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ON A METHOD OF PROVING THE HYERS-ULAM STABILITY OF FUNCTIONAL EQUATIONS ON RESTRICTED DOMAINS

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ABSTRACT. We show that generalizations of some (classical) results on the Hyers-Ulam stability of functional equations, in several variables, can be very easily derived from a simple result on stability of a functional equation in single variable.

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1. INTRODUCTION

The stability theory of functional equations (currently known as Hyers-Ulam stability) was essentially started in 1941 by D.H. HYERS [25], who provided a partial solution of a problem by S.M. ULAM, concerning approximate homomorphisms between groups. However, a special case of Ulam's problem, for real functions, was posed earlier by GY. PÓLYA and G. SZEGÖ [42, Teil I, Aufgabe 99] (cf. [22, p. 125]). Generalizations and further extensions of that problem have been proposed by several mathematicians (see, e.g., [4, 5, 6, 23, 26, 30, 31, 32]). Beginning in the year 1978, TH.M. RASSIAS'S research work has provided generalizations of D.H. HYERS'S theorem with a number of applications and stimulated further development in this field of research (see, e.g., [16, 27, 28, 29, 43, 44]). For more details, discussions and surveys on the results obtained so far we refer, e.g., to [2, 15, 17, 20, 27, 33, 36, 37, 44]; some very recent results can be found, for instance, in [3, 7, 9, 10, 11, 12, 13, 16, 21, 22, 24, 28, 45, 46].

Usually the problem is considered for functions with values in Banach spaces, but actually it had been stated for functions with values in metric spaces, e.g., in the following way (see [47]): *Given a group G_1, a metric group G_2 with metric d and a positive number ε, find a positive number δ such that, for every $f : G_1 \to G_2$ satisfying: $d(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in G_1$, there exists a homomorphism $h : G_1 \to G_2$ with $d(f(x), h(x)) \leq \varepsilon$ for all $x \in G_1$.* Motivated by this, we prove some stability results for functions with values in metric spaces. Moreover, inspired by G-L. FORTI [18] and the talk of P. VOLKMANN at the 42nd International Symposium on Functional Equations (Brno, The Czech Republic, 2004), as the main tool we use a result on the stability of a functional equation for a single variable. This approach enables us to show that it is quite natural to consider the stability of conditional versions of some well known functional equations on restricted domains (such an approach has already been applied in [40] and [8]).

2. THE MAIN TOOL

Let us start with the following theorem, concerning stability of the equation $\Psi \circ f \circ a = f$. For a function a mapping a nonempty set K into K in the sequel we write $a^0(x) = x$ for $x \in K$ and $a^n = a \circ a^{n-1}$ for $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers.

The theorem can be easily derived from [18], however, for the convenience of the readers we present it here with a proof.

Theorem 2.1. *Assume that (Y, d) is a complete metric space, K is a nonempty set, $f : K \to Y$, $\Psi : Y \to Y$, $a : K \to K$, $h : K \to [0, \infty)$, $\lambda \in [0, \infty)$, $d(\Psi \circ f \circ a(x), f(x)) \leq h(x)$ for $x \in K$,*

$$d(\Psi(x), \Psi(y)) \leq \lambda d(x, y) \quad \text{for } x, y \in Y, \tag{2.1}$$

and

$$H(x) := \sum_{i=0}^{\infty} \lambda^i h\left(a^i(x)\right) < \infty \quad \text{for } x \in K. \tag{2.2}$$

Then, for every $x \in K$, the limit $F(x) := \lim_{n\to\infty} \Psi^n \circ f \circ a^n(x)$ exists and $F : K \to Y$ is a unique function such that $\Psi \circ F \circ a = F$ and $d(f(x), F(x)) \leq H(x)$ for $x \in K$.

Proof. Note that

$$\begin{aligned} d\left(\Psi^n \circ f \circ a^n(x), f(x)\right) &\le \sum_{i=1}^{n} d\left(\Psi^i \circ f \circ a^i(x), \Psi^{i-1} \circ f \circ a^{i-1}(x)\right) \\ &\le \sum_{i=1}^{n} \lambda^{i-1} d\left(\Psi \circ f \circ a^i(x), f \circ a^{i-1}(x)\right) \\ &\le \sum_{i=0}^{n-1} \lambda^i h\left(a^i(x)\right) \end{aligned}$$

for $x \in K$ and $n \in \mathbb{N}$. Hence, for every $x \in K$ and $m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} & d\left(\Psi^{m+k} \circ f \circ a^{m+k}(x), \Psi^k \circ f \circ a^k(x)\right) \\ &\quad \le \lambda^k d\left(\Psi^m \circ f \circ a^{m+k}(x), f \circ a^k(x)\right) \\ &\quad \le \lambda^k \sum_{i=0}^{m-1} \lambda^i h\left(a^{k+i}(x)\right) \le \sum_{i=k}^{m+k-1} \lambda^i h\left(a^i(x)\right). \end{aligned}$$

Since $\sum_{i=k}^{m+k-1} \lambda^i h\left(a^i(x)\right) \to 0$ with $k \to \infty$, $F(x) := \lim_{n\to\infty} \Psi^n \circ f \circ a^n(x)$ exists and $d(F(x), f(x)) \le H(x)$ for every $x \in K$. Moreover Ψ is continuous (in view of (2.1)), whence for every $x \in K$ we have

$$F(x) = \lim_{n\to\infty} \Psi^{n+1} \circ f \circ a^{n+1}(x) = \Psi\left(\lim_{n\to\infty} \Psi^n \circ f \circ a^n(a(x))\right) = \Psi \circ F \circ a(x).$$

It remains to show the uniqueness of F. We suppose that $G : K \to Y$, $d(f(x), G(x)) \le H(x)$ for $x \in K$ and $\Psi \circ G \circ a = G$. By induction it is easy to show that $\Psi^n \circ G \circ a^n = G$ and $\Psi^n \circ F \circ a^n = F$ for $n \in \mathbb{N}$. Hence, for $x \in K$,

$$\begin{aligned} d(F(x), G(x)) &= d\left(\Psi^n \circ F \circ a^n(x), \Psi^n \circ G \circ a^n(x)\right) \\ &\le \lambda^n d\left(F \circ a^n(x), G \circ a^n(x)\right) \\ &\le \lambda^n d\left(F \circ a^n(x), f \circ a^n(x)\right) + \lambda^n d\left(f \circ a^n(x), G \circ a^n(x)\right) \\ &\le 2\lambda^n H\left(a^n(x)\right) \\ &= 2\sum_{i=n}^{\infty} \lambda^i h\left(a^i(x)\right). \end{aligned}$$

Since, for every $x \in K$, $\sum_{i=n}^{\infty} \lambda^i h\left(a^i(x)\right) \to 0$ with $n \to \infty$, this completes the proof. ∎

Now, in a series of corollaries we show applications of Theorem 2.1 to the problem of stability of functional equations in several variables, on restricted domains. For the sake of preservation of reasonable simplicity, we confine ourselves to some rather less general situations.

However, before we begin, let us recall that a groupoid $(G, +)$ (i.e., a nonempty set G endowed with a binary operation $+ : G^2 \to G$) is uniquely divisible by 2 provided, for each $x \in X$, there is a unique $y \in X$ with $x = 2y$ (we write $2x := x + x$ for $x \in G$); such an element y will be denoted by $\frac{x}{2}$ or $\frac{1}{2}x$ in the sequel. In what follows, we use the notion: $2^0 x := x$, $2^n x = 2(2^{n-1}x)$ and (only for groupoids uniquely divisible by 2) $2^{-n}x = \frac{1}{2}(2^{-n+1}x)$ for $x \in G$, $n \in \mathbb{N}$.

We say that a groupoid $(G, +)$ is square symmetric provided the operation $+$ is square symmetric, i.e., $2(x + y) = 2x + 2y$ for $x, y \in G$; it is easy to show by induction that, for each $n \in \mathbb{N}$ (and also for all integers n, if the groupoid is uniquely divisible by 2),

$$2^n(x + y) = 2^n x + 2^n y \quad \text{for } x, y \in G. \tag{2.3}$$

Such groupoids have been already considered in the theory of stability of functional equations, e.g., in [38] and [39] (see also [41]).

Clearly, every commutative semigroup is a square symmetric groupoid. Next, let X be a linear space over a field $\mathbb{K}$, $a, b \in \mathbb{K}$, $z \in X$ and define a binary operation $\star : X^2 \to X$ by: $x \star y := ax + by + z$ for $x, y \in X$. Then it is easy to check that $(X, \star)$ provides a simple example of a square symmetric groupoid.

Finally, we say that $(G, +, d)$ is a complete metric groupoid provided $(G, +)$ is a groupoid, (G, d) is a complete metric space and the operation $+$ is continuous with respect to the metric d.

3. THE CAUCHY EQUATION

In this part we assume that $(X, +)$ and $(Y, +)$ are square symmetric groupoids, $(Y, +, d)$ is a complete metric groupoid, $K \subset X$, and $\chi : X^2 \to [0, \infty)$. We start with the stability of the conditional Cauchy functional equation

$$F(x + y) = F(x) + F(y) \quad \text{for } x, y \in K \text{ with } x + y \in K. \tag{3.1}$$

Corollary 3.1. *Suppose that X is uniquely divisible by 2, $\frac{1}{2}K \subset K$ (i.e., $\frac{a}{2} \in K$ for $a \in K$), and there exist $\xi, \eta \in (0, \infty)$ such that $\xi\eta < 1$,*

$$\chi\left(\frac{x}{2}, \frac{y}{2}\right) \le \eta\chi(x, y) \quad \textit{for } x, y \in K, \tag{3.2}$$

$$d(2x, 2y) \le \xi d(x, y) \quad \textit{for } x, y \in Y. \tag{3.3}$$

Let $\varphi : K \to Y$ satisfy

$$d(\varphi(x + y), \varphi(x) + \varphi(y)) \le \chi(x, y) \quad \textit{for } x, y \in K \textit{ with } x + y \in K. \tag{3.4}$$

Then there is a unique solution $F : K \to Y$ of (3.1) with

$$d(\varphi(x), F(x)) \le \frac{\eta\chi(x, x)}{1 - \xi\eta} \quad \textit{for } x \in K.$$

Proof. From (3.4) we obtain $d\left(\varphi(x), 2\varphi\left(\frac{x}{2}\right)\right) \le \chi\left(\frac{x}{2}, \frac{x}{2}\right)$ for $x \in K$. Hence, by Theorem 2.1 (with $f = \varphi$, $\Psi(z) = 2z$, $\lambda = \xi$, $h(x) = \chi\left(\frac{x}{2}, \frac{x}{2}\right)$, and $a(x) = \frac{x}{2}$), for every $x \in K$ the limit $F(x)$ exists and $d(f(x), F(x)) \le H(x)$. Take $x, y \in K$ with $x + y \in K$. Since (2.3) and (3.4) yield

$$d\left(2^n\varphi\left(2^{-n}(x + y)\right), 2^n\varphi\left(2^{-n}x\right) + 2^n\varphi\left(2^{-n}y\right)\right) \le (\xi\eta)^n\chi(x, y)$$

for $n \in \mathbb{N}$, so letting $n \to \infty$ we obtain $F(x + y) = F(x) + F(y)$.

Thus we have shown that (3.1) holds. Suppose $F_0 : K \to Y$ also is a solution of (3.1) and $d(f(x), F_0(x)) \le H(x)$ for every $x \in K$. Then $\Psi \circ F_0 \circ a = F_0$, whence, by Theorem 2.1, $F = F_0$, which implies the uniqueness of F. ∎

Remark 3.1. For instance, if in Corollary 3.1 the metric d is invariant, i.e.,

$$d(x + z, y + z) = d(x, y) = d(z + x, z + y) \quad \text{for } x, y, z \in Y, \tag{3.5}$$

then

$$d(2x, 2y) \le d(2x, x + y) + d(x + y, 2y) = 2d(x, y) \quad \text{for } x, y \in Y,$$

whence (3.3) holds with $\xi = 2$. Moreover, (3.5) implies

$$d(b_n + c_n, b + c) \le d(b_n + c_n, b_n + c) + d(b_n + c, b + c) \le d(c_n, c) + d(b_n, b)$$

for every $b_n, c_n, b, c \in Y$, $n \in \mathbb{N}$, which yields continuity of $+$.

Note also that the classical Hyers result in [25] with constant χ cannot be derived from Corollary 3.1, because then condition (3.2) holds with $\eta \geq 1$. However, it can be deduced from the following.

Corollary 3.2. *Suppose that Y is uniquely divisible by 2, $2K \subset K$ (i.e., $2a \in K$ for $a \in K$), and there exist $\xi, \eta \in [0, \infty)$ such that $\xi\eta < 1$,*

$$d\left(\frac{1}{2}x, \frac{1}{2}y\right) \leq \xi d(x,y) \quad \text{for } x, y \in Y, \tag{3.6}$$

$$\chi(2x, 2y) \leq \eta\chi(x,y) \quad \text{for } x, y \in K. \tag{3.7}$$

Let $\varphi : K \to Y$ satisfy (3.4). Then there is a unique solution $G : K \to Y$ of (3.1) with

$$d(\varphi(x), G(x)) \leq \frac{\xi^2 \chi(x,x)}{1 - \xi\eta} \quad \text{for } x \in K.$$

Proof. From (3.4) and (3.6) we get

$$d\left(\frac{1}{2}\varphi(2x), \varphi(x)\right) \leq \xi d(\varphi(2x), 2\varphi(x)) \leq \xi\chi(x,x)$$

for $x \in K$. Now, we can use Theorem 2.1 analogously as in the proof of Corollary 3.1 (with $\lambda = \xi$, $f = 2\varphi$, $\Psi(z) = \frac{1}{2}z$, $h(x) = \xi\chi(x,x)$ and $a(x) = 2x$). Then, with $G = \frac{1}{2}F$, for every $x \in K$ we get

$$d(\varphi(x), G(x)) \leq \xi d(f(x), F(x)) \leq \xi^2 \chi(x,x) \sum_{i=0}^{\infty} (\xi\eta)^i.$$

Next, by (2.3) and (3.4), for every $x, y \in K$ with $x + y \in K$, we have

$$d\left(2^{-n}\varphi(2^n(x+y)), 2^{-n}\varphi(2^n x) + 2^{-n}\varphi(2^n y)\right) \leq (\xi\eta)^n \chi(x,y)$$

for $n \in \mathbb{N}$, which implies that F is a solution of (3.1) and so is G.

The proof of uniqueness is analogous as in the proof of Corollary 3.1. ∎

4. THE JENSEN EQUATION

Let $(X, +)$ and $(Y, +)$ be square symmetric groupoids, uniquely divisible by 2. Moreover, assume that $(X, +)$ has a neutral element 0_X, d is a complete metric in Y, (3.5) holds, $K \subset X$, and $\chi : X^2 \to [0, \infty)$. Now, we are in a position to consider stability of the conditional Jensen functional equation

$$F\left(\frac{x+y}{2}\right) = \frac{F(x) + F(y)}{2} \quad \text{for } x, y \in K \text{ with } \frac{x+y}{2} \in K. \tag{4.1}$$

Corollary 4.1. *Assume that $0_X \in K$, $2K \subset K$, and there exist $\xi, \eta \in [0, \infty)$ such that $\xi\eta < 1$, (3.6) holds and*

$$\chi(2x, 0_X) \leq \eta\chi(x, 0_X) \quad \text{for } x \in K. \tag{4.2}$$

Let $\psi : K \to Y$ satisfy

$$d\left(\psi\left(\frac{x+y}{2}\right), \frac{\psi(x) + \psi(y)}{2}\right) \leq \chi(x,y) \quad \text{for } x, y \in K \text{ with } \frac{x+y}{2} \in K. \tag{4.3}$$

Then there exists a unique solution $F : X \to Y$ of (4.1) such that

$$d(\psi(x), F(x)) \leq \frac{\eta\chi(x, 0_X)}{1 - \xi\eta} \quad \text{for } x \in K.$$

Proof. Replacing in (4.3) x by $2x$ and y by 0_X we obtain

$$d\left(\psi\left(x\right), \frac{\psi(2x)+\psi(0_X)}{2}\right) \le \chi(2x, 0_X)$$

for $x \in K$. In view of Theorem 2.1 (with $\lambda = \xi$, $f = \psi$, $a(x) = 2x$, $\Psi(y) = \frac{1}{2}(y + \psi(0_X))$, and $h(x) = \chi(2x, 0_X)$), for every $x \in K$ the limit $F(x)$ exists, $d(f(x), F(x)) \le H(x)$ and consequently, by (4.2),

$$d(F(x), \psi(x)) = d(F(x), f(x)) \le \sum_{i=0}^{\infty} \xi^i h(2^i x) = \eta\chi(x, 0_X) \sum_{i=0}^{\infty} (\xi\eta)^i.$$

Next, by induction, in view of (2.3), (3.5) and (3.6), we get

$$d\left(\Psi^n \circ f\left(2^n\left(\frac{x+y}{2}\right)\right), \frac{1}{2}\left[\Psi^n \circ f\left(2^n x\right) + \Psi^n \circ f\left(2^n y\right)\right]\right) \\ \le \xi^n d\left(\psi\left(2^n\left(\frac{x+y}{2}\right)\right), \frac{1}{2}\left[\psi\left(2^n x\right) + \psi\left(2^n y\right)\right]\right) \le (\xi\eta)^n \chi(x, y)$$

for every $x, y \in K$ with $\frac{x+y}{2} \in K$, whence F is a solution of (4.1) (because the mapping $Y \ni z \to \frac{z}{2} \in Y$ is continuous on account of (3.6)).

Analogously as in the proof of Corollary 3.1 we show that F is unique. ∎

Corollary 4.2. *Assume that $(Y, +)$ is a commutative group, $0_X \in K$, $\frac{1}{2}K \subset K$ and there exist $\eta, \xi \in [0, \infty)$ such that $\eta\xi < 1$, (3.3) holds and*

$$\chi\left(\frac{x}{2}, 0_X\right) \le \eta\chi(x, 0_X) \quad \textit{for } x \in K. \tag{4.4}$$

Let $\psi : K \to Y$ satisfy (4.3). Then there exists a unique solution $G : X \to Y$ of (4.1) such that

$$d(\psi(x), G(x)) \le \frac{\xi\chi(x, 0_X)}{1 - \eta\xi} \quad \textit{for } x \in K.$$

Proof. (4.3), (3.3) and (3.5) imply that, for every $x \in K$,

$$d\left(2\left(\psi\left(\frac{x}{2}\right) - \psi(0_X)\right), \psi(x) - \psi(0_X)\right) \le \xi d\left(\psi\left(\frac{x}{2}\right), \frac{\psi(x)+\psi(0_X)}{2}\right) \le \xi\chi(x, 0_X).$$

Hence, according to Theorem 2.1 (with $\Psi\left(z\right) = 2z$, $\lambda = \xi$, $f = \psi - \psi(0_X)$, $h(x) = \xi\chi\left(x, 0_X\right)$, and $a(x) = \frac{x}{2}$), for every $x \in K$ the limit $F(x)$ exists and $d(F(x), f(x)) \le H(x)$. Since one can easily show by induction that

$$d\left(2\Psi^n \circ f\left(\frac{2^{-n}x + 2^{-n}y}{2}\right), \Psi^n \circ f(2^{-n}x) + \Psi^n \circ f(2^{-n}y)\right) \\ \le \xi^n d\left(2\psi\left(\frac{2^{-n}x + 2^{-n}y}{2}\right), \psi(2^{-n}x) + \psi(2^{-n}y)\right) \le \xi^n \eta^n \chi(x, y)$$

for every $n \in \mathbb{N}$ and $x, y \in K$ with $x + y \in K$, letting $n \to \infty$ we obtain that F is a solution of (4.1). To complete, observe that $G := F + \psi(0_X)$ satisfies (4.1) as well and $d(G(x), \psi(x)) = d(F(x), f(x)) \le H(x)$ for every $x \in K$. The uniqueness of G results from Theorem 2.1. ∎

Remark 4.1. In the case where X is a commutative group, the assumption that $0_X \in K$ in Corollaries 4.1 and 4.2 is not very restrictive for we can always replace K by $K_0 := \{x - x_0 :$

$x \in K\}$, with any $x_0 \in K$, and then the functions $\psi_0 : K_0 \to Y$, $\psi_0(z) := \psi(z + x_0)$, $\chi_0 : X^2 \to [0, \infty)$, $\chi_0(z, w) = \chi(z + x_0, w + x_0)$, satisfy

$$d\left(\psi_0\left(\frac{z+w}{2}\right), \frac{\psi_0(z) + \psi_0(w)}{2}\right) \le \chi_0(z, w) \text{ for } z, w \in K_0 \text{ with } \frac{z+w}{2} \in K_0.$$

However we need then to reformulate conditions (4.2) and (4.4) in a suitable way.

5. THE QUADRATIC EQUATION

In this part we assume that $(X, +)$ is a square symmetric group with the neutral element 0_X, $(Y, +)$ is a square symmetric groupoid with the neutral element 0_Y, d is a complete metric in Y satisfying (3.5), $K \subset X$, $0_X \in K$, and $\chi : X^2 \to [0, \infty)$. Moreover, we write $4z := 2z + 2z$ for $z \in Y$. Now, we are in a position to study the stability of the conditional quadratic equation

$$(5.1) \qquad F(x + y) + F(x - y) = 2F(x) + 2F(y) \quad \text{for } x, y \in K \text{ with } x + y, x - y \in K.$$

Corollary 5.1. *Assume that X is uniquely divisible by 2, $\frac{1}{2}K \subset K$, and there exist $\xi, \eta \in (0, \infty)$ such that $\xi < 1$, $\xi^2\eta < 1$ and (3.2), (3.3) hold. Let $\gamma : K \to Y$ satisfy*

$$(5.2) \qquad d\left(\gamma(x + y) + \gamma(x - y), 2\gamma(x) + 2\gamma(y)\right) \le \chi(x, y)$$

for $x, y \in K$ with $x + y, x - y \in K$. Then there exists a unique solution $F : X \to Y$ of (5.1) such that

$$(5.3) \qquad d\left(\gamma(x), F(x)\right) \le d(\gamma(0_X), 0_Y) + \frac{\xi d(2\gamma(0_X), 0_Y)}{1 - \xi^2} + \frac{\eta\chi(x, x)}{1 - \xi^2\eta} \quad \textit{for } x \in K.$$

Moreover, in the case $\eta < 1$, $2\gamma(0_X) = 0_Y$.

Proof. (3.5) and (5.2) yield

$$d(0_Y, 2\gamma(0_X)) = d(2\gamma(0_X), 4\gamma(0_X)) \le \chi(0_X, 0_X)$$

and

$$\begin{aligned}
& d\left(\gamma(x) + \gamma(0_X), 4\left[\gamma\left(\frac{1}{2}x\right) + \gamma(0_X)\right]\right) \\
& \le d\left(\gamma(x) + \gamma(0_X), 4\gamma\left(\frac{1}{2}x\right)\right) + d\left(4\gamma\left(\frac{1}{2}x\right), 4\left[\gamma\left(\frac{1}{2}x\right) + \gamma(0_X)\right]\right) \\
& \le \eta\chi\left(x, x\right) + \xi d(0_Y, 2\gamma(0_X))
\end{aligned}$$

for every $x \in K$. Hence

$$d(0_Y, 2\gamma(0_X)) \le \chi\left(2^{-n}0_X, 2^{-n}0_X\right) \le \eta^n\chi(0_X, 0_X)$$

for $n \in \mathbb{N}$, whence $2\gamma(0_X) = 0_Y$ when $\eta < 1$. Next, by Theorem 2.1 (with $f = \gamma + \gamma(0_X)$, $\Psi(z) = 4z$, $\lambda = \xi^2$, $h(x) = \eta\chi\left(x, x\right) + \xi d(0_Y, 2\gamma(0_X))$, and $a(x) = \frac{x}{2}$), for every $x \in K$ the limit $F(x)$ exists and $d\left(F(x), f(x)\right) \le H(x)$, which implies (5.3). Since, for $x, y \in K$ with $x + y, x - y \in K$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
& d\left(\Psi^n \circ f\left(\frac{x+y}{2^n}\right) + \Psi^n \circ f\left(\frac{x-y}{2^n}\right), 2\Psi^n \circ f\left(\frac{x}{2^n}\right) + 2\Psi^n \circ f\left(\frac{y}{2^n}\right)\right) \\
& \qquad \le \xi^{2n} d\left(\gamma\left(\frac{x+y}{2^n}\right) + \gamma\left(\frac{x-y}{2^n}\right), 2\gamma\left(\frac{x}{2^n}\right) + 2\gamma\left(\frac{y}{2^n}\right)\right) \le \xi^{2n}\eta^n\chi(x, y),
\end{aligned}$$

letting $n \to \infty$ we obtain (5.1). The uniqueness of F results from Theorem 2.1. ∎

Corollary 5.2. *Assume that Y is uniquely divisible by 2, $2K \subset K$, and there are $\xi \in [0,1)$, $\eta \in [0,\infty)$ such that $\xi^2\eta < 1$ and (3.6), (3.7) hold. Let $\gamma : K \to Y$ satisfy (5.2) for $x, y \in K$ with $x+y, x-y \in K$. Then there exists a unique solution $F : X \to Y$ of (5.1) such that*

$$d\left(\gamma(x), F(x)\right) \le \xi^2 \left[\frac{\chi(x,x)}{1-\xi^2\eta} + \frac{d\left(0_Y, \gamma(0_X)\right)}{1-\xi^2}\right] \quad \text{for } x \in K. \tag{5.4}$$

Proof. From (3.5), (3.6) and (5.2), for every $x \in K$, we get

$$\begin{aligned} d\left(\frac{\gamma(2x)}{4}, \gamma(x)\right) &\le \xi^2[d\left(\gamma(2x), \gamma(2x)+\gamma(0_X)\right) + d\left(\gamma(2x)+\gamma(0_X), 4\gamma(x)\right)] \\ &\le \xi^2[d\left(0_Y, \gamma(0_X)\right) + d\left(\gamma(2x)+\gamma(0_X), 4\gamma(x)\right)] \\ &\le \xi^2[d\left(0_Y, \gamma(0_X)\right) + \chi(x,x)]. \end{aligned}$$

Hence, according to Theorem 2.1 (with $\lambda = \xi^2$, $f = \gamma$, $\Psi(z) = \frac{1}{4}z$, $h(x) = \xi^2[\chi(x,x) + d\left(0_Y, \gamma(0_X)\right)]$, and $a(x) = 2x$), for every $x \in K$ the limit $F(x)$ exists and $d\left(F(x), \gamma(x)\right) \le H(x)$, whence (5.4) holds. Since, for $x, y \in K$ with $x+y, x-y \in K$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} &d\left(4^{-n} f\left(2^n(x+y)\right) + 4^{-n} f\left(2^n(x-y)\right), 4^{-n}2f\left(2^n x\right) + 4^{-n}2f\left(2^n y\right)\right) \\ &\le \xi^{2n} d\left(\gamma\left(2^n(x+y)\right) + \gamma\left(2^n(x-y)\right), 2\gamma\left(2^n x\right) + 2\gamma\left(2^n y\right)\right) \\ &\le \xi^{2n}\chi\left(2^n x, 2^n y\right) \le (\xi^2\eta)^n \chi(x,y), \end{aligned}$$

letting $n \to \infty$ we obtain (5.1). The uniqueness of F results from Theorem 2.1. ∎

6. FINAL REMARKS

Let Y be a normed space and $d(x,y) = \|x-y\|$ for $x, y \in Y$. Then clearly (3.3), (3.5), (3.6) hold with $\xi \in \{\frac{1}{2}, 2\}$. Hence, in the case where either $\chi(x,y) = c_1\|x\|^p + c_2\|y\|^q$ or $\chi(x,y) = c\|x\|^p\|y\|^q$ with some real p, q and $c_1, c_2, c \in (0,\infty)$, the assumptions supposed on χ in the paper are fulfilled at least for some K, p, q. Consequently our corollaries generalize numerous results concerning the Hyers-Ulam-Rassias stability of the functional equations considered in this paper (cf. e.g. [15], [27] and [33]).

Some information concerning those functional equations and many further references can be found in [1].

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