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ON THE CONVERGENCE IN LAW OF ITERATES OF RANDOM–VALUED FUNCTIONS

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ABSTRACT. Given a probability space $(\Omega, \mathcal{A}, P)$, a separable and complete metric space X with the σ–algebra $\mathcal{B}$ of all its Borel subsets and a $\mathcal{B} \otimes \mathcal{A}$–measurable $f : X \times \Omega \to X$ we consider its iterates $f^n, n \in \mathbb{N}$, defined on $X \times \Omega^{\mathbb{N}}$ by $f^1(x, \omega) = f(x, \omega_1)$ and $f^{n+1}(x, \omega) = f(f^n(x, \omega), \omega_{n+1})$, provide a simple criterion for the convergence in law of $(f^n(x, \cdot))_{n \in \mathbb{N}}$ to a random variable independent of $x \in X$, and apply this criterion to linear functional equations in a single variable.

Key words and phrases: Random-valued functions, Iterates, Convergence in law, Linear iterative equations, Lipschitzian and bounded solutions.

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1. INTRODUCTION

Throughout the paper $(\Omega, \mathcal{A}, P)$ is a probability space and (X, ϱ) is a separable metric space.

Let $\mathcal{B}$ denote the σ–algebra of all Borel subsets of X. We say that $f : X \times \Omega \to X$ is a *random–valued* function (an *rv–function* for short) if it is measurable with respect to the product σ–algebra $\mathcal{B} \otimes \mathcal{A}$. The iterates of such an rv–function are given by

$$f^1(x, \omega_1, \omega_2, \ldots) = f(x, \omega_1), \quad f^{n+1}(x, \omega_1, \omega_2, \ldots) = f(f^n(x, \omega_1, \omega_2, \ldots), \omega_{n+1})$$

for x from X and $(\omega_1, \omega_2, \ldots)$ from Ω^∞ defined as $\Omega^{\mathbb{N}}$. Note that $f^n : X \times \Omega^\infty \to X$ is an rv–function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. More precisely, the n–th iterate f^n is $\mathcal{B} \otimes \mathcal{A}_n$–measurable, where $\mathcal{A}_n$ denotes the σ–algebra of all sets of the form

$$\{(\omega_1, \omega_2, \ldots) \in \Omega^\infty : (\omega_1, \ldots, \omega_n) \in A\}$$

with A from the product σ–algebra $\mathcal{A}^n$ (see [4, Sec. 1.4], [2]).

A result on the a.s. convergence of $(f^n(x, \cdot))_{n\in\mathbb{N}}$ for X the unit interval may found in [4, Sec. 1.4B]. The paper [2] by R. Kapica brings theorems on the convergence a.s. and in L^1 of those sequences of iterates in the case where X is a closed subset of a Banach lattice. It is the aim of this note to provide a simple criterion for the convergence in law of $(f^n(x, \cdot))_{n\in\mathbb{N}}$ to a random variable independent of $x \in X$ and to apply it to the iterative equations

$$\varphi(x) = \int_\Omega \varphi(f(x, \omega))P(d\omega) + F(x). \tag{1.1}$$

2. WASSERSTEIN METRIC

By a distribution (on X) we mean any probability measure defined on $\mathcal{B}$. Recall that a sequence $(\pi_n)_{n\in\mathbb{N}}$ of distributions converges weakly to a distribution π if

$$\lim_{n\to\infty} \int_X u(x)\pi_n(dx) = \int_X u(x)\pi(dx)$$

for any continuous and bounded function $u : X \to \mathbb{R}$. It is well known (see [1, Th. 11.3.3]) that this convergence is metrizable by the (Fortet–Mourier, Lévy–Prohorov, Wasserstein) metric

$$\|\pi_1 - \pi_2\|_W = \sup\left\{\left|\int_X u d\pi_1 - \int_X u d\pi_2\right| : \ u \in \mathrm{Lip}_1(X), \|u\|_\infty \le 1\right\},$$

where

$$\mathrm{Lip}_1(X) = \{u : X \to \mathbb{R} |\ |u(x) - u(z)| \le \varrho(x, z) \quad \text{for } x, z \in X\}$$

and $\|u\|_\infty = \sup\{|u(x)| : \ x \in X\}$ for a bounded $u : X \to \mathbb{R}$.

Following an idea of A. Lasota from [5], we will consider also the Hutchinson distance of distributions:

$$d_H(\pi_1, \pi_2) = \sup\left\{\left|\int_X u d\pi_1 - \int_X u d\pi_2\right| : \ u \text{ is in } \mathrm{Lip}_1(X) \text{ and bounded}\right\}$$

which may be infinite for some distributions. Clearly

$$\|\pi_1 - \pi_2\|_W \le d_H(\pi_1, \pi_2) \tag{2.1}$$

for any distributions π_1 and π_2 on X.

3. MAIN RESULT

Fix an rv–function $f : X \times \Omega \to X$ and let $\pi_n(x, \cdot)$ denote the distribution of $f^n(x, \cdot)$, i.e.,

$$\pi_n(x, B) = P^\infty(f^n(x, \cdot) \in B) \tag{3.1}$$

for $n \in \mathbb{N}, x \in X$ and $B \in \mathcal{B}$. Clearly $\pi_1(x, \cdot)$ is the distribution of $f(x, \cdot)$:

$$\pi_1(x, B) = P(f(x, \cdot) \in B) \quad \text{for } x \in X \text{ and } B \in \mathcal{B}. \tag{3.2}$$

Our main result reads as follows.

Theorem 3.1. *Assume that (X, ϱ) is complete and separable. If*

$$\int_\Omega \varrho(f(x, \omega), f(z, \omega))P(d\omega) \le \lambda\varrho(x, z) \quad \textit{for } x, z \in X \tag{3.3}$$

with a $\lambda \in (0, 1)$, and

$$\int_\Omega \varrho(f(x, \omega), x)P(d\omega) < \infty \quad \textit{for } x \in X, \tag{3.4}$$

then there exists a distribution π on X such that for every $x \in X$ the sequence $\pi_n(x, \cdot))_{n\in\mathbb{N}}$ converges weakly to π; moreover,

$$\|\pi_n(x, \cdot) - \pi\|_W \le \frac{\lambda^n}{1-\lambda}\int_\Omega \varrho(f(x, \omega), x)P(d\omega) \quad \textit{for } x \in X \textit{ and } n \in \mathbb{N}. \tag{3.5}$$

Proof. Fix a bounded function $u \in \mathrm{Lip}_1(X)$ and define $v : X \to \mathbb{R}$ by

$$v(x) = \int_\Omega u(f(x, \omega))P(d\omega).$$

Then, according to (3.3), $\frac{1}{\lambda}v \in \mathrm{Lip}_1(X)$. Hence and from (3.1) we infer that

$$\begin{aligned}
&\left|\int\limits_X u(y)\pi_{n+1}(x, dy) - \int\limits_X u(y)\pi_n(x, dy)\right| \\
&= \left|\int\limits_{\Omega^\infty} u\left(f^{n+1}(x, \omega)\right) P^\infty(d\omega) - \int\limits_{\Omega^\infty} u(f^n(x, \omega))P^\infty(d\omega)\right| \\
&= \left|\int\limits_{\Omega^\infty} u\left(f(f^n(x, \omega_1, \omega_2, \ldots), \omega_{n+1})\right) P^\infty(d(\omega_1, \omega_2, \ldots))\right. \\
&\quad \left. - \int\limits_{\Omega^\infty} u\left(f(f^{n-1}(x, \omega_1, \omega_2, \ldots), \omega_n)\right) P^\infty(d(\omega_1, \omega_2, \ldots))\right| \\
&= \left|\int\limits_{\Omega^\infty} v(f^n(x, \omega))P^\infty(d\omega) - \int\limits_{\Omega^\infty} v\left(f^{n-1}(x, \omega)\right) P^\infty(d\omega)\right| \\
&= \left|\int\limits_X v(y)\pi_n(x, dy) - \int\limits_X v(y)\pi_{n-1}(x, dy)\right| \\
&\le \lambda d_H(\pi_n(x, \cdot), \pi_{n-1}(x, \cdot))
\end{aligned}$$

and

$$d_H(\pi_{n+1}(x,\cdot),\pi_n(x,\cdot)) \le \lambda d_H(\pi_n(x,\cdot),\pi_{n-1}(x,\cdot))$$

for $x \in X$ and $n \in \mathbb{N}$, where $\pi_0(x,\cdot)$ is the point mass at x:

$$\pi_0(x,\cdot) = \delta_x \quad \text{for } x \in X.$$

Consequently

$$d_H(\pi_{n+m}(x,\cdot),\pi_n(x,\cdot)) \le \frac{\lambda^n}{1-\lambda}(1-\lambda^m) d_H(\pi_1(x,\cdot),\pi_0(x,\cdot)) \tag{3.6}$$

for $x \in X$ and $m, n \in \mathbb{N}$. Moreover, taking (3.2) into account,

$$\begin{aligned}
&d_H(\pi_1(x,\cdot),\pi_0(x,\cdot)) \\
&\quad = \sup\left\{\left|\int\limits_X u(y)\pi_1(x,dy) - \int\limits_X u(y)\delta_x(dy)\right| : \ u \text{ is in } \mathrm{Lip}_1(X) \text{ and bounded}\right\} \\
&\quad = \sup\left\{\left|\int\limits_\Omega (u(f(x,\omega)) - u(x))P(d\omega)\right| : \ u \text{ is in } \mathrm{Lip}_1(X) \text{ and bounded}\right\} \\
&\quad \le \int\limits_\Omega \varrho(f(x,\omega),x)P(d\omega)
\end{aligned}$$

for $x \in X$. Hence and from (2.1) and (3.6) we infer that

$$\|\pi_{n+m}(x,\cdot) - \pi_n(x,\cdot)\|_W \le \frac{\lambda^n}{1-\lambda}(1-\lambda^m)\int_\Omega \varrho(f(x,\omega),x)P(d\omega)$$

for $x \in X$ and $m, n \in \mathbb{N}$. This and the Prohorov theorem on the completeness of the space of all distributions on X with the Wasserstein metric (see [1, Cor. 11.5.5]) prove the weak convergence of $(\pi_n(x,\cdot))_{n\in\mathbb{N}}$ to a distribution $\pi(x,\cdot)$ for every $x \in X$ and gives

$$\|\pi(x,\cdot) - \pi_n(x,\cdot)\|_W \le \frac{\lambda^n}{1-\lambda}\int_\Omega \varrho(f(x,\omega),x)P(d\omega) \quad \text{for } x \in X \text{ and } n \in \mathbb{N}.$$

It remains to show that $\pi(x,\cdot) = \pi(z,\cdot)$ for $x, z \in X$. To this end, fix a bounded u in $\mathrm{Lip}_1(X)$. Since, from (3.3) by induction,

$$\int_{\Omega^\infty} \varrho(f^n(x,\omega), f^n(z,\omega))P^\infty(d\omega) \le \lambda^n \varrho(x,z) \quad \text{for } x, z \in X \text{ and } n \in \mathbb{N}, \tag{3.7}$$

according to (3.1) we have

$$\begin{aligned}
&\left|\int\limits_X u(y)\pi_n(x,dy) - \int\limits_X u(y)\pi_n(z,dy)\right| \\
&\quad = \left|\int\limits_{\Omega^\infty} u(f^n(x,\omega))P^\infty(d\omega) - \int\limits_{\Omega^\infty} u(f^n(z,\omega))P^\infty(d\omega)\right| \\
&\quad \le \int\limits_{\Omega^\infty} \varrho(f^n(x,\omega), f^n(z,\omega))P^\infty(d\omega) \le \lambda^n \varrho(x,z)
\end{aligned}$$

for $x, z \in X$ and $n \in \mathbb{N}$. Passing to the limit we get

$$\int_X u(y)\pi(x,dy) = \int_X u(y)\pi(z,dy) \quad \text{for } x, z \in X.$$

This ends the proof. ∎

Remark 3.1. If (3.3) holds with a $\lambda \in (0, \infty)$ and

$$\int_\Omega \varrho(f(x_0, \omega), x_0)P(d\omega) < \infty \quad \text{for an } x_0 \in X,$$

then we have also (3.4).

4. APPLICATIONS AND EXAMPLES

In what follows π denotes the limit distribution obtained from Theorem 3.1.

Corollary 4.1. *Assume that (X, ϱ) is complete and separable, (3.3) holds with a $\lambda \in (0, 1)$ and (3.4) is satisfied.*

(i) *If $F : X \to \mathbb{R}$ is Borel and bounded, then any continuous and bounded solution $\varphi : X \to \mathbb{R}$ of (1.1) has the form*

$$\varphi(x) = c + \sum_{n=1}^{\infty} \int_{\Omega^\infty} F(f^n(x, \omega))P^\infty(d\omega) + F(x) \quad \textit{for } x \in X \tag{4.1}$$

with a real constant c; in particular, if (1.1) has a continuous and bounded solution $\varphi : X \to \mathbb{R}$, then

$$\lim_{n\to\infty} \int_{\Omega^\infty} F(f^n(x_0, \omega))P^\infty(d\omega) = 0 \tag{4.2}$$

for any $x_0 \in X$.

(ii) *Let $F : X \to \mathbb{R}$ be continuous and bounded. If (1.1) has a continuous and bounded solution $\varphi : X \to \mathbb{R}$, then*

$$\int_X F(y)\pi(dy) = 0; \tag{4.3}$$

in particular, if in addition F is nonnegative, then

$$\pi(F^{-1}(\{0\}) = 1, \tag{4.4}$$

and if F is nonnegative and $F^{-1}(\{0\})$ is a singleton $\{x_0\}$, then $\pi = \delta_{x_0}$,

$$\lim_{n\to\infty} P^\infty(\{\omega \in \Omega^\infty : \varrho(f^n(x, \omega), x_0) \geq \varepsilon\}) = 0 \quad \textit{for } \varepsilon \in (0, \infty) \tag{4.5}$$

and $x \in X$, and this convergence is uniform on every bounded subset of X.

(iii) *If $F : X \to \mathbb{R}$ is bounded,*

$$|F(x) - F(z)| \leq L\varrho(x, z) \quad \textit{for } x, z \in X \tag{4.6}$$

with an $L \in [0, \infty)$, and (4.2) holds for an $x_0 \in X$, then for any $c \in \mathbb{R}$, formula (4.1) defines a solution $\varphi : X \to \mathbb{R}$ of (1.1) and

$$|\varphi(x) - \varphi(z)| \leq \frac{L}{1 - \lambda}\varrho(x, z) \quad \textit{for } x, z \in X. \tag{4.7}$$

Proof. Fix a Borel and bounded $F : X \to \mathbb{R}$ and let $\varphi : X \to \mathbb{R}$ be a continuous and bounded solution of (1.1). It follows from (1.1) and (3.1) that

$$\begin{aligned}\varphi(x) &= \int_{\Omega^\infty} \varphi(f^n(x, \omega))P^\infty(d\omega) + \sum_{k=1}^{n-1} \int_{\Omega^\infty} F(f^k(x, \omega))P^\infty(d\omega) + F(x) \\ &= \int_X \varphi(y)\pi_n(x, dy) + \sum_{k=1}^{n-1} \int_{\Omega^\infty} F(f^k(x, \omega))P^\infty(d\omega) + F(x)\end{aligned}$$

for $x \in X$ and $n \in \mathbb{N}$. Moreover, since $(\pi_n(x,\cdot))$ converges weakly to π,

$$\lim_{n\to\infty} \int_X \varphi(y)\pi_n(x,dy) = \int_X \varphi(y)\pi(dy) \quad \text{for } x \in X.$$

Consequently, for every $x \in X$ the series occurring in (4.1) converges and we have (4.1) with

$$c = \int_X \varphi(y)\pi(dy).$$

Passing to the proof of (ii), assume that F is continuous. Then, as follows from (3.1) and (4.2),

$$\int_X F(y)\pi(dy) = \lim_{n\to\infty} \int_X F(y)\pi_n(x_0,dy) = \lim_{n\to\infty} \int_{\Omega^\infty} F(f^n(x_0,\omega))P^\infty(d\omega) = 0, \tag{4.8}$$

and it remains to consider the case where $F \geq 0$ and $F^{-1}(\{0\}) = \{x_0\}$ with an $x_0 \in X$. In this case (4.4) means that $\pi = \delta_{x_0}$ and applying [1, Prop. 11.1.3] we see that for every $x \in X$ the sequence $(f^n(x,\cdot))_{n\in\mathbb{N}}$ converges to x_0 in probability, i.e., (4.5) holds. To show that the convergence in (4.5) is uniform on bounded subsets of X, it is enough to observe that on making use of the Markov inequality (see, e.g., [6, Sec. 9.3.A]) and (3.7) for every $\varepsilon \in (0,\infty)$, $x \in X$ and $n \in \mathbb{N}$ we get

$$\begin{aligned}
&P^\infty(\{\omega \in \Omega^\infty :\ \varrho(f^n(x,\omega),x_0) \geq \varepsilon\}) \\
&\leq P^\infty\left(\left\{\omega \in \Omega^\infty : \varrho(f^n(x,\omega), f^n(x_0,\omega)) \geq \frac{\varepsilon}{2}\right\}\right) \\
&\quad + P^\infty\left(\left\{\omega \in \Omega^\infty :\ \varrho(f^n(x_0,\omega),x_0) \geq \frac{\varepsilon}{2}\right\}\right) \\
&\leq \frac{2}{\varepsilon}\int\limits_{\Omega^\infty} \varrho(f^n(x,\omega), f^n(x_0,\omega))P^\infty(d\omega) + P^\infty\left(\left\{\omega \in \Omega^\infty :\ \varrho(f^n(x_0,\omega),x_0) \geq \frac{\varepsilon}{2}\right\}\right) \\
&\leq \frac{2}{\varepsilon}\lambda^n\varrho(x,x_0) + P^\infty\left(\left\{\omega \in \Omega^\infty :\ \varrho(f^n(x_0,\omega),x_0) \geq \frac{\varepsilon}{2}\right\}\right).
\end{aligned}$$

To prove (iii), define $M : X \to [0,\infty)$ by

$$M(x) = (L + \|F\|_\infty)\frac{1}{1-\lambda}\int_\Omega \varrho(f(x,\omega),x)P(d\omega) \tag{4.9}$$

and observe that by applying (3.1) and (4.2) we have (4.8). Hence (4.3) holds and taking into account (3.1), (4.3), (4.6), (3.5) and (4.9) we see that

$$\begin{aligned}
\left|\int\limits_{\Omega^\infty} F(f^n(x,\omega))P^\infty(d\omega)\right| &= \left|\int\limits_X F(y)\pi_n(x,dy) - \int\limits_X F(y)\pi(dy)\right| \\
&\leq (L + \|F\|_\infty)\|\pi_n(x,\cdot) - \pi\|_W \leq M(x)\lambda^n
\end{aligned} \tag{4.10}$$

for $x \in X$ and $n \in \mathbb{N}$. This shows that for every $x \in X$ the series in (4.1) converges. Fix a $c \in \mathbb{R}$ and define $\varphi : X \to \mathbb{R}$ by (4.1). Making use of (4.6) and (3.7) we easily get (4.7).

It remains to show that φ solves (1.1). To this end, note that by applying (4.1) and (4.10) we have

$$|\varphi(x)| \leq |c| + \|F\|_\infty + \frac{\lambda}{1-\lambda}M(x) \quad \text{for } x \in X.$$

Moreover, according to the Fubini theorem, the function M given by (4.9) is Borel and an obvious application of (3.3), (3.4) and (4.9) gives

$$M(x) \leq c_1\varrho(x,x_0) + c_2 \quad \text{for } x \in X$$

with some constants $c_1, c_2 \in [0, \infty)$. Consequently, taking (3.4) and (4.7) into account, we obtain in turn the integrability of $M \circ f(x, \cdot)$ and of $\varphi \circ f(x, \cdot)$ for every $x \in X$. Finally, making use of (4.10), the integrability of $M \circ f(x, \cdot)$ and the Lebesgue dominated convergence theorem we see that

$$\int_{\Omega} \left(\sum_{n=1}^{\infty} \int_{\Omega^\infty} F(f^n(f(x, \omega_0), \omega_1, \omega_2, \ldots)) P^\infty(d(\omega_1, \omega_2, \ldots)) \right) P(d\omega_0)$$
$$= \sum_{n=1}^{\infty} \int_{\Omega} \left(\int_{\Omega^\infty} F(f^n(f(x, \omega_0), \omega_1, \omega_2, \ldots)) P^\infty(d(\omega_1, \omega_2, \ldots)) \right) P(d\omega_0)$$
$$= \sum_{n=1}^{\infty} \int_{\Omega^\infty} F(f^{n+1}(x, \omega)) P^\infty(d\omega)$$

whence

$$\int_{\Omega} \varphi(f(x, \omega)) P(d\omega) = c + \sum_{n=1}^{\infty} \int_{\Omega^\infty} F(f^{n+1}(x, \omega)) P^\infty(d\omega) + \int_{\Omega} F(f(x, \omega)) P(d\omega)$$
$$= c + \sum_{n=1}^{\infty} \int_{\Omega^\infty} F(f^n(x, \omega)) P^\infty(d\omega) = \varphi(x) - F(x)$$

for every $x \in X$. ∎

Example 4.1. *Let $\xi : \Omega \to \mathbb{R}$ be an integrable random variable, fix an $\alpha \in (-1, 1)$ and consider the rv–function $f : \mathbb{R} \times \Omega \to \mathbb{R}$ given by*

$$f(x, \omega) = \alpha x + \xi(\omega).$$

According to Theorem 3.1 for every $x \in \mathbb{R}$ the sequence $(f^n(x, \cdot))_{n \in \mathbb{N}}$ of its iterates converges in law and the limit distribution π is independent of x. Note that if ξ is not a.s. constant, then this sequence does not converge in probability. In fact, if $x \in \mathbb{R}$, then for every $n \in \mathbb{N}$ we have

$$f^n(x, \cdot) = \alpha f^{n-1}(x, \cdot) + \xi_n,$$

where

$$\xi_n(\omega_1, \omega_2, \ldots) = \xi(\omega_n) \quad \text{for } (\omega_1, \omega_2, \ldots) \in \Omega^\infty. \tag{4.11}$$

Hence, supposing that $(f^n(x, \cdot))_{n \in \mathbb{N}}$ converges in probability we obtain the convergence in probability of $(\xi_n)_{n \in \mathbb{N}}$. Since it is a sequence of independent and identically distributed random variables, this implies that they are a.s. constant.

It follows from Corollary 4.1(i) that every continuous and bounded solution $\varphi : \mathbb{R} \to \mathbb{R}$ of the equation

$$\varphi(x) = \int_{\Omega} \varphi(\alpha x + \xi(\omega)) P(d\omega) \tag{4.12}$$

is a constant function. Observe, however, that if $\alpha \in \mathbb{Q} \setminus \{0\}$ and $\xi(\Omega) \subset \mathbb{Q}$, then $\mathbb{1}_{\mathbb{Q}}$ is a (bounded and nonconstant) solution of (4.12).

The following modification of [3, Example 2.7] by R. Kapica and J. Morawiec shows that the assumption on the boundedness of solutions also cannot be omitted in Corollary 4.1(i).

Example 4.2. *Let* p_1, p_2 *be positive reals with* $p_1 + p_2 = 1$ *and let* L_1 *be a real number such that*

$$p_1 L_1^2 < 1 \quad \text{and} \quad p_1|L_1| + \big(p_2(1 - p_1 L_1^2)\big)^{1/2} < 1.$$

Putting $\Omega = \{1, 2\}$ *and* $P(\{j\}) = p_j$ *for* $j \in \{1, 2\}$, *consider the* rv*–function* $f : \mathbb{R} \times \Omega \to \mathbb{R}$ *defined by*

$$f(x, j) = L_j x,$$

where

$$|L_2| = \big((1 - p_1 L_1^2)/p_2\big)^{1/2}.$$

Then

$$\int_\Omega |f(x,\omega) - f(z,\omega)| P(d\omega) = (p_1|L_1| + p_2|L_2|)|x - z| \quad \text{for } x, z \in \mathbb{R},$$

$$p_1|L_1| + p_2|L_2| = p_1|L_1| + \big(p_2(1 - p_1 L_1^2)\big)^{1/2} < 1,$$

$$\int_\Omega |f(0,\omega)| P(d\omega) = 0$$

and (1.1) takes the form

$$\varphi(x) = p_1\varphi(L_1 x) + p_2\varphi(L_2 x) + F(x). \tag{4.13}$$

Since

$$p_1 L_1^2 + p_2 L_2^2 = 1,$$

the function $x \mapsto x^2, x \in \mathbb{R}$, *solves (4.13) with* $F = 0$.

Note that in the case considered we have $f(0,\omega) = 0$ *for* $\omega \in \Omega$, *whence also* $f^n(0,\omega) = 0$ *for* $\omega \in \Omega^\infty$ *and* $n \in \mathbb{N}$. *Consequently, any* $F : \mathbb{R} \to \mathbb{R}$ *vanishing at zero satisfies (4.2) with* $x_0 = 0$. *According to Corollary 4.1(iii) for any Lipschitzian, bounded and vanishing at zero* $F : \mathbb{R} \to \mathbb{R}$ *equation (4.13) has a Lipschitzian solution* $\varphi : \mathbb{R} \to \mathbb{R}$.

We end with an example showing that (3.3) with $\lambda = 1$ (and (3.4)) does not force the convergence in law of $(f^n(x,\cdot))_{n\in\mathbb{N}}$.

Example 4.3. *Let* $\xi : \Omega \to \mathbb{R}$ *be an (integrable) random variable and consider the* rv*–function* $f : \mathbb{R} \times \Omega \to \mathbb{R}$ *given by*

$$f(x,\omega) = x + \xi(\omega).$$

Then

$$f^n(x,\cdot) = x + \sum_{k=1}^n \xi_k \quad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{N}, \tag{4.14}$$

where $(\xi_n)_{n\in\mathbb{N}}$ *is defined by (4.11). Fix an* $x \in \mathbb{R}$. *We will show that* $(f^n(x,\cdot))_{n\in\mathbb{N}}$ *converges in law if and only if* $\xi = 0$ *a.s.*

Denote by φ_n *the characteristic function of* $f^n(x,\cdot)$ *and by* φ *the characteristic function of* ξ. *According to (4.14) and (4.11) we have*

$$\varphi_n(t) = e^{itx}\varphi(t)^n \quad \text{for } t \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

Hence, assuming that $(f^n(x,\cdot))_{n\in\mathbb{N}}$ *converges in law, we see that the sequence of powers* $(\varphi^n)_{n\in\mathbb{N}}$ *converges pointwise to a continuous function mapping* $\mathbb{R}$ *into* $\mathbb{C}$. *Consequently (cf.* [6, Sec. 14.1]*)* ξ *is a.s. constant, which jointly with (4.11) and (4.14) gives* $\xi = 0$ *a.s.*

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