left(1 - \sum_{i=1}^n |c_i|\right)} < 1,$$

we get

$$(Th)(t) \le (Ph)(t) \le \frac{M\left(1 - \sum_{i=1}^n |c_i|\right)}{m - (M+m)\sum_{i=1}^n |c_i|}\|Th\|.$$

∎

We define an operator $Q : X \to X$ by

$$Qy(t) = P(\lambda b(t)f((A^{-1}y)(t - \tau(t)))). \tag{3.13}$$

Lemma 3.4. $Q(K) \subset K$.

Proof. From the definition of Q, it is easy to verify that $Qy(t+\omega) = Qy(t)$. For $y \in K$, we have from Lemma 3.2 that

$$\begin{aligned} Qy(t) &= P(\lambda b(t)f((A^{-1}y)(t - \tau(t)))) \\ &\ge T(\lambda b(t)f((A^{-1}y)(t - \tau(t)))) \\ &= \lambda \int_t^{t+\omega} G(t,s)b(s)f[(A^{-1}y)(s - \tau(s))]\mathrm{d}s \\ &\ge \lambda l \int_0^\omega b(s)f[(A^{-1}y)(s - \tau(s))]\mathrm{d}s. \end{aligned}$$

On the other hand,

$$
\begin{aligned}
Qy(t) &= P(\lambda b(t) f((A^{-1}y)(t-\tau(t)))) \\
&\leq \frac{M\left(1-\sum_{i=1}^{n}|c_i|\right)}{m-(M+m)\sum_{i=1}^{n}|c_i|}\|T(\lambda b(t) f((A^{-1}y)(t-\tau(t))))\| \\
&= \lambda \frac{M\left(1-\sum_{i=1}^{n}|c_i|\right)}{m-(M+m)\sum_{i=1}^{n}|c_i|}\max_{t\in[0,\omega]}\int_t^{t+\omega} G(t,s)b(s)f((A^{-1}y)(s-\tau(s)))\mathrm{d}s \\
&\leq \lambda \frac{M\left(1-\sum_{i=1}^{n}|c_i|\right)}{m-(M+m)\sum_{i=1}^{n}|c_i|} L\int_0^{\omega} b(s)f((A^{-1}y)(s-\tau(s)))\mathrm{d}s.
\end{aligned}
$$

Therefore

$$
Qy(t) \geq \frac{l\left[m-(M+m)\sum_{i=1}^{n}|c_i|\right]}{LM\left(1-\sum_{i=1}^{n}|c_i|\right)}\|Qy\| = \alpha\|Qy\|,
$$

i.e., $Q(K)\subset K$. ∎

From the continuity of P, it is easy to verify that Q is completely continuous in X. Comparing (3.2) with (3.6), it is obvious that the existence of periodic solutions for equation (3.2) is equivalent to the existence of fixed-points for the operator Q on X. Recalling Lemma 3.4, the existence of positive periodic solutions for (3.2) is equivalent to the existence of fixed-points of Q on K. Furthermore, if Q has a fixed-point y in K, it means that $(A^{-1}y)(t)$ is a positive $\omega-$periodic solution of (1.1).

Lemma 3.5. *If there exists $\eta > 0$ such that*

$$
f((A^{-1}y)(t-\tau(t))) \geq (A^{-1}y)(t-\tau(t))\eta, \text{ for } t\in[0,\omega] \text{ and } y\in K,
$$

then

$$
\|Qy\| \geq \lambda l\eta \frac{\alpha-\sum_{i=1}^{n}|c_i|}{1-\left(\sum_{i=1}^{n}c_i\right)^2}\int_0^{\omega} b(s)\mathrm{d}s\|y\|,\ y\in K.
$$

Proof. By the assumption, we have for $y\in K$ that

$$
\begin{aligned}
Qy(t) &= P\left(b(t)f((A^{-1}y)(t-\tau(t)))\right) \\
&\geq T\left(b(t)f((A^{-1}y)(t-\tau(t)))\right) \\
&= \lambda\int_t^{t+\omega} G(t,s)b(s)f((A^{-1}y)(s-\tau(s)))\mathrm{d}s \\
&\geq \lambda l\eta\int_0^{\omega} b(s)(A^{-1}y)(s-\tau(s))\mathrm{d}s \\
&\geq \lambda l\eta \frac{\alpha-\sum_{i=1}^{n}|c_i|}{1-\left(\sum_{i=1}^{n}c_i\right)^2}\int_0^{\omega} b(s)\mathrm{d}s\|y\|.
\end{aligned}
$$

Hence

$$
\|Qy\| \geq \lambda l\eta \frac{\alpha-\sum_{i=1}^{n}|c_i|}{1-\left(\sum_{i=1}^{n}c_i\right)^2}\int_0^{\omega} b(s)\mathrm{d}s\|y\|,\ y\in K.
$$

∎

Lemma 3.6. *If there exists $\varepsilon > 0$ such that*

$$
f((A^{-1}y)(t-\tau(t))) \leq (A^{-1}y)(t-\tau(t))\varepsilon, \text{ for } t\in[0,\omega] \text{ and } y\in K,
$$

then

$$
\|Qy\| \leq \lambda\varepsilon \frac{LM\int_0^{\omega} b(s)\mathrm{d}s}{m-(M+m)\sum_{i=1}^{n}|c_i|}\|y\|,\ y\in K.
$$

Proof. By Lemma 2.2, Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned}\|Qy(t)\| &\leq \lambda \frac{M\,(1-\sum_{i=1}^{n}|c_i|)}{m-(M+m)\sum_{i=1}^{n}|c_i|} L\int_0^{\omega} b(s)f((A^{-1}y)(s-\tau(s)))\mathrm{d}s \\ &\leq \lambda \frac{M\,(1-\sum_{i=1}^{n}|c_i|)}{m-(M+m)\sum_{i=1}^{n}|c_i|} L\varepsilon\int_0^{\omega} b(s)(A^{-1}y)(s-\tau(s))\mathrm{d}s \\ &\leq \lambda\varepsilon \frac{LM\int_0^{\omega} b(s)\mathrm{d}s}{m-(M+m)\sum_{i=1}^{n}|c_i|}\|y\|.\end{aligned}$$

∎

Lemma 3.7. *If $y \in \partial K_r$, then*

$$\|Qy\| \geq \lambda l f_1(r)\int_0^{\omega} b(s)\mathrm{d}s.$$

Proof. By Lemma 2.2, we obtain

$$\frac{\alpha-|\sum_{i=1}^{n}c_i|}{1-(\sum_{i=1}^{n}c_i)^2}r \leq (A^{-1}y)(t-\tau(t)) \leq \frac{r}{1-\sum_{i=1}^{n}|c_i|}$$

for $y \in \partial K_r$, which yields $f((A^{-1}y)(t-\tau(t))) \geq f_1(r)$. The lemma now follows by imitating the proof of Lemma 3.5. ∎

Lemma 3.8. *If $y \in \partial K_r$, then*

$$\|Qy\| \leq \lambda \frac{LM\,(1-\sum_{i=1}^{n}|c_i|)\,F(r)}{m-(M+m)\sum_{i=1}^{n}|c_i|}\int_0^{\omega} b(s)\mathrm{d}s.$$

Proof. By Lemma 2.2, we have

$$0 \leq (A^{-1}y)(t-\tau(t)) \leq \frac{r}{1-\sum_{i=1}^{n}|c_i|}$$

for $y \in \partial K_r$, which yields $f((A^{-1}y)(t-\tau(t))) \leq F(r)$. Using a process similar to the proof of Lemma 3.6, we obtain the conclusion. ∎

We now quote the fixed point theorem which our results will be based on.

Lemma 3.9 ([2]). *Let X be a Banach space and K a cone in X. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|u\| = r\}$.*

(i) *If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$;*
(ii) *If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.*

Now we give our main results on positive periodic solutions for (1.1).

Theorem 3.10.

(a) *If $i_0 = 1$ or 2, then (1.1) has i_0 positive ω-periodic solution(s) for*

$$\lambda > \frac{1}{f_1(1)l\int_0^{\omega} b(s)\mathrm{d}s} > 0;$$

(b) *If $i_\infty = 1$ or 2, then (1.1) has i_∞ positive ω-periodic solution(s) for*

$$0 < \lambda < \frac{m-(M+m)\sum_{i=1}^{n}|c_i|}{LM\,(1-\sum_{i=1}^{n}|c_i|)\,F(1)\int_0^{\omega} b(s)\mathrm{d}s};$$

(c) *If $i_\infty = 0$ or $i_0 = 0$, then (1.1) has no positive ω-periodic solution(s) for sufficiently small or sufficiently large $\lambda > 0$, respectively.*

Proof. (a). Choose $r_1 = 1$. Take

$$\lambda_0 = \frac{1}{f_1(r_1) l \int_0^\omega b(s)\mathrm{d}s},$$

then for all $\lambda > \lambda_0$, we have from Lemma 3.7 that

$$\|Qy\| > \|y\|, \text{ for } y \in \partial K_{r_1}. \tag{3.14}$$

Case 1. If $f_0 = 0$, we can choose $0 < \bar{r}_2 < r_1$, so that $f(u) \le \varepsilon u$ for $0 \le u \le \bar{r}_2$, where the constant $\varepsilon > 0$ satisfies

$$\lambda\varepsilon \frac{LM \int_0^\omega b(s)\mathrm{d}s}{m - (M+m)\sum_{i=1}^n |c_i|} < 1. \tag{3.15}$$

Let $r_2 = \left(1 - \sum_{i=1}^n |c_i|\right) \bar{r}_2$. By Lemma 2.2, we have

$$0 \le (A^{-1}y)(t - \tau(t)) \le \frac{\|y\|}{1 - \sum_{i=1}^n |c_i|} \le \bar{r}_2$$

for $y \in \partial K_{r_2}$, which yields

$$f((A^{-1}y)(t - \tau(t))) \le \varepsilon (A^{-1}y)(t - \tau(t)).$$

In view of Lemma 3.6 and (3.15), we have for $y \in \partial K_{r_2}$ that

$$\|Qy\| \le \lambda\varepsilon \frac{LM \int_0^\omega b(s)\mathrm{d}s}{m - (M+m)\sum_{i=1}^n |c_i|} \|y\| < \|y\|.$$

It follows from Lemma 3.9 and (3.14) that

$$i(Q, K_{r_2}, K) = 1, \; i(Q, K_{r_1}, K) = 0,$$

thus $i(Q, K_{r_1} \backslash \bar{K}_{r_2}, K) = -1$ and Q has a fixed point y in $K_{r_1} \backslash \bar{K}_{r_2}$, which means that $(A^{-1}y)(t)$ is a positive ω-positive solution of (1.1) for $\lambda > \lambda_0$.

Case 2. If $f_\infty = 0$, there exists a constant $\tilde{H} > 0$ such that $f(u) \le \varepsilon u$ for $u \ge \tilde{H}$, where the constant $\varepsilon > 0$ satisfies

$$\lambda\varepsilon \frac{LM \int_0^\omega b(s)\mathrm{d}s}{m - (M+m)\sum_{i=1}^n |c_i|} < 1. \tag{3.16}$$

Let

$$r_3 = \max\left\{ 2r_1, \frac{\tilde{H}\left[1 - \left(\sum_{i=1}^n c_i\right)^2\right]}{\alpha - \sum_{i=1}^n |c_i|} \right\}.$$

Since

$$(A^{-1}y)(t - \tau(t)) \ge \frac{\alpha - \left|\sum_{i=1}^n c_i\right|}{1 - \left(\sum_{i=1}^n c_i\right)^2} \|y\| \ge \tilde{H}$$

for $y \in \partial K_{r_3}$, we obtain

$$f((A^{-1}y)(t - \tau(t))) \le \varepsilon (A^{-1}y)(t - \tau(t)).$$

Thus by Lemma 3.6 and (3.16), we have for $y \in \partial K_{r_3}$ that

$$\|Qy\| \le \lambda\varepsilon \frac{LM \int_0^\omega b(s)\mathrm{d}s}{m - (M+m)\sum_{i=1}^n |c_i|} \|y\| < \|y\|.$$

Recalling from Lemma 3.9 and (3.14) that

$$i(Q, K_{r_3}, K) = 1, \; i(Q, K_{r_1}, K) = 0,$$

then $i(Q, K_{r_3}\backslash\bar{K}_{r_1}, K) = 1$ and Q has a fixed point y in $K_{r_3}\backslash\bar{K}_{r_1}$, which means that $(A^{-1}y)(t)$ is a positive ω-positive solution of (1.1) for $\lambda > \lambda_0$.

Case 3. If $f_0 = f_\infty = 0$, from the above arguments, there exist $0 < r_2 < r_1 < r_3$ such that Q has a fixed point $y_1(t)$ in $K_{r_1}\backslash\bar{K}_{r_2}$ and a fixed point $y_2(t)$ in $K_{r_3}\backslash\bar{K}_{r_1}$. Consequently, $(A^{-1}y_1)(t)$ and $(A^{-1}y_2)(t)$ are two positive ω-periodic solutions of (1.1) for $\lambda > \lambda_0$.

(b). Let $r_1 = 1$. Take

$$\lambda_0 = \frac{m - (M+m)\sum_{i=1}^n |c_i|}{LM\left(1 - \sum_{i=1}^n |c_i|\right) F(r_1) \int_0^\omega b(s)\mathrm{d}s},$$

then by Lemma 3.8, we know that if $\lambda < \lambda_0$ then

$$\|Qy\| < \|y\|,\ y \in \partial K_{r_1}. \tag{3.17}$$

Case 1. If $f_0 = \infty$, we can choose $0 < \bar{r}_2 < r_1$ so that $f(u) \geq \eta u$ for $0 \leq u \leq \bar{r}_2$, where the constant $\eta > 0$ satisfies

$$\lambda l \eta \frac{\alpha - \sum_{i=1}^n |c_i|}{1 - \left(\sum_{i=1}^n c_i\right)^2} \int_0^\omega b(s)\mathrm{d}s > 1. \tag{3.18}$$

Let $r_2 = \left(1 - \sum_{i=1}^n |c_i|\right)\bar{r}_2$. Since

$$0 \leq (A^{-1}y)(t - \tau(t)) \leq \frac{\|y\|}{1 - \sum_{i=1}^n |c_i|} \leq \bar{r}_2$$

for $y \in \partial K_{r_2}$, we obtain

$$f((A^{-1}y)(t - \tau(t))) \geq \eta (A^{-1}y)(t - \tau(t)).$$

Thus by Lemma 3.5 and (3.18),

$$\|Qy\| \geq \lambda l \eta \frac{\alpha - \sum_{i=1}^n |c_i|}{1 - \left(\sum_{i=1}^n c_i\right)^2} \int_0^\omega b(s)\mathrm{d}s \|y\| > \|y\|,\ y \in \partial K_{r_2}.$$

It follows from Lemma 3.9 and (3.17) that

$$i(Q, K_{r_2}, K) = 0,\ i(Q, K_{r_1}, K) = 1,$$

which implies that $i(Q, K_{r_1}\backslash\bar{K}_{r_2}, K) = 1$ and Q has a fixed point y in $K_{r_1}\backslash\bar{\Omega}_{r_2}$. Therefore $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.1) for $0 < \lambda < \lambda_0$.

Case 2. If $f_\infty = \infty$, there exists a constant $\tilde{H} > 0$ such that $f(u) \geq \eta u$ for $u \geq \tilde{H}$, where the constant $\eta > 0$ satisfies

$$\lambda l \eta \frac{\alpha - \sum_{i=1}^n |c_i|}{1 - \left(\sum_{i=1}^n c_i\right)^2} \int_0^\omega b(s)\mathrm{d}s > 1. \tag{3.19}$$

Let

$$r_3 = \max\left\{2r_1, \frac{\tilde{H}\left[1 - \left(\sum_{i=1}^n c_i\right)^2\right]}{\alpha - \sum_{i=1}^n |c_i|}\right\}.$$

By Lemma 2.2, we have

$$(A^{-1}y)(t - \tau(t)) \geq \frac{\alpha - \sum_{i=1}^n |c_i|}{1 - \left(\sum_{i=1}^n c_i\right)^2}\|y\| \geq \tilde{H}$$

for $y \in \partial K_{r_3}$ and then

$$f((A^{-1}y)(t - \tau(t))) \geq \eta (A^{-1}y)(t - \tau(t)).$$

Thus by Lemma 3.5 and (3.19), we have for $y \in \partial K_{r_3}$ that

$$\|Qy\| \geq \lambda l \eta \frac{\alpha - \sum_{i=1}^n |c_i|}{1 - \left(\sum_{i=1}^n c_i\right)^2} \int_0^\omega b(s)\mathrm{d}s \|y\| > \|y\|.$$

It follows from Lemma 3.9 and (3.17) that

$$i(Q, K_{r_3}, K) = 0, \ i(Q, K_{r_1}, K) = 1,$$

i.e., $i(Q, K_{r_3} \backslash \bar{K}_{r_1}, K) = -1$ and Q has a fixed point y in $K_{r_3} \backslash \bar{K}_{r_1}$. This means that $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.1) for $0 < \lambda < \lambda_0$.

Case 3. If $f_0 = f_\infty = 0$, from the above arguments, Q has a fixed point y_1 in $K_{r_1} \backslash \bar{K}_{r_2}$ and a fixed point y_2 in $K_{r_3} \backslash \bar{K}_{r_1}$. Consequently, $(A^{-1}y_1)(t)$ and $(A^{-1}y_2)(t)$ are two positive ω-periodic solutions of (1.1) for $0 < \lambda < \lambda_0$.

(c). By Lemma 2.2, if $y \in K$, then

$$(A^{-1}y)(t - \tau(t)) \geq \frac{\alpha - \sum_{i=1}^n |c_i|}{1 - \left(\sum_{i=1}^n c_i\right)^2} \|y\| > 0$$

for $t \in [0, \omega]$.

Case 1. If $i_0 = 0$, we have $f_0 > 0$ and $f_\infty > 0$. Let $b_1 = \min\left\{\frac{f(u)}{u}; u > 0\right\} > 0$. Then we obtain

$$f(u) \geq b_1 u, \ u \in [0, +\infty).$$

Assume that $y(t)$ is a positive ω-periodic solution of (1.1) for $\lambda > \lambda_0$, where

$$\lambda_0 = \frac{1 - \left(\sum_{i=1}^n c_i\right)^2}{l b_1 \left(\alpha - \sum_{i=1}^n |c_i|\right) \int_0^\omega b(s)\mathrm{d}s}.$$

Since $Qy(t) = y(t)$ for $t \in [0, \omega]$, then by Lemma 3.5, if $\lambda > \lambda_0$, we have

$$\|y\| = \|Qy\| \geq \lambda l b_1 \frac{\alpha - \sum_{i=1}^n |c_i|}{1 - \left(\sum_{i=1}^n c_i\right)^2} \int_0^\omega b(s)\mathrm{d}s \|y\| > \|y\|,$$

which is a contradiction.

Case 2. If $i_\infty = 0$, we have $f_0 < \infty$ and $f_\infty < \infty$. Let $b_2 = \max\left\{\frac{f(u)}{u}; u > 0\right\} > 0$. Then we obtain

$$f(u) \leq b_2 u, \ u \in [0, \infty).$$

Assume that $y(t)$ is a positive ω-periodic solution of (1.1) for $0 < \lambda < \lambda_0$, where

$$\lambda_0 = \frac{m - (M + m)\sum_{i=1}^n |c_i|}{b_2 L M \int_0^\omega b(s)\mathrm{d}s}.$$

Since $Qy(t) = y(t)$ for $t \in [0, \omega]$, it follows from Lemma 3.6 that

$$\|y\| = \|Qy\| \leq \lambda b_2 \frac{LM \int_0^\omega b(s)\mathrm{d}s}{m - (M + m)\sum_{i=1}^n |c_i|} \|y\| < \|y\|,$$

which is a contradiction. ∎

Theorem 3.11.

(a) *If there exists a constant $b_1 > 0$ such that $f(u) \geq b_1 u$ for $u \in [0, +\infty)$, then (1.1) has no positive ω-periodic solution for*

$$\lambda > \frac{1 - \left(\sum_{i=1}^n c_i\right)^2}{l b_1 \left(\alpha - \sum_{i=1}^n |c_i|\right) \int_0^\omega b(s)\mathrm{d}s}.$$

(b) *If there exists a constant* $b_2 > 0$ *such that* $f(u) \le b_2 u$ *for* $u \in [0, +\infty)$*, then (1.1) has no positive* ω*-periodic solution for*

$$0 < \lambda < \frac{m - (M+m)\sum_{i=1}^n |c_i|}{b_2 LM \int_0^\omega b(s)\mathrm{d}s}.$$

Proof. From the proof of (c) in Theorem 3.10, we immediately obtain this theorem. ∎

Theorem 3.12. *If*

$$\frac{1 - \left(\sum_{i=1}^n c_i\right)^2}{l\left(\alpha - \sum_{i=1}^n |c_i|\right)\int_0^\omega b(s)\mathrm{d}s \max\{f_0, f_\infty\}} < \lambda < \frac{m - (M+m)\sum_{i=1}^n |c_i|}{LM\int_0^\omega b(s)\mathrm{d}s \min\{f_0, f_\infty\}},$$

then (1.1) has one positive ω*-periodic solution.*

Proof. **Case 1**. If $f_0 \le f_\infty$, then

$$\frac{1 - \left(\sum_{i=1}^n c_i\right)^2}{f_\infty l\left(\alpha - \sum_{i=1}^n |c_i|\right)\int_0^\omega b(s)\mathrm{d}s} < \lambda < \frac{m - (M+m)\sum_{i=1}^n |c_i|}{f_0 LM\int_0^\omega b(s)\mathrm{d}s}.$$

It is easy to see that there exists an $0 < \varepsilon < f_\infty$ such that

$$\frac{1 - \left(\sum_{i=1}^n c_i\right)^2}{(f_\infty - \varepsilon) l\left(\alpha - \sum_{i=1}^n |c_i|\right)\int_0^\omega b(s)\mathrm{d}s} < \lambda < \frac{m - (M+m)\sum_{i=1}^n |c_i|}{(f_0 + \varepsilon) LM\int_0^\omega b(s)\mathrm{d}s}.$$

For the above ε, we choose $\bar{r}_1 > 0$ such that $f(u) \le (f_0 + \varepsilon)u$ for $0 \le u \le \bar{r}_1$. Let $r_1 = \left(1 - \sum_{i=1}^n |c_i|\right)\bar{r}_1$. By Lemma 2.2, we have

$$0 \le (A^{-1}y)(t - \tau(t)) \le \frac{\|y\|}{1 - \sum_{i=1}^n |c_i|} \le \bar{r}_1,$$

and then

$$f((A^{-1}y)(t - \tau(t))) \le (f_0 + \varepsilon)(A^{-1}y)(t - \tau(t)).$$

Thus by Lemma 3.6 we have for $y \in \partial K_{r_1}$ that

$$\|Qy\| \le \lambda(f_0 + \varepsilon)\frac{LM\int_0^\omega b(s)\mathrm{d}s}{m - (M+m)\sum_{i=1}^n |c_i|}\|y\| < \|y\|.$$

On the other hand, there exists a constant $\tilde{H} > 0$ such that $f(u) \ge (f_\infty - \varepsilon)u$ for $u \ge \tilde{H}$. Let

$$r_2 = \max\left\{2r_1, \frac{\tilde{H}\left[1 - \left(\sum_{i=1}^n c_i\right)^2\right]}{\alpha - \sum_{i=1}^n |c_i|}\right\}.$$

By Lemma 2.2, we have

$$(A^{-1}y)(t - \tau(t)) \ge \frac{\alpha - \left|\sum_{i=1}^n c_i\right|}{1 - \left(\sum_{i=1}^n c_i\right)^2}\|y\| \ge \tilde{H}$$

for $y \in \partial K_{r_2}$ and then

$$f((A^{-1}y)(t - \tau(t))) \ge (f_\infty - \varepsilon)(A^{-1}y)(t - \tau(t)).$$

Thus by Lemma 3.5, for $y \in \partial K_{r_2}$

$$\|Qy\| \ge \lambda l(f_\infty - \varepsilon)\frac{\alpha - \sum_{i=1}^n |c_i|}{1 - \left(\sum_{i=1}^n c_i\right)^2}\int_0^\omega b(s)\mathrm{d}s\|y\| > \|y\|.$$

It follows from Lemma 3.9 that

$$i(Q, K_{r_1}, K) = 1,\ i(Q, K_{r_2}, K) = 0,$$

thus $i(Q, K_{r_2}\backslash\bar{K}_{r_1}, K) = -1$ and Q has a fixed point y in $K_{r_2}\backslash\bar{K}_{r_1}$. So $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.1).

Case 2. If $f_0 > f_\infty$, in this case, we have

$$\frac{1-\left(\sum_{i=1}^n c_i\right)^2}{f_0 l(\alpha - \sum_{i=1}^n |c_i|)\int_0^\omega b(s)\mathrm{d}s} < \lambda < \frac{m-(M+m)\sum_{i=1}^n |c_i|}{f_\infty LM\int_0^\omega b(s)\mathrm{d}s}.$$

It is easy to see that there exists an $0 < \varepsilon < f_0$ such that

$$\frac{1-\left(\sum_{i=1}^n c_i\right)^2}{(f_0-\varepsilon) l(\alpha - \sum_{i=1}^n |c_i|)\int_0^\omega b(s)\mathrm{d}s} < \lambda < \frac{m-(M+m)\sum_{i=1}^n |c_i|}{(f_\infty+\varepsilon) LM\int_0^\omega b(s)\mathrm{d}s}.$$

For the above ε, we choose $\bar{r}_1 > 0$ such that $f(u) \geq (f_0-\varepsilon)u$ for $0 \leq u \leq \bar{r}_1$. Let $r_1 = (1-\sum_{i=1}^n |c_i|)\,\bar{r}_1$. By Lemma 2.2 we have

$$0 \leq (A^{-1}y)(t-\tau(t)) \leq \frac{\|y\|}{1-\sum_{i=1}^n |c_i|} \leq \bar{r}_1$$

for $y \in \partial K_{r_1}$ and then

$$f((A^{-1}y)(t-\tau(t))) \geq (f_0-\varepsilon)(A^{-1}y)(t-\tau(t)).$$

Thus we have by Lemma 3.5 that for $y \in \partial K_{r_1}$

$$\|Qy\| \geq \lambda l(f_0-\varepsilon)\frac{\alpha-\sum_{i=1}^n |c_i|}{1-\left(\sum_{i=1}^n c_i\right)^2}\int_0^\omega b(s)\mathrm{d}s\|y\| > \|y\|.$$

On the other hand, there exists a constant $\tilde{H} > 0$ such that $f(u) \leq (f_\infty+\varepsilon)u$ for $u \geq \tilde{H}$. Let

$$r_2 = \max\left\{2r_1, \frac{\tilde{H}\left[1-\left(\sum_{i=1}^n c_i\right)^2\right]}{\alpha-\sum_{i=1}^n |c_i|}\right\}.$$

By Lemma 2.2 we have

$$(A^{-1}y)(t-\tau(t)) \geq \frac{\alpha-\sum_{i=1}^n |c_i|}{1-\left(\sum_{i=1}^n c_i\right)^2}\|y\| \geq \tilde{H}$$

for $y \in \partial K_{r_2}$ and then

$$f((A^{-1}y)(t-\tau(t))) \leq (f_\infty+\varepsilon)(A^{-1}y)(t-\tau(t)).$$

Thus by Lemma 3.6, for $y \in \partial K_{r_2}$,

$$\|Qy\| \leq \lambda(f_\infty+\varepsilon)\frac{LM\int_0^\omega b(s)\mathrm{d}s}{m-(M+m)\sum_{i=1}^n |c_i|}\|y\|.$$

It follows from Lemma 3.9 that

$$i(Q, K_{r_1}, K) = 0 \; i(Q, K_{r_2}, K) = 1.$$

Thus $i(Q, K_{r_2}\backslash\bar{K}_{r_1}, K) = 1$ and Q has a fixed point y in $K_{r_2}\backslash\bar{K}_{r_1}$. This means that $(A^{-1}y)(t)$ is a positive ω-periodic solution of (1.1). ∎

Remark 3.2. When $n = 1$, (1.1) degenerates to

$$(x(t)-cx(t-\delta))'' + a(t)x(t) = \lambda b(t)f(x(t-\tau(t))),$$

and Theorems 3.10 – 3.12 still hold.

3.2. **Case II.** $c_i > 0$ for all $i = 1, 2, \dots, n$ and $\sum_{i=1}^{n} c_i < \min\left\{\frac{m}{M+m}, \frac{LM-lm}{(L-l)M-lm}\right\}$.

In this case, obviously we have $\alpha < 1$. We denote

$$f_2(r) = \min\left\{f(t) : \frac{\alpha}{1-\sum_{i=1}^{n} c_i} r \le t \le \frac{r}{1-\sum_{i=1}^{n} c_i}\right\}.$$

In a similar manner to Subsection 3.1, we obtain the following results.

Theorem 3.13.

(a) *If $i_0 = 1$ or 2, then (1.1) has i_0 positive ω-periodic solution(s) for*

$$\lambda > \frac{1}{f_2(1) l \int_0^{\omega} b(s)\mathrm{d}s} > 0.$$

(b) *If $i_\infty = 1$ or 2, then (1.1) has i_∞ positive ω-periodic solution(s) for*

$$0 < \lambda < \frac{m - (M+m)\sum_{i=1}^{n} c_i}{LM\left(1-\sum_{i=1}^{n} c_i\right) F(1) \int_0^{\omega} b(s)\mathrm{d}s}.$$

(c) *If $i_\infty = 0$ or $i_0 = 0$, then (1.1) has no positive ω-periodic solution for sufficiently small or large $\lambda > 0$, respectively.*

Theorem 3.14.

(a) *If there exists a constant $b_1 > 0$ such that $f(u) \ge b_1 u$ for $u \in [0, +\infty)$, then (1.1) has no positive ω-periodic solution for*

$$\lambda > \frac{1-\sum_{i=1}^{n} c_i}{l\alpha b_1 \int_0^{\omega} b(s)\mathrm{d}s}.$$

(b) *If there exists a constant $b_2 > 0$ such that $f(u) \le b_2 u$ for $u \in [0, +\infty)$, then (1.1) has no positive ω-periodic solution for*

$$0 < \lambda < \frac{m - (M+m)\sum_{i=1}^{n} c_i}{b_2 LM \int_0^{\omega} b(s)\mathrm{d}s}.$$

Theorem 3.15. *If*

$$\frac{1-\sum_{i=1}^{n} c_i}{l\alpha \int_0^{\omega} b(s)\mathrm{d}s \max\{f_0, f_\infty\}} < \lambda < \frac{m - (M+m)\sum_{i=1}^{n} c_i}{LM \int_0^{\omega} b(s)\mathrm{d}s \min\{f_0, f_\infty\}},$$

then (1.1) has one positive ω-periodic solution.

Finally, we give an example to illustrate our results.

Example 3.1. *Consider the following neutral functional differential equation:*

$$\left[u(t) + \frac{1}{12}u\left(t+\frac{\pi}{3}\right) + \frac{1}{20}u\left(t-\frac{\pi}{2}\right) + \frac{1}{10}u\left(t-\frac{\pi}{5}\right)\right]'' + \frac{1}{16}u(t) = \lambda(1-\sin t)u(t-\tau(t))a^{u(t-\tau(t))}, \tag{3.20}$$

where λ and $0 < a < 1$ are two positive parameters, $\tau(t+2\pi) = \tau(t)$. We see that $\delta_1 = -\frac{\pi}{3}$, $\delta_2 = \frac{\pi}{2}$, $\delta_3 = \frac{\pi}{5}$, $c_1 = -\frac{1}{12}$, $c_2 = -\frac{1}{20}$, $c_3 = -\frac{1}{10}$, $a(t) \equiv \frac{1}{16}$, $b(t) = 1 - \sin t$, $\omega = 2\pi$, $f(u) = ua^u$. Additionally, $\max_{u\in[0,\infty)} f(u) = f\left(-\frac{1}{\ln a}\right)$.

Clearly, $M = \frac{1}{16} < (\frac{\pi}{2\pi})^2 = \frac{1}{4}$, $f_0 = 0$, $f_\infty = 0$. Then we easily obtain:

Conclusion 1. *The eq. (3.20) has two positive ω-periodic solutions for $\lambda > \frac{1}{4\pi r_1}$, where $r_1 = \min\left\{f(0.27), f\left(\frac{30}{23}\right)\right\}$.*

In fact, by simple computations, we have

$$M = m = \frac{1}{16}, \qquad \beta = \frac{1}{4}, \qquad L = \frac{1}{2\beta \sin\frac{\beta 2\pi}{2}} = 2\sqrt{2}, \qquad l = \frac{\cos\frac{\beta 2\pi}{2}}{2\beta\sin\frac{\beta 2\pi}{2}} = 2,$$

$$k = \frac{2+\sqrt{2}}{8}, \qquad k_1 = \frac{\sqrt{2}+1-\sqrt{3}}{2}, \qquad \alpha = \frac{8}{23}\sqrt{2},$$

$$\sum_{i=1}^{3} |c_i| = \frac{7}{30} < \min\left\{k_1, \frac{m}{M+m}\right\} = \frac{\sqrt{2}+1-\sqrt{3}}{2}, \qquad \sum_{i=1}^{3}|c_i| = \frac{7}{30} < \frac{8}{23}\sqrt{2} = \alpha,$$

and

$$M(1) = \max\left\{f(t) : 0 \le t \le \frac{30}{23}\right\} = \max\left\{f\left(\frac{30}{23}\right), f\left(-\frac{1}{\ln a}\right)\right\} = r_0,$$

$$\begin{aligned} m(1) &= \min\left\{f(t) : 0.27 \approx \frac{\frac{8}{23}\sqrt{2} - \frac{7}{30}}{1-\left(\frac{7}{30}\right)^2} \le t \le \frac{30}{23}\right\} \\ &= \min\left\{f(0.27), f\left(\frac{30}{23}\right)\right\} = r_1, \end{aligned}$$

$$\frac{1}{m(1) l \int_0^{\omega} b(s)\mathrm{d}s} = \frac{1}{4\pi r_1}.$$

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